Cost Allocation

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Introduction – General Aspects

What are typical fair division problems?

land division

cake cutting

cost/surplus sharing

dividing sets of items

Fairness – General Aspects

"Equals should be treated equally, and unequals unequally, in proportion to relevant similarities and differences." (Aristoteles – Nicomachean Ethics)

- Minimal Fairness-Test ("equal treatment of equals")
  - two individuals with same characteristics in all dimensions relevant to the allocation problem at hand, should receive the same treatment (i.e. the same share in whatever is distributed)
  - treating unequal individuals unequally is a vague principle
- **4 elementary principles of distributive justice**
  - compensation
  - reward
  - exogenous right
  - fitness

Plato's story about the flute that has to be given to one of 4 children.

Fairness – General Aspects

- **Procedural Justice**
  - If the procedure is fair, the outcome is fair!

> *"… pure procedural justice obtains when there is no independent criterion for the right result: instead there is a correct or fair procedure such that the outcome is likewise correct or fair, whatever it is, provided that the procedure has been properly followed. This situation is illustrated by gambling. If a number of persons engage in a series of fair bets, the distribution of cash after the last bet is fair, or at least not unfair, whatever this distribution is."*
>
> (John Rawls, A Theory of Justice, 1971)

- **Endstate Justice**
  - focus on the outcome of the procedure
  - consequences important, but not necessarily the properties of the procedure
    - collective welfare approach with benevolent dictator (e.g. state)

Fairness – General Aspects

- What is to be divided?
  - costs, cakes, indivisible goods, etc.
  - possible restriction, e.g. in form of network structures, etc.
- What do agents' preferences look like?
  - depends on the information acceptable in the division process
  - claims, rankings of items, cardinal value functions, etc.
- How are we dividing? What do we want to achieve?
  - define rules of a fair division procedure
    - what are the informational and/or computational requirements
  - what properties do such procedures satisfy
    - used to define fairness

Formal Framework

- formal structure (sharing fixed costs/resources)
  - set of n agents, N
  - resource (or cost), r
  - claims vector, $x = (x_1,\dots,x_n)$
  - sharing problem: (r,x)
    - $r \leq \sum_{i \in N} x_i$ or $r \geq \sum_{i \in N} x_i$
    - sharing a deficit or surplus
  - A procedure/rule F assigns to each fair division problem (r,x) a solution F(r,x)=y, where $y = (y_1,\dots,y_n)$ with
    $r = \sum_{i \in N} y_i$
- applications
  - bankruptcy
  - rationing problems
  - mergers

Example

- 2 agents: Anna (Piano) and Bob (Violin)
- stand-alone salary: x_A = 100000; x_B = 50000
- a joint net revenue of r = 210000 possible
  - how should they share the surplus?

- 3 major division rules
  - **proportional rule**
  - **constrained equal-awards rule**
  - **constrained equal-losses rule**

Major Rules

- proportional rule, P

  Given fair division problem (r, x), $y_i = \frac{x_i}{x_N} r$, where $x_N = \sum_{i \in N} x_i$.

- constrained equal awards rule, CEA

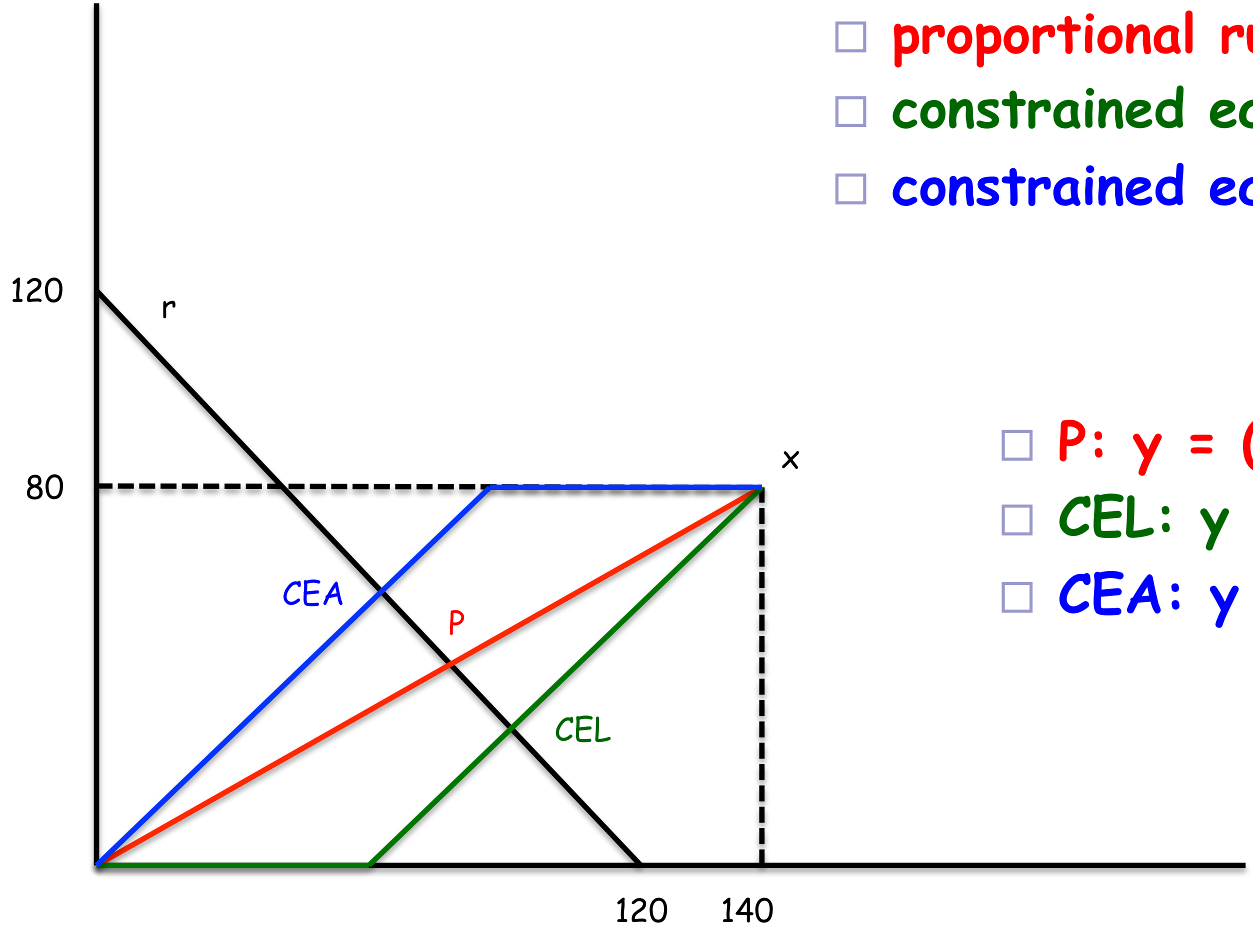
  Given (r, x), $y_i = \min\{\lambda, x_i\}$, where $\sum_{i \in N} \min\{\lambda, x_i\} = r$.

- constrained equal losses rule, CEL

  Given (r, x), $y_i = \max\{0, x_i - \lambda\}$, where $\sum_{i \in N} \max\{0, x_i - \lambda\} = r$.

Example

- x = (140,80); r = 120



- proportional rule
- constrained equal-losses rule
- constrained equal-awards rule

- P: y = (76.4,43.6)
- CEL: y = (90,30)
- CEA: y = (60,60)

Major Rules - Algorithms

how do you calculate the solutions?

- algorithm for CEA
  - divide r in equal shares – identify agents whose claims are on the "wrong" side of r/n, i.e., $x_i \leq r/n$ (in the deficit case).
  - give those agents their claim, decrease the resource accordingly, and repeat among remaining agents

- algorithm for CEL
  - use formula $y_i = x_i + \frac{1}{n}(r - x_N)$
  - identify agents with $y_i \leq 0$, assign 0 to them, repeat algorithm among remaining agents

- numerical example: |N|= 5; x = (20, 16, 10, 8, 6)
  - r = 50
  - CEA: y = (13,13,10,8,6)
  - CEL: y = (18,14,8,6,4)

Major Properties of Rules

- how "good" are the above rules?
- use axiomatic approach

- equal treatment of equals

For each (r, x). If, for any $i, j \in N$, $x_i = x_j$, then $F_i(r, x) = F_j(r, x)$.

- minimal rights first

For each (r, x), $F(r, x) = m(r, x) + F(r - \sum_{i \in N} m_i(r, x), x - m(r, x))$.

where $m_i(r, x) = \max\{0, r - \sum_{j \neq i} x_j\}$

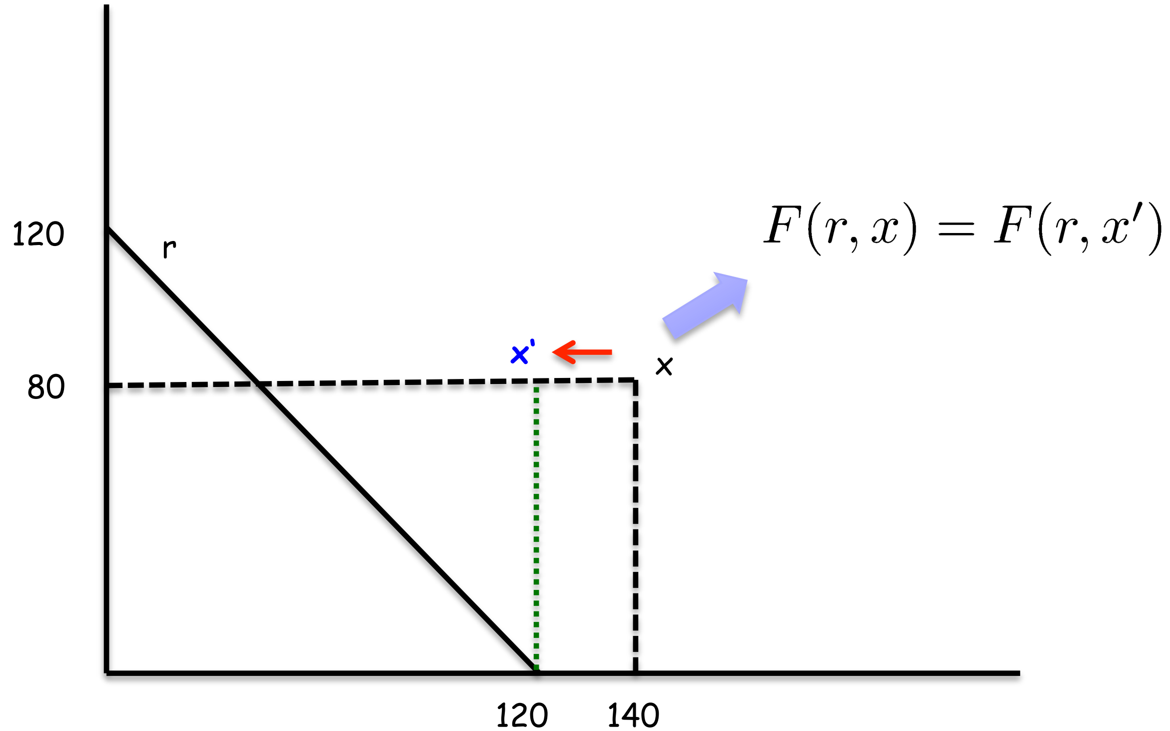
  - how much others concede to a player
  - what are the minimal rights for x = (100,50) and r = 90?

Major Properties of Rules

- invariance under claims truncation

For each (r, x), $F(r, x) = F(r, (\min\{r, x_i\})_{i \in N})$.

  - any claim above the amount to be divided should be ignored.

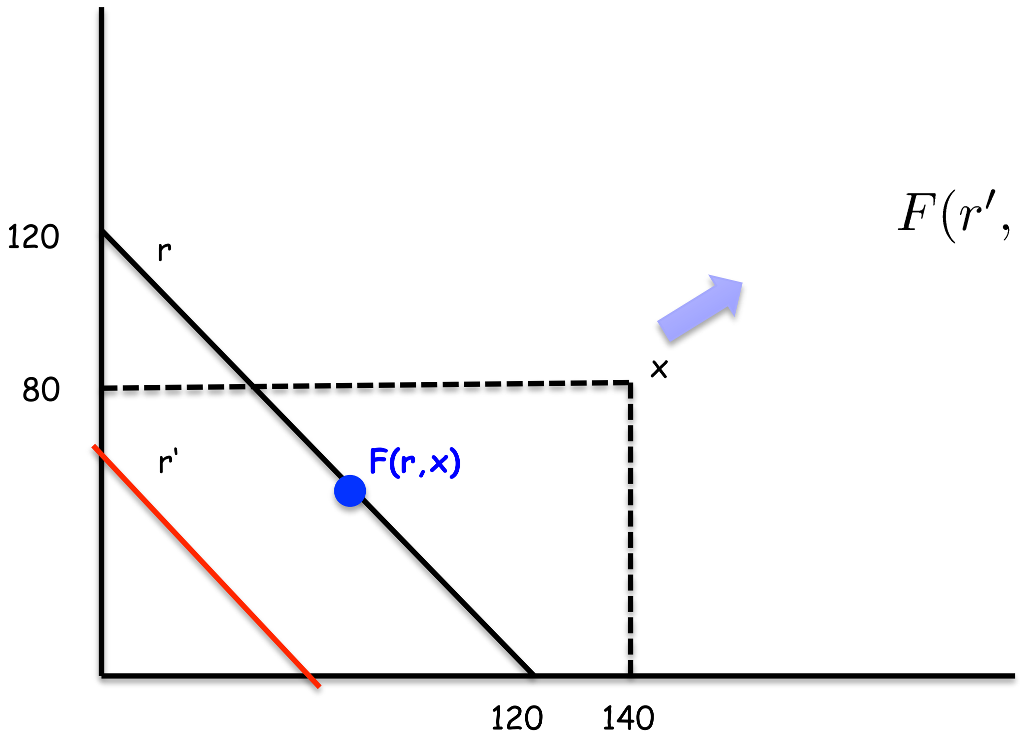


Major Properties of Rules

- composition down

For each (r, x) and each $r' < r$, $F(r', x) = F(r', F(r, x))$.

  - if resource allocation has been made, but resource decreases before final allocation, it is irrelevant whether original claims or previous allocation is used.



$$F(r', x) = F(r', F(r, x))$$

Major Properties of Rules

- composition up

  For each (r, x) and each $r' > r$. If $x_N \geq r' > r$, then $F(r', x) = F(r, x) + F(r' - r, x - F(r, x))$.

  - if resource allocation has been made, but resource increases before final allocation, it is irrelevant whether original claims are used or previous allocation is implemented and remaining resource distributed according to adjusted claims.

- no advantageous transfer

  For each (r, x), each $M \subset N$ and each $(x'_i)_{i \in M}$. If $\sum_M x_i = \sum_M x'_i$, then $\sum_M F_i(r, x) = \sum_M F_i(r, (x'_i)_{i \in M}, x_{N \setminus M})$.

  - no group of agents receives more by transferring claims among themselves
  - no merging – no splitting

Major Properties of Rules

- many other properties used in the literature
  - monotonicity properties
    - what happens if resource or claims change?
  - independence, additivity
    - minimal rights, merging fair division problems
  - variable population properties
    - consistency

For each pair (N, N') such that $N' \subset N$ and each (r, x), if $y = F(r, x)$, then $y_{N'} = F(\sum_{N'} y_i, x_{N'})$.

    - if rule is applied and some agents leave with their shares, by re-evaluating the situation from the viewpoint of the remaining agents, the rule should award to each of them the same amount as it did initially
    - important property (see Thomson, 2011)

Characterization Results

- previous properties used to characterize rules

The proportional rule is the only rule satisfying no advantageous transfer. (Moulin, 1985)

The constrained equal-losses rule is the only rule satisfying equal treatment of equals, minimal rights first and composition down. (Herrero, 2001)

The constrained equal-awards rule is the only rule satisfying equal treatment of equals, invariance under claims truncation and composition up. (Dagan, 1996)

- however, many other characterization results, using other properties, possible

Other Interesting Rules

- many rules discussed in the Talmud

- **contested garment rule**
  - for n = 2: each gets concessions, rest is distributed equally

$$y_1 = \tfrac{1}{2}(r + \min\{x_1, r\} - \min\{x_2, r\})$$
$$y_2 = \tfrac{1}{2}(r - \min\{x_1, r\} + \min\{x_2, r\})$$

- Example: r = 120; x = (140,80)
  - concessions: (40,0)
  - allocation: (80,40)

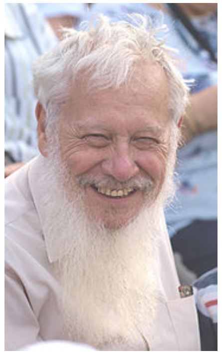
Other Interesting Rules

- **random-priority-rule**
  - randomly order the individuals and let them take from r until r = 0
  - do this for all possible orders and take the average for each i
  - Example: r = 120; x = (140,80)

Other Interesting Rules

Claims \ Estate	100	200	300
100	33 1/3	50	50
200	33 1/3	75	100
300	33 1/3	75	150

- **Talmud-rule**
  - order according to claims, $x_1 \leq x_2 \leq \ldots \leq x_n$
  - share r equally until ind. 1 gets $x_1/2$
    - eliminate ind. 1
  - share equally until ind. 2 gets $x_2/2$
    - eliminate ind. 2
    - etc.
  - if each has received half of claim and $r - x_N/2 > 0$, continue with increase of share for ind. n up to $x_n - y_n = x_{n-1} - y_{n-1}$.
  - etc.
  - Aumann and Maschler (1985)



Robert Aumann

Fairness - Algorithms

- division of variable costs/resources
  - cost/resource determined by individual demands
  - e.g. division of costs of a common facility determined by individual demands
    - cost function: $c(\sum_{i \in N} x_i)$
- **Average-cost method**

$$F_i^{ac}(c,x) = \frac{x_i}{x_N} c(x_N)$$

- costs shared proportional to individual demands
- example: |N|=3; x = (1,2,3); z = $x_1 + x_2 + x_3$; c(z) = max{0, z-4}
- y = (1/3, 2/3, 1)
- is the division fair according to the average-cost method?

Fairness - Algorithms

- **Serial cost-sharing method**
  - order $x_1 \leq x_2 \leq \ldots \leq x_n$ and define
  - $x^1 = nx_1$, $x^2 = x_1 + (n-1)x_2$; …; $x^i = (n-i+1)x_i + \sum_{j=1}^{i-1} x_j$
  - cost-shares are:

$$F_1^s(c,x) = \frac{c(x^1)}{n};\ F_2^s(c,x) = F_1^s(c,x) + \frac{c(x^2)-c(x^1)}{n-1};$$

$$F_i^s(c,x) = F_{i-1}^s(c,x) + \frac{c(x^i)-c(x^{i-1})}{n-i+1}$$

- example: $|N|=3$; $x = (1,2,3)$; $z = x_1 + x_2 + x_3$; $c(z) = \max\{0, z-4\}$
- $y = (0, 1/2, 3/2)$
  - ind. with smallest demand prefers serial-cost to average-cost method if marginal costs are increasing
- vice versa with decreasing marginal costs
- e.g. $c'(z) = \min\{z/2, 1 + z/6\}$

Coalitional Games

- A **coalitional game** (cooperative game) is a model of interacting decision-makers with a focus on the behavior of groups of players
  - a set of actions for every group of players
  - and not only for individual players as so far
  - every group of players is called coalition
  - the coalition of ALL players is the grand coalition
- The outcome of a coalitional game consists of a *partition of the players into groups together with an action for each group*
  - often each coalition is associated with a single number
    - interpreted as the payoff
    - which can usually be freely divided among the members of the coalition
      - transferable payoff

Coalitional Games

- Definition: A coalitional game with transferable payoff consists of
  - finite set N of players
  - characteristic function v assigning to every coalition S (subset of N) a real number v(S), the total payoff available to S

- models especially situations in which the actions of the players not in S have no influence on v(S)

- Property: A coalitional game (N,v) is cohesive if

$$v(N) \geq \sum_{k=1}^{K} v(S_k) \text{ for every partition } \{S_1, ..., S_k\} \text{ of } N$$

  - what does this condition tell us?
  - is a special case of superadditivity

$$v(S \cup T) \geq v(S) + v(T) \text{ for all disjoint } S, T \subset N$$

Coalitional Games

- coalitional game designed to model games in which players are better off forming groups than acting individually
  - often this incentive is extreme in the sense that a grand coalition is formed
  - happens if we have a cohesive game
- Example
  - group of 3 players has access to one unit of a (divisible) good; each majority can control the allocation of this unit
  - N = {1,2,3}
  - v(i) = 0 for i = 1,2,3
  - v(S) = 1 for all other coalitions S
- So what action (allocation, distribution) are we somehow expecting from the grand coalition?
  - one that is stable w.r.t. *pressure imposed by the possibility of forming other coalitions*

The Core

- *idea similar to Nash equilibrium*
  - only that now outcome must be stable w.r.t. deviations of any coalition

<u>Definition</u>: The **core** of a coalitional game (N,v) is the set of all feasible payoff profiles $(x_i)_{i \in N}$ such that there is no coalition S with a payoff profile $(y_i)_{i \in S}$ such that $y_i > x_i$ for all $i \in S$.

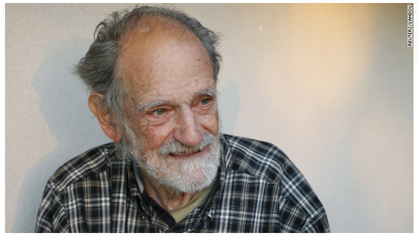
- Equivalently, an allocation $(x_i)_{i \in N}$ is in the core if no coalition S can improve upon it.

The Core

- Example 1
  - N = {1,2,3}
  - v(N) = 1, v(S) = a for |S|=2, and v(i) = 0 for all i ∈ S.
  - what are core allocations? or when do they exist?

- Example 2
  - N = {1,2,3}
  - v(N) = 60; v(i) = 10 for all i ∈ S; v(12) = 30; v(13) = 40; v(23) = 50
  - what are core allocations?

Fairness – Shapley Value



Lloyd Shapley

- **Shapley Value**
  - is an axiomatic solution to a simple model of the commons
  - focus on reward aspect
  - distributional justice needs to correctly evaluate the different production capabilities of the agents
    - **stand-alone-costs/benefits**
- stand alone test
  - C subadditive implies $y_i \leq C(i)$
  - C superadditive implies $y_i \geq C(i)$
- **stand alone core**
  - C subadditive implies $\sum_{i \in S} y_i \leq C(S), \forall S \subseteq N$
  - C superadditive implies $\sum_{i \in S} y_i \geq C(S), \forall S \subseteq N$

Fairness – Shapley Value

- division of costs of a jointly usable good
  - division of costs of building an elevator
  - stand alone costs: $c_1 = 5$, $c_2 = 10$, $c_3 = 40$
  - who should pay how much if they want to build only one elevator?

Fairness – Shapley Value

- how can we think of it formally?
- consider marginal cost/contribution

$$y_i = \sum_{s=0}^{n-1} \sum_{S \in \mathcal{A}_i(s)} \frac{s!(n-s-1)!}{n!} \{C(S \cup \{i\}) - C(S)\}$$

Shapley Value

- Shapley value as expected marginal cost
  - *reward the responsibility of the various agents in the total cost*
  - translates the reward principle into an explicit division of C(N) based on the 2^{n-1} numbers C(S), for all nonempty coalitions

Fairness – Shapley Value

- example
  - C(123) = 36; C(1) = C(2) = 20; C(3) =36; C(12) = 29; C(13) = C(23) = 36
  - what is the core?
  - what is the Shapley value?

$$\begin{matrix} 1 & 1 & 2 & 2 & 3 & 3 \\ 2 & 3 & 1 & 3 & 1 & 2 \\ 3 & 2 & 3 & 1 & 2 & 1 \end{matrix}$$

- P1: 20 + 20 + 9 + 0 + 0 + 0
- P2: 9 + 0 + 20 + 20 + 0 + 0
- P3: 7 + 16 + 7 + 16 + 36 + 36

Fairness – Shapley Value

- example
  - $C(123) = 120$; $C(i) = 60$; $C(12) = 120$; $C(13) = C(23) = 60$
  - what is the core?
  - what is the Shapley value?

$$
\begin{matrix}
1 & 1 & 2 & 2 & 3 & 3 \\
2 & 3 & 1 & 3 & 1 & 2 \\
3 & 2 & 3 & 1 & 2 & 1
\end{matrix}
$$

- Shapley value does not have to lie in the core!
- Shapley value always exists, even if core is empty!

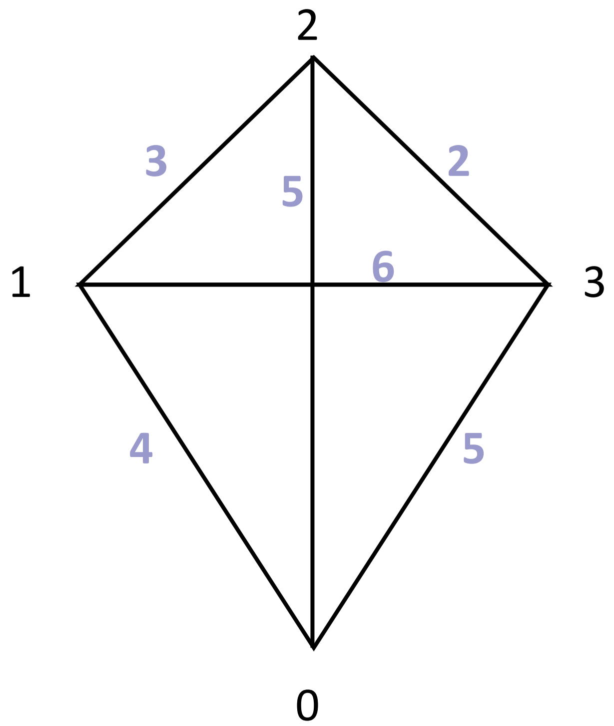
Fairness – Shapley Value (Characterization)

- What is a "good" solution?
  - **axiomatic analysis (characterization)**
- axioms (properties)
  - Equal treatment of equals
    - if i,j are equal relative to (N,C), then $y_i = y_j$
  - Dummy
    - if $C(S \cup \{i\}) - C(S) = 0$ for all S, then $y_i = 0$
  - Additivity
    - assume $C(S) = C^1(S) + C^2(S)$ [e.g. installation- and variable costs], then $y(N,C^1 + C^2) = y(N,C^1) + y(N,C^2)$

Shapley Value is the only solution for cooperative games satisfying equal treatment of equals, dummy and additivity. (Shapley, 1953)

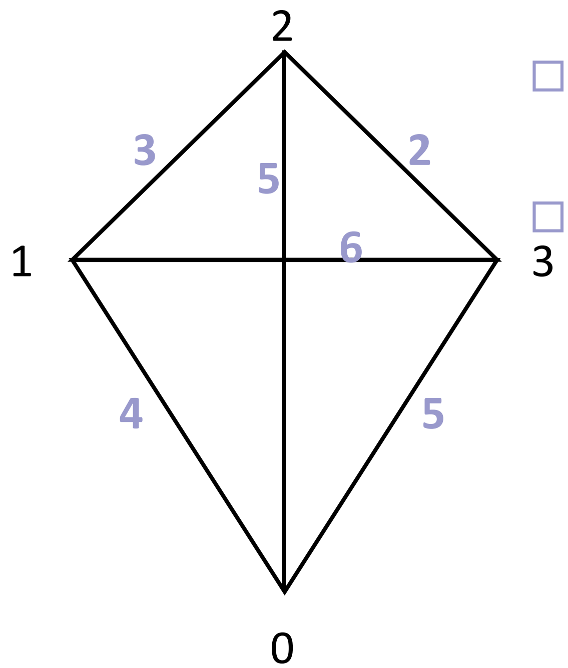
Fairness – Graph Structures

- can analyse different structures
  - e.g. cost sharing in the construction of networks
  - use graph $G(N \cup \{0\}, E)$ and cost function c



- if all nodes have to be connected to source 0, what are the costs and how should they be distributed?

Fairness – Graph Structures

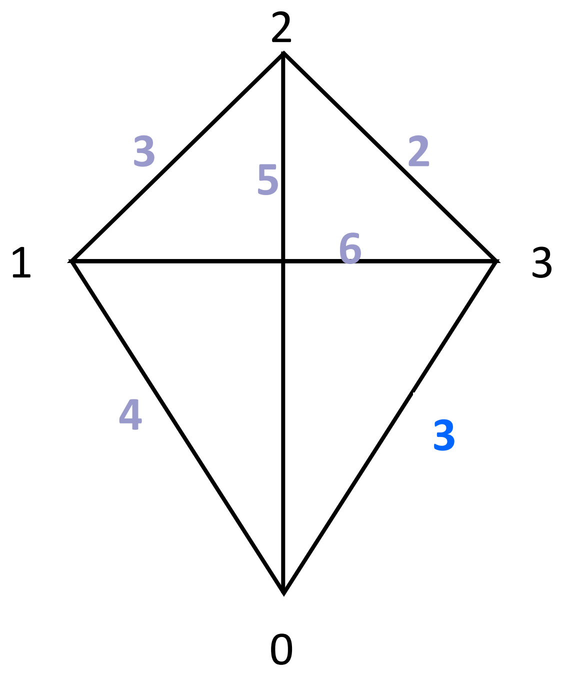


- look for a minimum cost spanning tree (Kruskal)
- possible algorithm: **Bird-Rule** (1976)
  - starting with source, every agent pays cost from predecessor to herself
  - what properties does this rule satisfy?

- reasonable property: **the core**
  - no coalition can block by connecting to the source at lower cost
  - do we always find a core?
  - does the Bird-rule always lie in the core?

Fairness – Graph Structures

- other reasonable property: **cost monotonicity**
  - whenever the cost of only one edge between two agents i and j, c(ij), decreases, then neither i nor j should have a larger cost share in the new network
  - does the Bird rule satisfy this property?



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