Protocols for Allocating Indivisible Goods

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Introduction – General Aspects

What are typical fair division problems?

land division

cake cutting

cost/surplus sharing

dividing sets of items

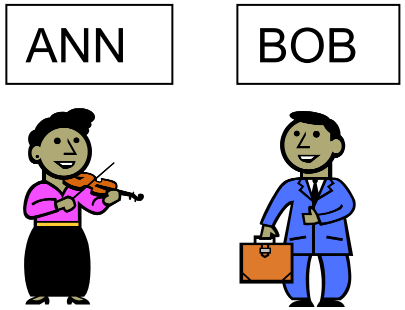
Introduction

Most of the fair division models are built on answers to the following questions:

- What is to be divided?
  - costs, cakes, indivisible goods, etc.
  - possible restriction, e.g. in form of network structures, etc.
- What do agents' preferences look like?
  - depends on the information acceptable in the division process
  - claims, rankings of items, cardinal value functions, etc.
- How are we dividing? What do we want to achieve?
  - define rules of a fair division procedure
  - what properties do such procedures satisfy
    - used to define fairness

Indivisible Goods

Here we consider the problem of **fairly dividing** a set of **indivisible items** between **two (or more) players**.



Examples:

- divorce settlement
- inheritance problems
- allocations of tasks to workers/machines

Outlook

- In this presentation the focus will be on
  - practical protocols/procedures/algorithms
  - fairness properties

  - hence we are not only concerned with the actual allocation by (possibly/hopefully) a benevolent dictator
    - elicitation process might be difficult/expensive
    - agents might be reluctant to accept a solution out of a "black box"
  - but with the design of practical procedures
    - what can actually be achieved by such a procedure?

Formal Framework

- Assume set $\mathcal{O}$ of p items ranked by the players
- $P_i \subseteq \mathcal{O} \times \mathcal{O}$ as player i's strict preference over $\mathcal{O}$
  - ordinal vs. cardinal
- $\mathcal{X}$ denotes the set of all subsets of $\mathcal{O}$
- π(i) as i's bundle or share
- $\succsim_i$ as i's preference over $\mathcal{X}$

- Example:

$$\mathcal{O} = \{1, 2, 3, 4\};\ N = \{A, B\}$$

A	B
1	2
2	3
3	4
4	1

allocation e.g. π(A) = {1,3}; π(B) = {2,4}

Ranking Sets of Items

- But how to compare different sets of items when only a linear order over the set of items is given?

(1) is {1,2} better for A than {3,4}?
(2) is {1,3} better for A than {2,4}?
(3) is {1,4} better for A than {2,3}?

A	B
1	2
2	3
3	4
4	1

- (1) and (2) seem plausible
  - pairwise dominance
- various axiomatic approaches to justify (3) and other comparisons
- Barbera, Bossert & Pattanaik (2004)

Ranking Sets of Items

- Two simple axioms:
- Simple Dominance

$$\text{For all } x, y \in \mathcal{O},$$
$$xPy \Rightarrow [\{x\} \succ \{x,y\} \text{ and } \{x,y\} \succ \{y\}]$$

- Independence

$$\text{For all } S, T \in \mathcal{X}, \text{ for all } x \in \mathcal{O} \backslash (S \cup T),$$
$$S \succ T \Rightarrow S \cup \{x\} \succeq T \cup \{x\}$$

- Result (Bossert, Pattanaik and Xu, 2000):

$$\text{If } \succeq \text{ satisfies simple dominance and independence, then}$$
$$S \sim \{\max(S), \min(S)\} \text{ for all } S \in \mathcal{X}.$$

  - By adding further axioms, certain lexicographic and/or max and min based preferences on set of items can be characterized.

Ranking Sets of Items

- or we could request more information from the agents
  - cardinal approach
- $w_i(x)$ as the value attached to item x by player i
- $w_i(S) = \Sigma_{x \in S}\, w_i(x)$
  - additive preferences
- what would be the ordinal counterpart of this property?

- separability

For every pair of bundles (S, S'), and every bundle S'' such that $(S \cup S') \cap S'' = \emptyset$, we have: $S \succeq S' \Rightarrow S \cup S'' \succeq S' \cup S''$.

Ranking Sets of Items

Example

- x_1 = voucher for flight to Grenoble
- x_2 = voucher for the summer school
- x_3 = camera
- if my preference is $x_1 > x_2 > x_3$ then separability implies that I prefer $\{x_1,x_2\}$ to $\{x_1,x_3\}$
  - seems reasonable
- now what if x_2 = train ticket to Grenoble

- complementarity
  - flight and summer school are complements
- substitutability
  - flight and train are substitutes

- hence additivity/separability implies no synergies

Envy-freeness, Efficiency, Equitability

- How to decide which allocation to choose or which algorithm to use?
  - normative/axiomatic approach
  - what is it that we want to achieve?

A	B
1	2
2	3
3	4
4	1

- envy-freeness (EF)
  - an allocation (π(A),π(B)) is EF if for all i ∈ N, π(i) $\succsim_i$ π(j), for all j ∈ N
  - this is the case for allocation (13,24), i.e., π(A) = {1,3} and π(B) = {2,4}

- efficiency (PO)
  - an allocation (π(A),π(B)) is efficient if there exists no other allocation (π′(A),π′(B)) such that π′(i) $\succsim_i$ π(i) for all i and π′(j) $\succ_j$ π(j) for some j.

- equitability
  - an allocation (π(A),π(B)) is equitable if w_A(π(A)) = w_B(π(B)).
  - uses cardinal information

Envy-freeness, Efficiency, Equitability

- See immediately that most of the standard fairness axioms will not necessarily hold in case of indivisible items

- But let us try to find out what certain procedures are able to achieve when we use certain assumptions.

Adjusted Winner Procedure

- adjusted winner procedure (Brams & Taylor, 1996)
  - one item may have to be divided
  - assign (100) points to items
  - transfer items to equalize sum of points

Basic Steps in the Adjusted Winner Procedure

Step 1. As described earlier, each party distributes 100 points over the items in a way that reflects their relative worth to that party.

Step 2. Each item is initially given to the party that assigned it more points. Each party then assesses how many of his or her own points he or she has received. The party with the fewest points is now given each item on which both parties placed the same number of points.

Step 3. Since the point totals are most likely not equal, let *A* denote the party with the higher point total and *B* be the other party. Start transferring items from *A* to *B*, in a certain order, until the point totals are equal. The point at which equality is achieved may involve a fractional transfer of one item.

Step 4. The order in which this is done is extremely important and is determined by going through the items in order of increasing **point ratio.**

An item's point ratio is the fraction

$$\frac{A\text{'s point value of the item}}{B\text{'s point value of the item}}$$

where *A* is the party with the higher point total.

Adjusted Winner Procedure

- Example

$item$	w_A	w_B
1	5	10
2	25	10
3	35	20
4	15	35
5	20	25

← step 1

- Step 2: A → {2,3}; B → {1,4,5}
  - total points: A → 60; B → 70

- Steps 3/4: point ratio B/A (as B has the higher total so far)

	1	2	3	4	5
$ratio(\frac{w_B}{w_A})$	$\frac{10}{5}$	$\frac{10}{25}$	$\frac{20}{35}$	$\frac{35}{15}$	$\frac{25}{20}$

  - consider the item of B with lowest ratio to be transferred first
    - try to equalize points

$$10 + 35 + 25x = 25 + 35 + 20(1 - x)$$

$x = \frac{7}{9}$ → transfer 2/9 of item 5 to A

Adjusted Winner Procedure

Theorem: The adjusted winner procedure leads to an allocation which is
- envy-free
- efficient (Pareto optimal)
- equitable

- but what if cardinal information is not available and all items are definitely indivisible?
  - Adjusted winner can not be used
  - no simple procedure will guarantee those fairness criteria

- Alternatives?
  - incomplete allocations
  - different fairness criteria

Maximin shares

- maximin shares (Procaccia and Wang, 2014)
  - cut and choose as convincing and simple method in cake cutting
  - but leads to problems in the division of indivisible items with cardinal valuations
  - no guarantee of 1/n – share (proportionality)
  - maximin share as what a player can guarantee herself by dividing the items in n piles

Theorem: There exists an allocation (π(A),π(B),…,π(N)) such that $u_i(\pi(i)) \geq 2/3\ MMS_i$

- can be found in polynomial time
- www.spliddit.org

$item$	w_A	w_B
1	5	10
2	25	10
3	35	20
4	15	35
5	20	25

- if A first: A offers split (34,125) with values 50-50 for A
- B chooses 34 with value 55 for B

- if B first: B has various options, e.g. (14,235) with 45-55
- A chooses 235 with value 80 for A

Minimizing Envy

- Lipton et al. (2014)
  - as envy can not be eliminated in general their focus is on minimizing envy
  - minimum envy
    - $\max_{i,j}\{0, w_i(\pi(j)) - w_i(\pi(i))\}$
  - minimum envy ratio
    - $\max_{i,j}\{1, \frac{w_i(\pi(j))}{w_i(\pi(i))}\}$

- They show that in general it is very difficult to find the exact allocation based on those envy-concepts.

Approximate Fair Allocations

- Graham's algorithm (1969)
  - based on additive utilities
  - all players have the same utility function
  - equivalent to scheduling problem for identical machines

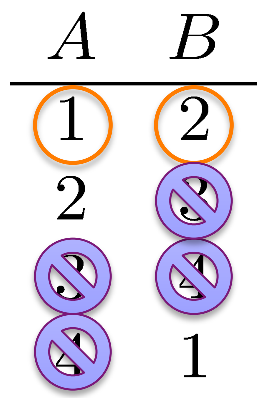
- sort the items in decreasing order of values and allocate them one by one in that order
- allocate the next item to the player whose current value of the bundle is lowest

Theorem: Graham's algorithm achieves an approximation factor of 1.4 for the envy-ratio problem, i.e., is at most 40% above the optimal envy-ratio for any such fair division problem.

Brams & Taylor Procedure (1999)

- now we turn to procedures using only ordinal information
- Consider the following simple procedure (BT-procedure):
  - ask players to name the item they want to have next
  - if they name different items allocate them
  - if they name the same item put it into a contested pile

- Problem: might lead only to partial allocation



- but does it satisfy previous desirable properties?
  - envy-freeness
  - efficiency
  - equitability
  - new property: completeness
  - new property: maximin

Allocation:
$\pi(A) = \{1\}$; $\pi(B) = \{2\}$;
$CP = \{3,4\}$

Envy-freeness

A	B
1	2
2	3
3	4
4	1

- previous definition of envy-freeness
  - an allocation ($\pi(A)$,$\pi(B)$) is EF if for all $i \in N$, $\pi(i) \succsim_i \pi(j)$, for all $j \in N$
  - difficult to apply without detailed ranking information over sets of items
- use a stronger definition
  - as we use no information other than the players' rankings over items

An allocation ($\pi(A)$,$\pi(B)$) is EF iff there exist an injection g_A: $\pi(A) \rightarrow \pi(B)$ and an injection g_B: $\pi(B) \rightarrow \pi(A)$ such that for each $x \in \pi(A)$, $x \succ_A g_A(x)$ and for each $x \in \pi(B)$, $x \succ_B g_B(x)$.

- hence we have EF if there is pairwise dominance (Bouveret, Endriss and Lang (2010))
- implies that the allocation must assign sets of items of equal size
- possible and necessary envy-freeness
- in above example: (13,24) is necessarily EF whereas (14,23) is possibly EF

Completeness

- when can we be sure that a complete EF allocation (π(A),π(B)) does exist, i.e., all items can be allocated in an envy-free way?

Condition C(k): A set consisting of i's k-most preferred items is equal to the set consisting of j's most preferred items.

  - only concerned with equality of sets not with their rankings
  - it will be important whether this condition holds for odd k

Condition D: Condition C(k) fails for all odd values of k ≤ p.

A	B
1	2
2	4
3	6
4	1
5	3
6	5

k=1: {1} vs {2}
k=3: {1,2,3} vs {2,4,6}
k=5: {1,2,3,4,5} vs {2,4,6,1,3}

Maximin

one fairness condition could be to maximize the lowest ranked item any player receives in the allocation

Depth of complete allocations
The depth of any complete allocation is the rank of the least preferred item assigned to either player.

A	B
1	8
2	7
3	6
4	1
5	2
6	3
7	4
8	5

$(\pi(A),\pi(B)) = (1347,8625)$ => depth 8
$(\pi(A),\pi(B)) = (1456,8723)$ => depth 6

maximin depth: least integer f such that every item is ranked f^{th} or higher => $f = 5$

Lemma: Any complete allocation has a depth of at least f.

Singles - Doubles

Let f < n. An item is called single if it is a top f item for only one player. It is called double if it is a top f item for both players.

singles: {3,4,5} for A; {6,7,8} for B
doubles: {1,2}

A	B
1	8
2	7
3	6
4	1
5	2
6	3
7	4
8	5

it follows that a player's top f items must include p – f singles and 2f – p doubles

singles can now be used to simplify the test for complete EF allocations!

Condition DS

Condition C(k): A set consisting of i's k-most preferred items is equal to the set consisting of j's most preferred items.

Condition DS: Condition C(k) fails for all odd values of k < s, where s is the rank where the first single appears in either player's ranking.

A	B
1	8
2	7
3	6
4	1
5	2
6	3
7	4
8	5

- k=1: {1} vs {8}
- k=3: {1,2,3} vs {8,7,6}
- singles: {3,4,5,6,7,8}
- the first single that appears is item 8 in ranking B; s=1
- condition DS is satisfied
- can be used to determine the existence of complete EF-MX allocations

Definitions/Results

Theorem (Brams, Kilgour & Klamler, 2015): Assume A and B strictly rank p items, where p is even. Then the following are equivalent:

1. Condition D holds.
2. Condition DS holds.
3. There exists a complete EF-MX allocation.

Corollary: If A's and B's rankings of an even number of items admit a complete EF allocation, then they admit a complete EF-MX allocation.

Singles-Doubles-Algorithm

(assume condition D being satisfied)

SD-Algorithm

Input: A's and B's rankings of p > 0 items, where p is even.
Output: A complete MX-EF allocation.

1. Determine *f*.
2. Identify A's singles, and assign them to A. Identify B's singles, and assign them to B. Stop if all items have been allocated.
3. Assign doubles using the following iterative procedure: identify each player's most preferred unassigned double. If they are different, assign them accordingly. If they are the same, identify the player who can be assigned its second-most preferred unassigned double guaranteeing EF, and assign the items accordingly (multiple outcomes possible). Repeat until all doubles are assigned.

EF identified by checking for any depth k whether $|\pi^k(i)| \geq \frac{k}{2}$

Example

Consider the following preferences:

A	B
1	8
2	7
3	6
4	1
5	2
6	3
7	4
8	5

* there are six singles, {3,4,5} for A and {6,7,8} for B
* the two doubles are {1,2}
* as they are ranked the same we have to identify whether we can assign item 2 to one of the players and still satisfy EF; possible for player B
* final allocation: {1,3,4,5} to A and {2,6,7,8} to B

Iterated SD-algorithm

Do not use the singles-assignment stage once but repeat it as long as possible.

A	B
1	8
2	7
3	6
4	3
5	2
6	1
7	5
8	4
9	0
0	9

* assign singles 9 to A and 0 to B

A	B
1	8
2	7
3	6
4	3
5	2
6	1
7	5
8	4

* assign singles {1,5,4} to A and {6,7,8} to B

A	B
2	3
3	2

* assign singles 2 to A and 3 to B

Final ISD allocation: (12459, 87630)

SD-allocation: (12349,87650)

(I)SD-Algorithm

The (I)SD-algorithm finds at least one PO allocation.

The (I)SD-algorithm is vulnerable to manipulation by a player's misrepresenting its sincere preference ranking.

- but: hardly any algorithm would be strategy-proof
  - dictator rule
  - constant rule
  - mechanism design

Sequential Procedures

- picking sequences (Bouveret and Lang, 2011)
  - use only partial elicitation of players' preferences
  - define a sequence for the n players
  - e.g. for N = {A,B,C} and p = 5 a possible sequence is ABCCB.
    - very simple to implement (even for many players)
    - little information necessary
    - very low complexity
  - (as usual our question is) what is the fairest sequence?
    - full independence of preferences (impartial culture assumption)
    - agents have additive utilities, where utilities are based on the same scoring function but rankings might be different
    - arbitrator does not know the agents' preferences but has probability distribution on possible profiles
  - goal is to maximize expected collective utility
    - what is this?

Sequential Procedures

- utilitarianism
  - maximize sum of Borda scores
- egalitarianism
  - maximize the minimum Borda score of an agent

A	B
1	1
2	2
3	3
4	4
5	5
6	6

utilitarian: π(A) = {1,3,5}
→ BS = 9; π(B) = {2,4,6}
→ BS = 6

egalitarian: π(A) = {1,3,6}
→ BS = 8; π(B) = {2,4,5}
→ BS = 7

egalitarian	utilitarian
ABBA	ABAB
ABABBA	ABABAB
ABBABAAB	ABABABAB
ABBAABABBA	ABABABABAB
ABABABABBABA	ABABABABABAB

- comparisons w.r.t. envy based on Borda scores could also be made
  - e.g. for 6 items the sequence ABBABA creates (on average) slightly less instances of envy than the egalitarian optimum

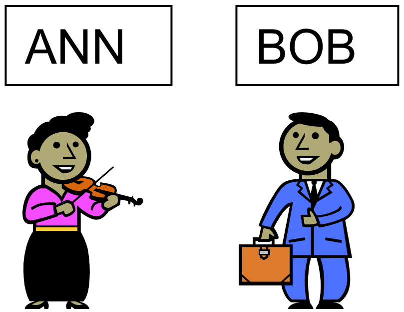
Descending Demand Procedure

- descending demand procedure (Herreiner and Puppe, 2002)
  - players rank all their bundles
    - only monotonicity assumption used
  - descend in their rankings until PO and set-maximin-optimal allocation is found
    - idea of fallback bargaining (Brams and Kilgour, 2001)
  - does not guarantee EF but produces "balanced" allocations
    - natural counterpart of egalitarian social welfare
  - but: is ranking all bundles realistic?
    - 10 items lead to 1023 non-empty sets to be ranked

A	B
1234	1234
124	123
134	234
14	23
1	134
.	.

Contested Piles

- now what if complete EF allocation is not possible according to previous procedures
  - items might be in a contested pile or unassigned
  - is there still hope for EF?
- what could we do with such items in the contested pile?
- before going in detail, consider the ultimatum game of dividing a single divisible good



  - in a first step Ann suggests a division of the dollar to Bob which he can accept (hence division implemented) or reject (no payoff to either player).
    - what is the rational proposal by Ann?
  - now – in a second stage – allow Bob to undercut Ann's proposal by 1 cent and implement the resulting division
    - what will Ann do in the first stage given this additional step?

Contested Pile

- are we able to divide the items in a contested pile even if both players rankings of the items are the same?
  - as is the case e.g. for the items in contested pile of BT-procedure
  - conflict often occurs when items are ranked the same

Is there a fair division procedure that leads to an envy-free division?
(at least under certain restrictions)

Definitions

- Preference $\succeq$ on $\mathcal{X}$ satisfies responsiveness if for all $S \in \mathcal{X}$ and all $x \in S$ and $y \in O \setminus S$

$$xRy \Leftrightarrow S \succ S \setminus \{x\} \cup \{y\} \quad \text{and} \quad S \succ S \setminus \{x\}$$

A
1
2
3
4
5

→ $\{1,2,5\} \succ \{2,3,5\}$

- Let $S,T \in \mathcal{X}$. T is said to be ordinally less than S, denoted by $T \leq_{OL} S$, if there exists an injective function $\sigma_{T,S}: T \setminus S \rightarrow S \setminus T$ such that for all $x \in T \setminus S$, $\sigma_{T,S}(x)Px$.

Definitions

- $S \in \mathcal{X}$ is a **minimal bundle** for player i if $S \succeq_i -S$ and, for any $T \leq_{OL} S$, it holds that $-T \succ_i T$
- Player i regards set $S \in \mathcal{X}$ as worth at least 50 percent if $S \succeq_i -S$

➔ Hence, a player regards a subset S as a **minimal bundle** if S is worth at least 50 percent AND any subset T that is ordinally less than S is worth less than 50 percent.

if {1,3,5} is a minimal bundle, then {1,4,5} must be worth less than 50%

A
1
2
3
4
5
6

- For any $S \in \mathcal{X}$, the split (S,-S) is envy-free if $S \succeq_A -S$ and $-S \succeq_B S$
- An envy-free split of X, (S,-S), is trivial if $S \sim_A -S$ and $-S \sim_B S$

Undercut Procedure

Undercut Procedure (Brams, Kilgour and Klamler, 2012)

- Players state items, if different, assign them, if the same, put in CP
- Given the CP, players state their sets of minimal bundles (MB_i)
- $MB_A \neq MB_B$: randomly choose a player (say player A) and let her propose a minimal bundle $S \in MB_A$ such that $S \notin MB_B$
- $MB_A = MB_B$: if there exists an S such that S, -S $\in MB_i$ then S becomes the proposal; if no such minimal bundle exists, choose one at random as the proposal
- Given the proposal, the other player (say player B) can either
  - accept the complement of the proposal, or
  - reject and undercut, i.e., take a set which is ordinally less than the proposal in which case its complement is assigned to the other player

- for other contested pile procedures see Vetschera & Kilgour (2014)

Undercut Procedure

A	B
1	1
2	2
3	3
4	4
5	5

- assume $\{1,2\} \in MB_A$ and $\{2,3,4,5\} \in MB_B$ but not vice versa
- assume A makes the proposal

- A proposes {1,2}
- B can do the following
  - accept: she gets {3,4,5}
  - undercut: she takes {1,3} and A gets {2,4,5}
- B: {3,4,5} must be worth less than 50% as $\{2,3,4,5\} \in MB_B$
- B: {1,3} worth at least 50% as {2,4,5} ordinally less than {2,3,4,5}
- A: {2,4,5} worth at least 50% as $\{1,2\} \in MB_A$ and therefore {1,3} less than 50% which makes the complement {2,4,5} worth more than 50%
- allocation ({2,4,5},{1,3}) is envy-free

Undercut Procedure - Result

Theorem

There is a nontrivial envy-free split of the contested pile if and only if one player has a minimal bundle that is not a minimal bundle of the other player. If so, UP implements an envy-free split.

- Different sets of MBs necessary, otherwise more information about players' preferences required.
- Definition (extension monotonicity):

Preference $\succsim$ on $\mathcal{X}$ satisfies extension monotonicity if for all $S, T \in \mathcal{X}$, all preferences P, and all $x, y \in \mathcal{O} \backslash (S \cup T)$, $S \succsim T$ and xPy imply $S \cup \{x\} \succ T \cup \{y\}$.

Proposition

Given responsive and extension monotonic preferences of the players and an envy-free division of the contested pile, the final division of O under UP and any previous procedure is envy-free.

Feasible Subsets

- recall that items in the contested pile are ranked the same by the players
- Question: Can we identify ALL possible **envy-free splits** of items when rankings are the same?
- Definition (feasibility):

A set S is feasible if there exists a responsive preference $\succsim$ such that $S \succ -S$

Feasible Subsets

- Example: $O = \{1,2,3,4,5\}$.
  - {1,2}, {3,4,5} are feasible sets
  - but not {4,5}

- We can guarantee an envy-free split if both, a subset and its complement are feasible.

- Some feasible sets may be part of envy-free splits, but others are not because their complements are not feasible.

p=1 ➔ no envy-free split possible
p=2 ➔ no envy-free split possible (exception indifference)
p=3 ➔ the only possible envy-free split is 1/23
p=4 ➔ the possible splits are 1/234 and 14/23

A	B
1	1
2	2
3	3
.	.
.	.
p	p

Feasible Subsets

- Let for any positive integer k, $\mathcal{O}^k = \{1,2,\ldots,k\}$. The following theorem gives a *necessary and sufficient condition* for *S being feasible*.
- Theorem:

Let $S \subseteq \mathcal{O}$ and $S \neq \emptyset$. Then S is feasible if and only if $|\mathcal{O}^k \cap S| > \frac{k}{2}$ for some $k = 1, 2, \ldots, p$.

Example: S={2,3}

$$k = 1 : \{1\} \cap \{2,3\} = \emptyset$$

$$k = 2 : \{1,2\} \cap \{2,3\} = \{2\}$$

$$k = 3 : \{1,2,3\} \cap \{2,3\} = \{2,3\}$$

$$k = 4 : \{1,2,3,4\} \cap \{2,3\} = \{2,3\}$$

A
1
2
3
4

Feasible Subsets

- Brams and Fishburn (2000) derived the following formula for h(p), the number of possible envy-free splits (S,-S) of a CP with p items:

$$h(p) = \begin{cases} 2^{p\text{-}1} - \binom{p}{(p\text{-}1)/2} & \text{if p is odd} \\ 2^{p\text{-}1} - \binom{p}{p/2} & \text{if } p \text{ is even.} \end{cases}$$

- This number increases exponentially in p

p	1	2	3	4	5	6	7	8	9
EF-splits	0	0	1	2	6	12	29	58	130

Properties of UP - manipulability

Proposition

A player's (say A) **lexicographic maximin strategy** under UP is to name all his or her minimal bundles. Whatever B's strategy, A's worst outcome (in expectation) is never lower than her worst outcome for any other strategy, and in case that $MB_A \neq MB_B$, A cannot receive less than 50 percent from CP by proposing a minimal bundle, whereas proposing a non-minimal bundle may lead to her receiving less that 50 percent.

Example:

- CP={1,2,3,4,5}
- MB_A={1,2}; MB'_A={1,2,5}; MB_B={2,3,4,5}

- A's offer of a split ({1,2,5},{3,4}) will be undercut by B's counterproposal of ({3,4,5},{1,2}).
- As {1,2} was in MB_A, {3,4,5} must be less than 50 percent to A.

Properties of UP - efficiency

- Can envy-free splits under UP fail to be Pareto-optimal?
  - if we assume cardinal information

Items:	1	2	3	4	5	6
A's utility:	27	26	15	13	11	8
B's utility:	30	19	15	13	12	11

- A's minimal bundles:
  - {1,2} (value 53), {1,3,6} (value 50), {2,4,5} (value 50)
- B's minimal bundles:
  - {1,5,6} (value 53), {3,4,5,6} (value 51)
- If A is proposer: split ({1,2},{3,4,5,6}) (value 53:51)
- Pareto-preferred split ({2,3,4},{1,5,6}) (value 54:53)
- Envy-freeness comes at an efficiency cost.

Properties of UP – size of CP

Expected size of CP

- p items, A's strict ranking is 1>2>…>p
- B's p! possible rankings are assumed to be equiprobable (compare to Impartial Culture assumption in voting)
- for p=1, this single item must be in the CP, hence c(1)=1
- for p=2, B's preferences are either $1P_B2$ or $2P_B1$ each with probability 0.5. If it is $2P_B1$, A gets item 1 and B item 2, if it is $1P_B2$ it is the same as A's ranking and both items go into the contested pile. The expected number therefore is c(2)=1.

Properties of UP - size of CP

Theorem 3:

If $k \geq 1$, then

$$c(2k+1) = c(2k+2) = 1 + \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{2k+1}$$

k=0	c(1)	c(2)	1
k=1	c(3)	c(4)	1.33
k=2	c(5)	c(6)	1.533
k=3	c(7)	c(8)	1.676
k=4	c(9)	c(10)	1.787

Selected Literature

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