Strategic Social Choice

Hans Peters

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Overview of the course

Part I Classical results and domain restrictions

Part II Probabilistic approaches and minimal manipulability

Part III Voting equilibria

Part I: Classical results and domain restrictions

I.1 The basic model

I.2 Classical results: Arrow and Gibbard-Satterthwaite

I.3 Domain restrictions: single-peaked preferences

I.4 Domain restrictions: single-dipped preferences

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No agent's preference between a_1 and a_4 has changed, but society's preference has!

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Agent 1 prefers a_5 over a_3 and thus has successfully manipulated!

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- A simple and elegant proof of the theorem jointly with the Theorem of Gibbard and Satterthwaite (later), can be found in:
  Reny (2001) Arrow's theorem and the Gibbard-Satterthwaite theorem: a unified approach. Economics Letters 70:99–105
- See also Chapter 11 in:
  Peters (2008) Game theory: a multi-leveled approach. Springer, Berlin Heidelberg

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- Ethical reasons

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Consider the following condition within the same framework:

- F is (*Maskin*) *monotonic* if for all $R^N, Q^N \in L^N$ such that $F(R^N)R^i a \Rightarrow F(R^N)Q^i a$ for all $a \in A$, we have $F(Q^N) = F(R^N)$

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- Muller E, Satterthwaite MA (1977) The equivalence of strong positive association and strategy-proofness. Journal of Economic Theory 14:412–418
- Reny (2001) and Chapter 11 in Peters (2008)

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In what follows we restrict attention to social choice functions (as opposed to social welfare functions)

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The idea of considering 'single-peaked' preferences goes back to at least:

Black D (1948) On the rationale of group-decision-making. Journal of Political Economy 56:23–34

Formal treatment

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- Strategy-proofness and Pareto optimality of F are defined as before

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(a) F is peaks-only, anonymous and strategy-proof if and only if there are $b^0, \ldots, b^n \in A$ with $b^n \leq \ldots \leq b^0$ such that for every profile $R^N \in \mathcal{S}^N$, we have

$$F(R^N) = \mathrm{median}\{x^1, \ldots, x^n, b^0, \ldots, b^n\}$$

where x^i is the peak of R^i for each $i \in N$.

Theorem (Moulin 1980)

Let $F : \mathcal{S}^N \to A$ be a social choice function.

(a) F is peaks-only, anonymous and strategy-proof if and only if there are $b^0, \ldots, b^n \in A$ with $b^n \leq \ldots \leq b^0$ such that for every profile $R^N \in \mathcal{S}^N$, we have

$$F(R^N) = \mathrm{median}\{x^1, \ldots, x^n, b^0, \ldots, b^n\}$$

where x^i is the peak of R^i for each $i \in N$.

(b) F is peaks-only, anonymous, strategy-proof, and Pareto optimal if and only if there are $b^1, \ldots, b^{n-1} \in A$ with $b^{n-1} \leq \ldots \leq b^1$ such that for every profile $R^N \in \mathcal{S}^N$, we have

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- For an extension to a continuum of agents (voters) with $A = [0,1] \subseteq \mathbb{R}$ see Maus et al (2006). There a fixed ballot takes the form of a (decumulative) distribution function
- The theorem can be extended to non-anonymous social choice functions: then we have a fixed ballot for every coalition, and coalitions of the same size can have different fixed ballots

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Theorem: Border and Jordan 1983

The social choice function $F : \mathcal{Q}^N \to A$ is strategy-proof and unanimous if and only if there are peaks-only strategy-proof unanimous social choice functions $F_1, \ldots, F_k : \mathcal{S} \to \mathbb{R}$ such that $F(R^N) = (F_1(R_1^N), \ldots, F_m(R_k^N))$ for every $R^N \in \mathcal{Q}^N$.

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- What can we get under Pareto optimality? We consider a further domain restriction

Let $\widetilde{\mathcal{Q}}$ be the set of all Euclidian preferences on $A = \mathbb{R}^k$ (e.g., circular if $k = 2$)

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Let $F : \widetilde{\mathcal{Q}}^N \to A$ be a social choice function.

(a) Let $k = 2$ and let n be odd. Then F is anonymous, Pareto optimal, and strategy-proof, if and only if there are orthogonal axes in $\mathbb{R}^2$ such that $F(R^N) = (F_1(R_1^N), F_2(R_2^N))$, where $F_j(R_j^N)$ is the median of the peaks of the profile R_j^N induced by R^N on axis $j = 1, 2$.

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- References: Kim and Roush (MASS 1984); Peters et al (IJGT 1992)

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- F is *dictatorial* if there is an agent $i \in N$ (the *dictator*) such that F assigns to each profile some point at maximal distance from agent i's dip

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- See: Öztürk et al (ET 2014)

Part II: Probabilistic approaches and minimal manipulability

II.1 Decision schemes and random dictatorship

II.2 Decision schemes and single-peaked preferences

II.3 Cardinally strategy-proof decision schemes

II.4 Minimal manipulability

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- $\varphi : \mathcal{L}^N \to L(A)$ is a *decision scheme* (probabilistic social choice function)
- φ is *ordinally strategy-proof* (OSP) if for each $R^N \in L^N$, each $i \in N$, each function $u^i : A \to \mathbb{R}$ representing R^i, and each $Q^N \in L^N$ with $Q^j = R^j$ for all $j \in N \setminus \{i\}$, we have $Eu^i(\varphi(R^N)) \geq Eu^i(\varphi(Q^N))$. Here $Eu^i(\cdot)$ denotes expected utility

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- In other words, if agent i deviates to Q^i, then the result is a lottery which is (weakly) stochastically dominated by the lottery obtained when i is truthful

- φ is *ex post Pareto optimal* if, for all $R^N \in L^N$ and all $x, y \in A$ such that xR^iy for every $i \in N$, $\varphi(R^N)(y) = 0$ (i.e., the probability assigned to y by the lottery $\varphi(R^N)$ is zero)

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Theorem: Random dictatorship

Let $\varphi : \mathcal{L}^N \to L(A)$ be an ordinally strategy-proof and ex post Pareto optimal decision scheme. Then φ is a random dictatorship.

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- A decision scheme φ is a *random dictatorship* if there are probabilities $\lambda_1, \ldots, \lambda_n$ such that for each $R^N \in L^N$ and each $x \in A$, we have $\varphi(R^N)(x) = \sum_{i \in N(x)} \lambda_i$, where $N(x) = \{i \in N : xR^iy \text{ for all } y \in A\}$

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Proved in:

Gibbard A (1977) Manipulation of schemes that mix voting with chance. Econometrica 45:665–681

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- However, consider a profile of the kind

R^1	a	d	$\cdots$	$(bc\ldots)$
R^2	b	d	$\cdots$	$(ac\ldots)$
R^3	c	d	$\cdots$	$(ab\ldots)$
$\cdots$	$\cdots$	d	$\cdots$	$\cdots$

Then d seems to be a better compromise than a random dictatorship

II.2 Decision schemes and single-peaked preferences

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  there is an $a_p \in A$ (the *peak*) such that $a_p P^i a_\ell R^i a_k$ whenever $p > \ell > k$ or $p < \ell < k$
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Let $\varphi : \mathcal{S}^N \to L(A)$ be a decision scheme

- Ordinal strategy-proofness of φ was defined above
- Peaks-onliness of φ is defined in the obvious way

Theorem: decision schemes and single-peaked preferences

Every peaks-only and ordinally strategy-proof decision scheme $\varphi : \mathcal{S}^N \to \mathcal{L}$ is a probability mixture of peaks-only and strategy-proof deterministic social choice functions

- These deterministic social choice functions follow from Moulin (1980), discussed in part I

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- For instance, that question is open for the case that A is not a finite set but a real interval or the real line

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- A *fixed probabilistic ballot* (over the extended real line) $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ is a probability distribution (measure) on $\bar{\mathbb{R}}$

- A collection of fixed probabilistic ballots $(D_S)_{S \subseteq N}$ is *admissible* if
  - $D_\emptyset(\{-\infty\}) = 0$
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- With an admissible collection of fixed probabilistic ballots $\Delta = (D_S)_{S \subseteq N}$ we associate a decision scheme Φ^Δ as follows
- Let $R^N \in \mathcal{L}^N$ with distinct peaks $p^1 < \ldots < p^k$ and associated coalitions $S_j = \{i \in N \mid \text{peak}(R^i) \leq p^j\}$. So $\emptyset =: S_0 \subsetneq S_1 \subsetneq \ldots \subsetneq S_{k-1} \subsetneq S_k = N$. Also let $p^0 = -\infty$ and $p^{k+1} = +\infty$

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  - on each peak p^{ℓ} for $\ell = 1, \ldots, k$ the probability distribution μ puts $D_{S_{\ell}}([-\infty, p^{\ell}]) - D_{S_{\ell-1}}([-\infty, p^{\ell}))$

Theorem: OSP decision schemes on the real line

The decision schemes Phi^{Δ} for Δ an admissible collection of fixed probabilistic ballots, are the only decision schemes for single-peaked preference profiles on the real line that are ordinally strategy-proof and peaks-only. Moreover, the collections Δ are uniquely determined.

(Ehlers et al, JET 2002)

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- A decision scheme $\varphi : U^N \to L(A)$ assigns to each n-tuple a lottery on A
- φ is *cardinally strategy-proof* (CSP) if for each $u^N \in U^N$, each $i \in N$, and each $v^N \in U^N$ with $v^j = u^j$ for each agent $j \neq i$, we have $Eu^i(\varphi(u^N)) \geq Eu^i(\varphi(v^N))$

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- Further see Barberà et al (MASS 1998) and Nandeibam (RED 2013)

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- References include Kelly (SCW 1988, 1989); Maus et al (JME 2007, JET 2007); Campbell and Kelly (ET 2009); Diss et al (2010); Gehrlein and Lepelley (JME 1998); Pritchard and Wilson (MASS 2009); Peters et al (SCW 2012); Arribillaga and Massó (2014)

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- Hence, F is strategy-proof iff $M_F = \emptyset$
- F is *almost dictatorial* if there is (i) an agent $d \in N$, (ii) a profile Q^N, and (iii) an alternative x with xQ^iz for all $i \neq d$ where z is the top alternative of Q^d, such that

$$F(R^N) = \begin{cases} \text{top alternative of } R^d & \text{if } R^N \neq Q^N \\ x & \text{if } R^N = Q^N \end{cases}$$

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This result was proved by Maus et al (JME 2007), building on earlier partial results by Kelly (SCW 1988) and Fristrup and Keiding (SCW 1998)

Anonymous, surjective and peaks-only social choice functions

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- A social choice function F is a *unanimity rule with status quo* if there is a fixed alternative $a \in A$ (the status quo) such that $F(R^N) = a$ unless R^N is a unanimous profile (i.e., $R^1 = \ldots = R^n$): in that case $F(R^N)$ is the top alternative of (each) R^i

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Theorem: Anonymity, surjectivity, peaks-onliness

Let $n > m \geq 3$. Let $F : \mathcal{L}^N \to A$ be anonymous, surjective and peaks-only. Then $|M_F| \leq |M_G|$ for all anonymous, surjective and peaks-only social choice functions G if and only if F is a unanimity rule with status quo.

See Maus et al (JET 2007). Unanimity with status quo is applied in the European union!

Some remarks on approval voting

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- *k-approval voting* means that each agent approves of exactly k alternatives ($k = 1, \ldots, m-1$): this is an example of a *scoring rule*
- If the number of agents n becomes large, then $k \approx m/2$ approval voting is minimally manipulable among *all* scoring rules (Pritchard and Wilson, MASS 2009; Peters et al, SCW 2012)

Part III: Voting equilibria

III.1 Exactly and strongly consistent social choice functions

III.2 Effectivity functions and Nash consistent representation

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- Given a profile $R^N \in \mathcal{L}^N$, we can regard (F, R^N) as an ordinal noncooperative game with player set N, strategy set $\mathcal{L}$ for each player $i \in N$, outcome $F(Q^N)$ for each strategy profile Q^N, evaluated by each player i according to R^i

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- Then strategy-proofness of F is equivalent to the statement that R^N is a Nash equilibrium in (F, R^N) for each $R^N \in \mathcal{L}^N$
- If F is manipulable (not strategy-proof) then we could impose the weaker requirement that there should be a Nash equilibrium $Q^N \in \mathcal{L}^N$ in the game (F, R^N) such that $F(Q^N) = F(R^N)$, for each $R^N \in \mathcal{L}^N$

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- A *social choice correspondence* is a map $H : \mathcal{L}^N \to P_0(A)$, where $P_0(A)$ is the set of all nonempty subsets of A.

Example $A = \{a, b, c\}$; $N = \{1, \ldots, 5\}$; $\beta(a) = \beta(b) = \beta(c) = 2$. Consider R^N in the following table.

R^1	R^2	R^3	R^4	R^5
b	c	a	c	a
c	b	b	a	c
a	a	c	b	b

Then there exist two f.e.p.'s: $(a, \{1, 2\}; b, \{4, 5\}; c)$ and $(b, \{4, 5\}; a, \{1, 2\}; c)$. So $M(R^N) = \{c\}$.

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Then $M(R'^N) = \{b, c\}$.

Theorem: selections from M and ESC

Let the social choice function F be a selection from M, i.e., $F(R^N) \in M(R^N)$ for each $R^N \in \mathcal{L}^N$. Then F is ESC.

- See Peleg (1978) or Peleg and Peters (2010)
- A selection F from M is also Pareto optimal
- An anonymous selection F can easily be constructed (for instance, select from M according to a fixed ordering $R \in \mathcal{L}$)

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A strong Nash equilibrium Q^N for R^N with $F(Q^N) = c$ is:

Q^1	Q^2	Q^3	Q^4	Q^5
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c	c	a	c	a
a	a	b	b	c

(This is also the basic idea of the proof of the theorem)

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- With a surjective social choice function $F : \mathcal{L}^N \to A$ we associate an *effectivity function* $E^F : P(N) \to P(P_0(A))$ as follows: for every $S \in P_0(N)$

$$B \in E^F(S) \Leftrightarrow \exists R^S \in \mathcal{L}^S \, [F(R^S, Q^{N \setminus S}) \in B \; \forall Q^{N \setminus S} \in \mathcal{L}^{N \setminus S}]$$

and $E^F(\emptyset) = \emptyset$

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- Observe: $E^F(N) = P_0(A)$ by surjectivity of F
- Let $n + 1 \geq m$ and let $\beta(x)$ be positive integer weights with $\sum_{x \in A} \beta(x) = n + 1$. We define an *effectivity function* $E_\beta : P(N) \to P(P_0(A))$ as follows: for every $S \in P(N)$

$$B \in E_\beta(S) \Leftrightarrow |S| \geq \sum_{x \notin B} \beta(x)$$

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Theorem: more on ESC social choice functions

Let $n+1 \geq m$. The following statements are equivalent:

(a) $F : \mathcal{L}^N \to A$ is an ESC social choice function with anonymous E^F such that $E^F(\{i\}) = \{A\}$ for each $i \in N$

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- See Polishchuk (1978), Peleg (1984), Peleg and Peters (SCW 2006), Peleg and Peters (2010)

Computation of R^N-maximal alternatives

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- **Lemma** $x \in M(R^N)$ if and only if there exist pairwise disjoint coalitions $S(y)$, $y \in A \setminus \{x\}$, such that
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- Difference with *implementation*: for implementation the representation issue is not important, but all "equilibria" should result in the "desired" payoff

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- Consider the game form

$$\begin{array}{cc} & \begin{array}{cc} b & w \end{array} \\ \begin{array}{c} b \\ w \end{array} & \begin{pmatrix} (b,b) & (b,w) \\ (w,b) & (w,w) \end{pmatrix} \end{array}$$

where person 1 is the row player and person 2 the column player

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- "Rights" (effectiveness) of the individuals:
  Person 1 : $\{(b, b), (b, w)\}$, $\{(w, b), (w, w)\}$
  Person 2 : $\{(b, b), (w, b)\}$, $\{(b, w), (w, w)\}$
- Can we find a "game form" that preserves these rights? For this example there is an easy answer
- Consider the game form

$$\begin{array}{cc} & \begin{array}{cc} b & w \end{array} \\ \begin{array}{c} b \\ w \end{array} & \begin{pmatrix} (b, b) & (b, w) \\ (w, b) & (w, w) \end{pmatrix} \end{array}$$

  where person 1 is the row player and person 2 the column player
- For every preference profile, we would like the resulting game to have a "stable" outcome, i.c., a Nash equilibrium

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- This example is based on Gibbard (JET 1974)

Game form and effectivity function

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Definition: game form

A *game form* is an object $\Gamma = (\Sigma^1, \ldots, \Sigma^n; \pi; A)$, where Σ^i is the *strategy set* of player $i \in N$ and $\pi : \Sigma^1 \times \ldots \times \Sigma^n \rightarrow A$ is the surjective *outcome function*

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- With a game form Γ we associate an effectivity function E^Γ as follows: for $S \in P_0(N)$ and $B \in P_0(A)$

$$B \in E^\Gamma(S) \Leftrightarrow \exists \sigma^S \in \Sigma^S \, [\pi(\sigma^S, \tau^{N \setminus S}) \in B \; \forall \tau^{N \setminus S} \in \Sigma^{N \setminus S}]$$

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  and $E^\Gamma(\emptyset) = \emptyset$
- We say that a game form Γ *represents* an effectivity function E if $E^\Gamma = E$
- We call Γ a *Nash consistent representation* of E if
  (a) Γ represents E, that is, $E^\Gamma = E$
  (b) The game (Γ, R^N) has a Nash equilibrium for each $R^N \in \mathcal{L}^N$

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- For any game form Γ, the associated effectivity function E^{Γ} is monotonic and superadditive. Hence, these conditions are necessary for the existence of a Nash consistent representation Γ of an effectivity function E
- For an effectivity function E and an agent $i \in N$ we define

$$E^*(\{i\}) = \{B \in P_0(A) \mid B \cap B' \neq \emptyset \ \forall B' \in E(N \setminus \{i\})\}$$

$E^*(\{i\})$ contains the sets of alternatives that i cannot be "kept from"

("α-effectivity" versus "β-effectivity"; observe that $E(\{i\}) \subseteq E^*(\{i\})$ by superadditivity of E)

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- Suppose that a strategy profile (σ^1, σ^2) results in a, so $\pi(\sigma^1, \sigma^2) = a$. Note that in reaction to σ^1, player 2 can make sure that the outcome is in $B_2 = \{b, c\}$, which he prefers. So a Nash equilibrium cannot result in a. Etc.

Theorem: Nash consistent representation

Let $E : P(N) \to P(P_0(A))$ be a monotonic and superadditive effectivity function. Then E has a Nash consistent representation if and only if

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- Only-if direction basically as in the example. For the if-direction a special game form is constructed

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  - the outcome is $\varphi^{i_0}(B) \in A$, where $i_0 = (t^1 + \ldots + t^n) \mod n$

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- The game form used above has the additional feature that there always exists a Pareto optimal Nash equilibrium
- For specific effectivity functions E there can be simpler and more natural game forms

- Strong Nash consistent representation: an additional necessary condition for existence is maximality of E, that is $E = E^*$. A sufficient condition is stability of E (non-emptiness of the core of E for every preference profile). See P & P (2010), also for relevant references

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- See Abdou and Keiding (1991) on effectivity functions in general
- What happens if we allow lotteries over A as outcomes? See Peleg and Peters (2009)

THE END

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