COST Action IC1205
Summer School on Computational Social Choice
Miramar Palace, San Sebastián, 18-22 July 2016

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1) Using Gale-Shapley algorithm find the man-optimal and the women optimal stable matching for the following instance of SM:

$P(m_1):$	w_3, w_4, w_2, w_5, w_1	$P(w_1):$	m_1, m_3, m_4, m_5, m_2
$P(m_2):$	w_1, w_4, w_2, w_3, w_5	$P(w_2):$	m_2, m_3, m_5, m_4, m_1
$P(m_3):$	w_3, w_4, w_1, w_2, w_5	$P(w_3):$	m_5, m_2, m_4, m_1, m_3
$P(m_4):$	w_1, w_3, w_2, w_5, w_4	$P(w_4):$	m_4, m_2, m_3, m_1, m_5
$P(m_5):$	w_1, w_4, w_5, w_2, w_3	$P(w_5):$	m_1, m_2, m_5, m_3, m_4

2) Consider the man-optimal stable matching in an instance of SM.

   a) Prove that at most one man is matched to his least preferred woman.

   b) Is it possible that no man is matched to his most preferred woman?

3) Prove that it is impossible in an instance of SM with n men and n women ($n \geq 3$) that its $n!$ perfect matchings are stable.

4) For the given instance of SMTI find all weakly stable matchings. Does this instance admit a super stable matching?

$P(m_1):$	$(w_4, w_1), w_3, w_2$	$P(w_1):$	(m_1, m_2)
$P(m_2):$	w_2, w_1	$P(w_2):$	m_1, m_4, m_2
$P(m_3):$	w_1, w_4, w_2, w_3	$P(w_3):$	m_1, m_2
$P(m_4):$	$(w_4, w_1), w_2$	$P(w_4):$	m_3, m_1

5) Find the resident and hospital optimal stable matchings for the following instance of HR. The capacities of hospitals are in brackets.

(3)	$P(h_1):$	r_3, r_5, r_4	$P(r_1):$	h_3, h_4, h_2
(2)	$P(h_2):$	r_1, r_4, r_2, r_3, r_6	$P(r_2):$	h_3, h_2, h_4
(1)	$P(h_3):$	r_6, r_1, r_2	$P(r_3):$	h_4, h_2, h_1
(1)	$P(h_4):$	r_2, r_3, r_1	$P(r_4):$	h_1, h_4, h_2
			$P(r_5):$	h_1, h_4, h_2, h_3
			$P(r_6):$	h_2, h_3

6) Show that the following instances of HR with couples do not admit a stable matching. Capacities of hospitals are in brackets.

a)

(1) $P(h_1):$	r_1, r_3
(1) $P(h_2):$	r_3, r_2
$P(r_1, r_2):$	(h_1, h_2)
$P(r_3):$	h_1, h_2

b)

(2) $P(h_1):$	r_1, r_3, r_2
$P(r_1, r_2):$	(h_1, h_1)
$P(r_3):$	h_1

7) Find a feasible matching for the following instance of the two-semester hospitals/residents problem with 7 residents, three medical and three surgical units. Preferences of all players, schedule constraints of residents and capacities of units (in both semesters) are given below.

$(2+1)$	$M_1 : r_3, r_2, r_5, r_4, r_1$	$(1+1)$	$S_1 : r_2, r_5, r_6, r_3$
$(1+0)$	$M_2 : r_4, r_1, r_5, r_3$	$(2+1)$	$S_2 : r_1, r_5, r_2$
$(0+2)$	$M_3 : r_1, r_4, r_5, r_2$	$(1+0)$	$S_3 : r_3, r_5, r_1, r_6, r_2$

$r_1 :$ M_1, M_2, M_3	$r_1 :$ S_3, S_2	$r_1 :$ SM
$r_2 :$ M_3, M_1	$r_2 :$ S_3, S_1, S_2	$r_2 :$ not defined
$r_3 :$ M_1, M_2	$r_3 :$ S_1, S_3	$r_3 :$ MS
$r_4 :$ M_1, M_2, M_3	$r_4 :$ not defined	$r_4 :$ $M-$
$r_5 :$ M_2, M_1	$r_5 :$ S_2, S_3, S_1	$r_5 :$ MS
$r_6 :$ not defined	$r_6 :$ S_1, S_2, S_3	$r_7 :$ $-S$

8) Research problem: Imagine that the residents have to spend **three** trimesters in a practical placement, say in a medical, surgical and pediatric unit. How would you find a feasible matching that does not assign any student to two different units in the same trimester?

9) Find a feasible matching in the following instance of the teachers allocation problem with 6 teachers and 4 schools. The subjects are Mathematics, English, Biology and Informatics.

(ME)	$a_1 :$	s_3, s_4
(MB)	$a_2 :$	s_1, s_4
(BE)	$a_3 :$	s_3, s_1, s_2
(MI)	$a_4 :$	s_3, s_4
(BI)	$a_5 :$	s_1, s_3, s_4
(MI)	$a_6 :$	s_2, s_4

	M	E	I	B
$s_1 :$	1	1	0	1
$s_2 :$	0	2	1	1
$s_3 :$	2	1	1	0
$s_4 :$	1	1	1	1

10) An instance of the stable roommates problem SR consists of a set of players A and preference profile. Any player can be matched to any other player, stability definition is the same as in SR. Show that the following instance does not admit any stable matching.

$$\begin{aligned} P(a_1) &: \ a_2, a_3, a_4 \\ P(a_2) &: \ a_3, a_1, a_4 \\ P(a_3) &: \ a_2, a_3, a_4 \\ P(a_4) &: \ a_1, a_2, a_3 \end{aligned}$$

11) An **exchange blocking pair** for a matching μ is a pair of players (a, a') such that a prefers $\mu(a')$ to $\mu(a)$ and vice versa, a' prefers $\mu(a)$ to $\mu(a')$. A matching is **exchange stable** if it admits no exchange blocking pair.

   $a)$ Show that the following instance of SM admits two stable matchings, but no exchange stable one.

$$\begin{aligned} P(m_1) &: \ w_1, w_2 \qquad & P(w_1) &: \ m_2, m_1 \\ P(m_2) &: \ w_2, w_1 \qquad & P(w_2) &: \ m_1, m_2 \end{aligned}$$

b) Suppose that in an instance of SM with n men and n women the preference list of each man has the form $w_1, w_2, \ldots, w_n$ and the preference list of each woman is $m_1, m_2, \ldots, m_n$. Show that each perfect matching is exchange stable.

A matching in an instance of SM is **men-exchange** stable if it admits no exchange blocking pair consisting of two men.

c) Show that if the preference lists in an instance of SM are consistent then there is at least one men exchange stable matching and it can be found by Serial dictatorship of men.

d) Consider the following instance of SM with inconsistent preference list. Show that there is no men exchange stable feasible perfect matching. (It is supposed that men can exchange their partners even if the concerned women do not agree. In fact, deciding the existence of a men exchange stable matching in this case is NP-complete, see Cechlárová and Manlove 2005.)

$P(m_1):$	w_1, w_2, w_3, w_4	$P(w_1):$	m_4, m_1
$P(m_2):$	w_2, w_1, w_3	$P(w_2):$	m_1, m_3
$P(m_3):$	w_1, w_4, w_2	$P(w_3):$	m_2, m_1, m_3
$P(m_4):$	w_2, w_3, w_4	$P(w_4):$	m_1, m_2, m_4

Solutions.

2a) Let m_1 be the first man who proposes to the last woman in his list. At the moment of this proposal, all the $n-1$ other women in his list are already engaged with $n-1$ men different from m_1. All these men are engaged to a better than their last woman. so in fact this is the last proposal during the Gale-Shapley algorithm and the engaged pairs become marriage partners.

2b) Yes, see the following preference profile. This is incorrect. Argument for No: take the bipartite graph where (m, w) is an edge iff w is the first choice of m. Of the men adjacent ot a woman w her partner in μ_M is the one whom she prefers most.

$$
\begin{array}{llll}
P(m_1): & w_1, w_2, w_3 & P(w_1): & m_3, m_2, m_1 \\
P(m_2): & w_1, w_3, w_2 & P(w_2): & m_1, m_3, m_2 \\
P(m_3): & w_2, w_1, w_3 & P(w_3): & m_1, m_2, m_3
\end{array}
$$

(3)	$P(h_1):$	r_3, r_6, r_5, r_4	$P(r_1):$	h_3, h_4, h_2
(2)	$P(h_2):$	$r_1, r_4, r_6, r_2, r_7, r_3$	$P(r_2):$	h_3, h_2, h_4
(1)	$P(h_3):$	r_7, r_1, r_2	$P(r_3):$	h_4, h_2, h_3, h_1
(1)	$P(h_4):$	r_2, r_3, r_6, r_4	$P(r_4):$	h_1, h_4, h_2
			$P(r_5):$	h_1, h_4, h_2, h_3
			$P(r_6):$	h_2, h_4, h_1
			$P(r_7):$	h_2, h_3

(3)	$P(h_1):$	r_3, r_6, r_5, r_4	$P(r_1):$	h_3, h_4, h_2
(2)	$P(h_2):$	$r_1, r_4, r_6, r_2, r_7, r_3$	$P(r_2):$	h_3, h_2, h_4
(1)	$P(h_3):$	r_7, r_1, r_2	$P(r_3):$	h_4, h_2, h_3, h_1
(1)	$P(h_4):$	r_2, r_3, r_6, r_4	$P(r_4):$	h_1, h_4, h_2
			$P(r_5):$	h_1, h_4, h_2, h_3
			$P(r_6):$	h_2, h_4, h_1
			$P(r_7):$	h_2, h_3

(3)	$P(h_1):$	r_3, r_6, r_5, r_4	$P(r_1):$	h_3, h_4, h_2
(2)	$P(h_2):$	$r_1, r_4, r_6, r_2, r_7, r_3$	$P(r_2):$	h_3, h_2, h_4
(1)	$P(h_3):$	r_7, r_1, r_2	$P(r_3):$	h_4, h_2, h_3, h_1
(1)	$P(h_4):$	r_2, r_3, r_6, r_4	$P(r_4):$	h_1, h_4, h_2
			$P(r_5):$	h_1, h_4, h_2, h_3
			$P(r_6):$	h_2, h_4, h_1
			$P(r_7):$	h_2, h_3