The nucleolus and other core allocations in assignment games

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Outline

- Assignment markets — games
    - Stable outcomes — Core
    - Special stable outcomes
    - Manipulability of stable mechanisms
- Assignment games
    - Core
    - Nucleolus (efficient computation)

Assignment market:

- two types of agents (sellers - buyers)
- the objects of trade are indivisible (houses)
- unit supplies, unit demands (houses are identified with sellers)
- money (perfectly divisible), side-payments are allowed
- utility is transferable (identified with money)

Outcome consists of

- an allocation of houses (sellers) to buyers (or to sellers, if not traded)
- monetary transfers among agents

Model (Shapley, Shubik, 1972)

Assignment market:

- set of sellers M, set of buyers M'
- each seller $i \in M$ has a reservation value r_i on his house
- each buyer $j \in M'$ has reservation values $\left(t_{ij}\right)_{i \in M}$ on the houses

$\Downarrow$

Pairwise value matrix: $A = \left[a_{ij} \geq 0\right]_{i \in M, j \in M'}$ with $\boxed{a_{ij} = \left(t_{ij} - r_i\right)^+}$

$\Downarrow$

Assignment game: player set $M \cup M'$, value of coalition S

$$\boxed{w_A(S) = \max_{\mu \in \mathcal{M}(S \cap M, S \cap M')} \sum_{(i,j) \in \mu} a_{ij}}$$

where $\mathcal{M}(S \cap M, S \cap M')$: set of seller-buyer matchings in S

Outcomes: feasibility

A **feasible outcome** $\big(X,(u;v)\big)$ consists of

- a **feasible assignment** $X=[x_{ij}]$ (of sellers (houses) to buyers)

$$\begin{aligned} \sum_{j\in M'} x_{ij} &\le 1 && \text{for all } i\in M \\ \sum_{i\in M} x_{ij} &\le 1 && \text{for all } j\in M' \\ x_{ij} &\in \{0,1\} && \text{for all } i\in M, j\in M' \end{aligned}$$

  - represents matching $\mu_X \in \mathcal{M}(M,M')$ by $\boxed{(i,j)\in\mu_X \Leftrightarrow x_{ij}=1}$

- a **feasible payoff vector** $\big(\ldots u_i \ldots ; \ldots v_j \ldots\big)_{i\in M, j\in M'}$ (w.r.to X)

$$\sum_{i\in M} u_i + \sum_{j\in M'} v_j \le \sum_{(i,j)\in(M,M')} a_{ij}x_{ij} = \sum_{(i,j)\in\mu_X} a_{ij}$$

Outcomes: stability

A feasible outcome $\big(X,(u;v)\big)$ is **stable**, if payoffs are **acceptable**:

- $u_i \geq 0\ ,\ \ v_j \geq 0$ for all $i \in M, j \in M'$ (individually acceptable)
- $u_i + v_j \geq a_{ij}$ for all $i \in M, j \in M'$ (pairwise acceptable)

Note: If payoffs are acceptable then for any feasible assignment X

$$\sum_{(i,j)\in\mu_X} a_{ij} + \sum 0 \leq \sum_{(i,j)\in\mu_X} (u_i + v_j) + \sum_{i\in M\setminus\mu_X} u_i + \sum_{j\in M'\setminus\mu_X} v_j$$

Recall: Feasibility implies the reverse inequality:

$$\sum_{(i,j)\in\mu_X} (u_i + v_j) + \sum_{i\in M\setminus\mu_X} u_i + \sum_{j\in M'\setminus\mu_X} v_j \leq \sum_{(i,j)\in(M,M')} a_{ij}x_{ij}$$

Stability results /1

Proposition

A feasible outcome $(X, (u; v))$ *is stable,* **if and only if** *at the same time*

- X *maximizes total value* $\sum_{ij} a_{ij} x_{ij}$ *over all feasible assignments*
- $(u; v)$ *minimizes total payoff* $\sum_i u_i + \sum_j v_j$ *over all acceptable payoffs*

In that case,

- $u_i = 0$ *for each unmatched seller* $i \in M \setminus \mu_X$ $(\sum_{j \in M'} x_{ij} < 1)$
- $v_j = 0$ *for each unmatched buyer* $j \in M' \setminus \mu_X$ $(\sum_{i \in M} x_{ij} < 1)$
- $u_i + v_j = a_{ij}$ *for each matched pair* $(i, j) \in \mu_X$ $(x_{ij} > 0)$

? **Existence** Is there a stable outcome for any assignment market?

Stability results /1

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- $u_i + v_j = a_{ij}$ *for each matched pair* $(i, j) \in \mu_X$ $(x_{ij} > 0)$

? **Existence** Is there a stable outcome for any assignment market?

Theorem (Egerváry, 1931)

For any nonnegative matrix $A = [a_{ij}]$,

$$\max_{X \text{ feasible}} \sum_{ij} a_{ij} x_{ij} = \min_{(u;v) \text{ acceptable}} \sum_i u_i + \sum_j v_j$$

Stability results /2

Theorem (Shapley, Shubik, 1972)

For any nonnegative matrix $A = [a_{ij}]$, *the* **core**

$$Co = \{x = (u, v) : x(M \cup M') = w(M \cup M'),\ x(S) \geq w(S) \quad \forall S \subset M \cup M'\}$$

of the induced assignment game $w = w_A$

- $=$ *the set of* **dual optimal** *solutions to the max assignment LP* (= *the set of stable payoff vectors*)
- *hence, it is a* **non-empty** *polytope (and 'easily' computable)*
- *has a* **lattice** *structure: if* $(.. \, u_i \, ..; .. \, v_j \, ..)$ *and* $(.. \, u'_i \, ..; .. \, v'_j \, ..) \in Co$, *then* $(.. \, u_i \vee u'_i \, ..; .. \, v_j \wedge v'_j \, ..)$ *and* $(.. \, u_i \wedge u'_i \, ..; .. \, v_j \vee v'_j \, ..) \in Co$
- *has a* **seller-optimal** $(.. \, \overline{u}_i \, ..; .. \, \underline{v}_j \, ..)$ *vertex*
  where $\overline{u}_i = \max\{u_i : (u, v) \in Co\}$ *and* $\underline{v}_j = \min\{v_j : (u, v) \in Co\}$
  has a **buyer-optimal** $(.. \, \underline{u}_i \, ..; .. \, \overline{v}_j \, ..)$ *vertex*
  where $\underline{u}_i = \min\{u_i : (u, v) \in Co\}$ *and* $\overline{v}_j = \max\{v_j : (u, v) \in Co\}$
- $=$ *the set of surplus vectors attainable at* **competitive equilibrium prices**

Example 3×3

market

	$t_{.1}$	$t_{.2}$	$t_{.3}$
$r_1 = 20$	27	21	24
$r_2 = 21$	29	27	28
$r_3 = 22$	27	20	25

$\Longrightarrow$ pairwise values

[7]	1	4
8	[6]	7
5	0	[3]

stable outcome

	$v_1 \geq 0$	$v_2 \geq 0$	$v_3 \geq 0$
$0 \leq u_1$	[$u_1 + v_1 = 7$]	$u_1 + v_2 \geq 1$	$u_1 + v_3 \geq 4$
$0 \leq u_1$	$u_2 + v_1 \geq 8$	[$u_2 + v_2 = 6$]	$u_2 + v_3 \geq 7$
$0 \leq u_2$	$u_3 + v_1 \geq 5$	$u_3 + v_2 \geq 0$	[$u_3 + v_3 = 3$]

seller payoffs

	$u_1 \leq 7$	$u_2 \leq 6$	$u_3 \leq 3$
$0 \leq u_1$	.	$u_1 - u_2 \geq -5$	$u_1 - u_3 \geq +1$
$0 \leq u_2$	$u_2 - u_1 \geq +1$	.	$u_2 - u_3 \geq +4$
$0 \leq u_3$	$u_3 - u_1 \geq -2$	$u_3 - u_2 \geq -6$	.

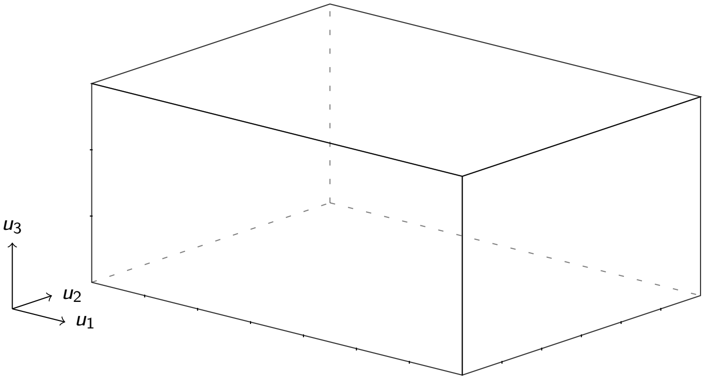
buyer payoffs $\quad v_1 = 7 - u_1 \quad v_2 = 6 - u_2 \quad v_3 = 3 - u_3$

Example 3 × 3: the core

7	1	4
8	**6**	7
5	0	**3**

$\Longrightarrow$

$0 \leq u_1 \leq 7$	$0 \leq u_2 \leq 6$	$0 \leq u_3 \leq 3$
.	$u_1 - u_2 \geq -5$	$u_1 - u_3 \geq 1$
$u_2 - u_1 \geq 1$	.	$u_2 - u_3 \geq 4$
$u_3 - u_1 \geq -2$	$u_3 - u_2 \geq -6$	.

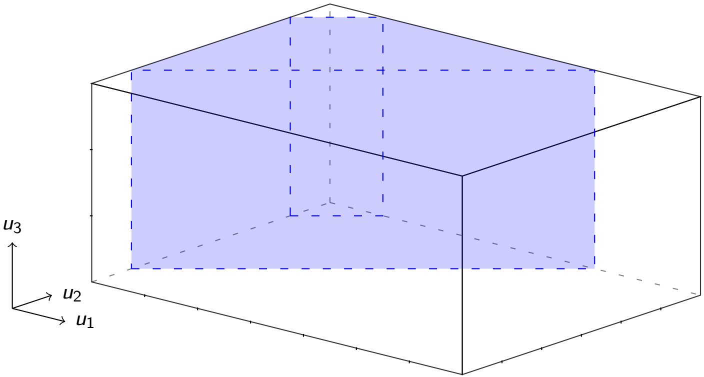


Example 3×3: the core

7	1	4
8	**6**	7
5	0	**3**

$\Longrightarrow$

$0 \le u_1 \le 7$	$0 \le u_2 \le 6$	$0 \le u_3 \le 3$
.	$u_1 - u_2 \ge -5$	$u_1 - u_3 \ge 1$
$u_2 - u_1 \ge 1$	.	$u_2 - u_3 \ge 4$
$u_3 - u_1 \ge -2$	$u_3 - u_2 \ge -6$	.

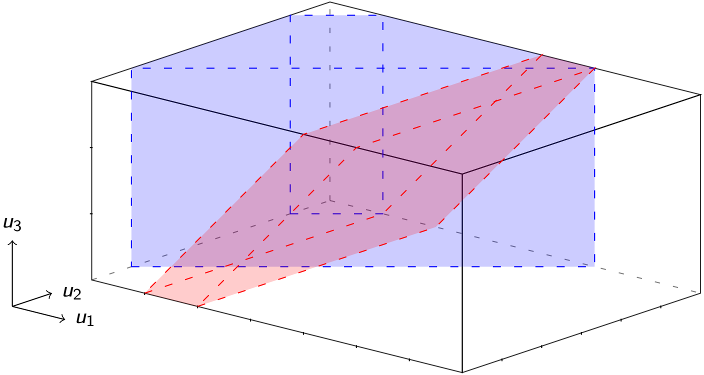


Example 3×3: the core

7	1	4
8	**6**	7
5	0	**3**

$\Longrightarrow$

$0 \leq u_1 \leq 7$	$0 \leq u_2 \leq 6$	$0 \leq u_3 \leq 3$
.	$u_1 - u_2 \geq -5$	$u_1 - u_3 \geq 1$
$u_2 - u_1 \geq 1$	.	$u_2 - u_3 \geq 4$
$u_3 - u_1 \geq -2$	$u_3 - u_2 \geq -6$	.



Example 3×3: the core

7	1	4
8	**6**	7
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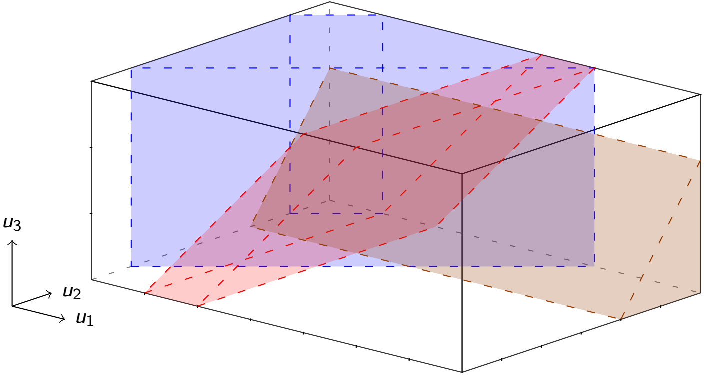
$\Longrightarrow$

$0 \le u_1 \le 7$	$0 \le u_2 \le 6$	$0 \le u_3 \le 3$
.	$u_1 - u_2 \ge -5$	$u_1 - u_3 \ge 1$
$u_2 - u_1 \ge 1$	.	$u_2 - u_3 \ge 4$
$u_3 - u_1 \ge -2$	$u_3 - u_2 \ge -6$	.

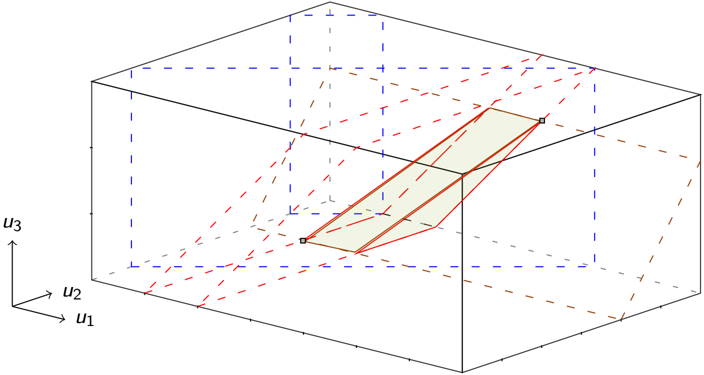


Example 3 × 3: the core

7	1	4
8	6	7
5	0	3

$\Longrightarrow$

$0 \le u_1 \le 7$	$0 \le u_2 \le 6$	$0 \le u_3 \le 3$
.	$u_1 - u_2 \ge -5$	$u_1 - u_3 \ge 1$
$u_2 - u_1 \ge 1$	.	$u_2 - u_3 \ge 4$
$u_3 - u_1 \ge -2$	$u_3 - u_2 \ge -6$	.



For sellers: worst corner $(1, 4, 0)$, best corner $(4, 6, 2)$

Example 2×2

Market

	$t_{.1}$	$t_{.2}$
$r_1 = 20$	26	22
$r_2 = 22$	26	25

$\Longrightarrow$ pairwise values $\begin{bmatrix} 6 & 2 \\ 4 & 3 \end{bmatrix}$

stable payoffs

	$v_1 \geq 0$	$v_2 \geq 0$
$0 \leq u_1$	$\boxed{u_1 + v_1 = 6}$	$u_1 + v_2 \geq 2$
$0 \leq u_2$	$u_2 + v_1 \geq 4$	$\boxed{u_2 + v_2 = 3}$

- for sellers

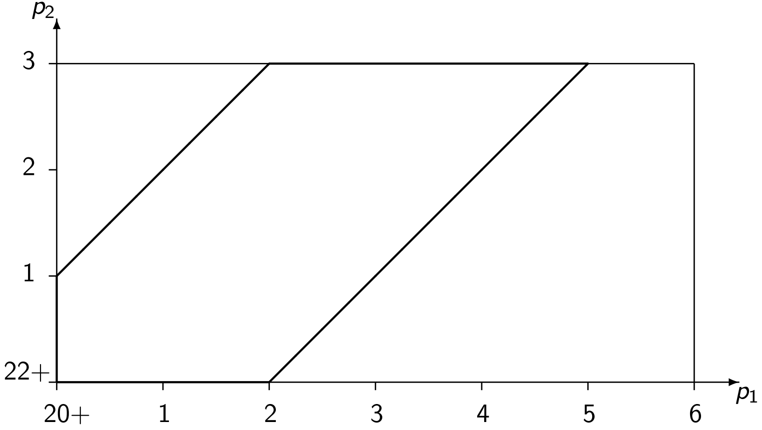
	$u_1 \leq 6$	$u_2 \leq 3$
$0 \leq u_1$	.	$u_1 - u_2 \geq -1$
$0 \leq u_2$	$u_2 - u_1 \geq -2$	.

- for buyers $\quad v_1 = 6 - u_1 \quad v_2 = 3 - u_2$

competitive equilibrium prices $p_1 = r_1 + u_1 = t_{11} - v_1$, $p_2 = r_2 + u_2 = t_{22} - v_2$
induce optimal seller-buyer assignment and stable payoffs

Core, competitive prices in Example 2×2

		$u_1 \leq 6$	$u_2 \leq 3$
20+	$u_1 \geq 0$	.	$u_1 - u_2 \geq -1$
22+	$u_2 \geq 0$	$u_2 - u_1 \geq -2$	.



Stable market mechanism

Stable mechanism in assignment markets:

- each seller reports his reservation price
- each buyer reports his monetary valuation on each house
- prices are determined (from stable outcomes) and announced
- each buyer demands the house which maximizes his surplus
- houses are allocated, payments are made

Theorem

No stable mechanism exists for which stating the true reservation prices is a dominant strategy **for every agent**.

Theorem (Demange, 1982 / Leonard, 1983)

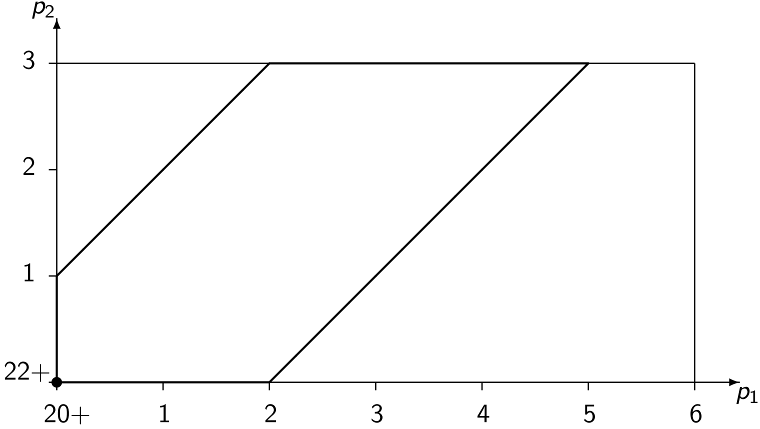
If the mechanism selects the minimum equilibrium prices then truthful reporting is a dominant strategy **for every buyer**.

Theorem (Demange, Gale, Sotomayor, 1986)

The minimum equilibrium prices can be achieved by ascending auctions.

Buyer-optimal competitive prices

$20+0$	$\boxed{6}$	2
$22+0$	4	$\boxed{3}$

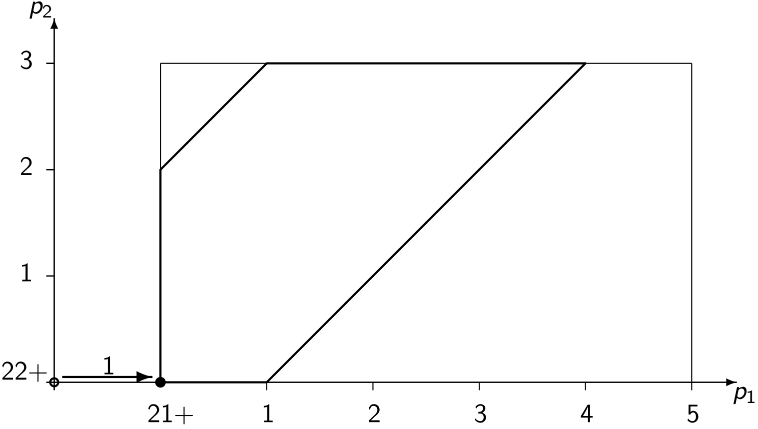


Buyer-optimal competitive prices

$20 + 0$	$\boxed{6}$	2
$22 + 0$	4	$\boxed{3}$

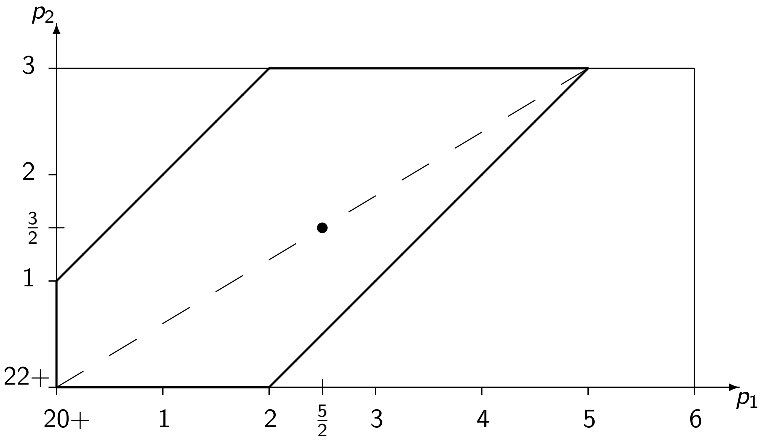
$\longrightarrow$

$1 + 20 + 0$	$\boxed{5}$	1
$22 + 0$	4	$\boxed{3}$



Fair prices (Thompson, 1981)

Fair prices = average of the buyer-optimal and seller-optimal prices
Núñez, Rafels (2002) Tau-value = payoffs at the fair prices



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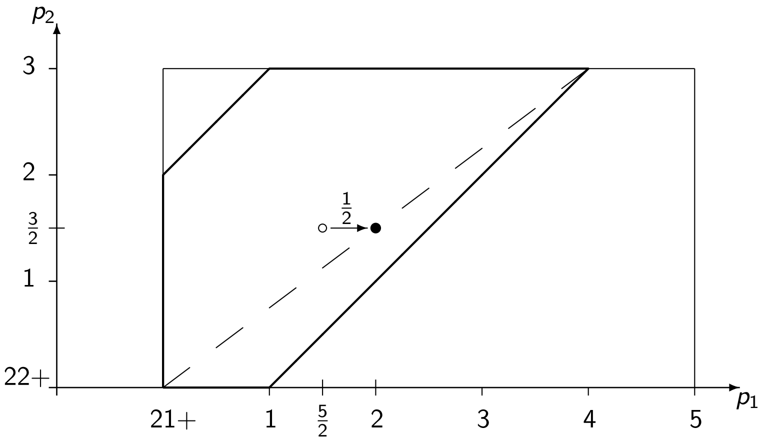
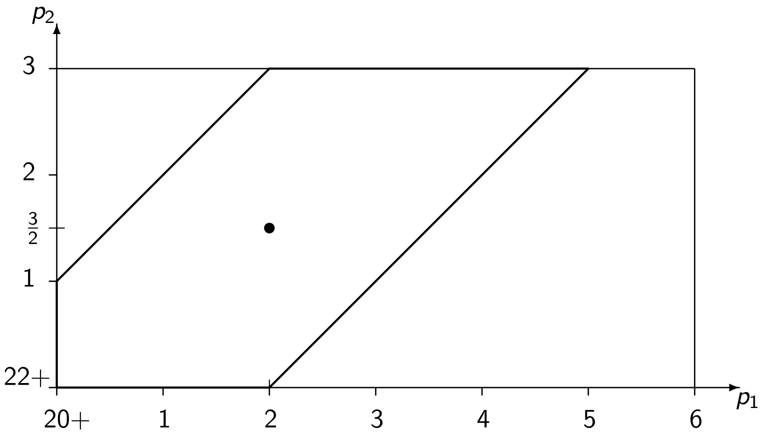


Nucleolus prices

The nucleolus is the lexicographically 'innermost' point of the core
Solymosi, Raghavan (1994) Efficiently computable from data matrix



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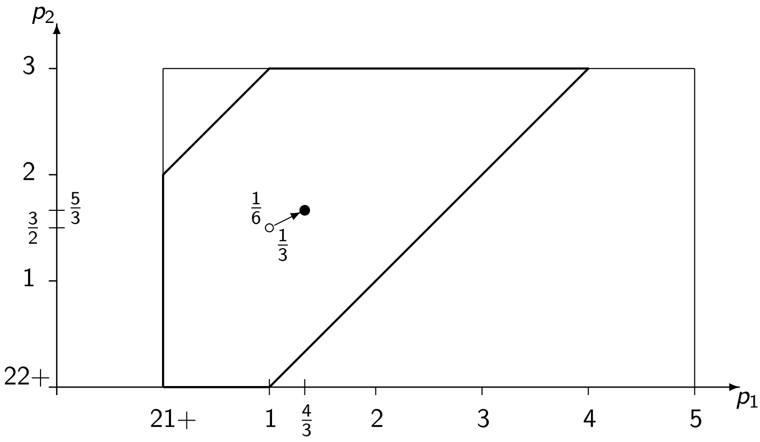
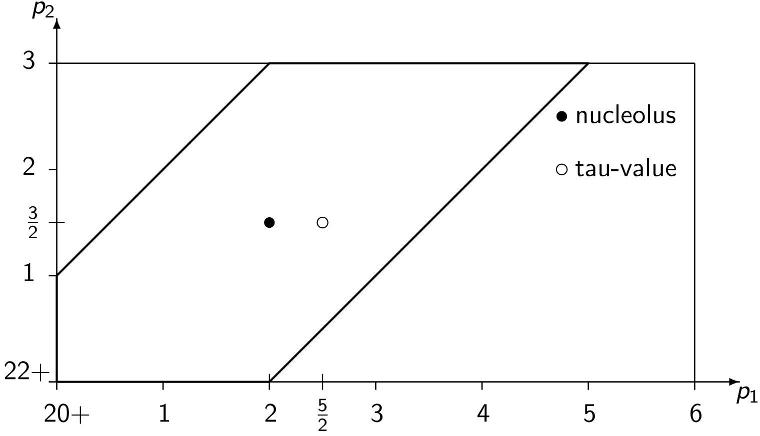


Example 2×2 continued

0	6	2
0	4	3

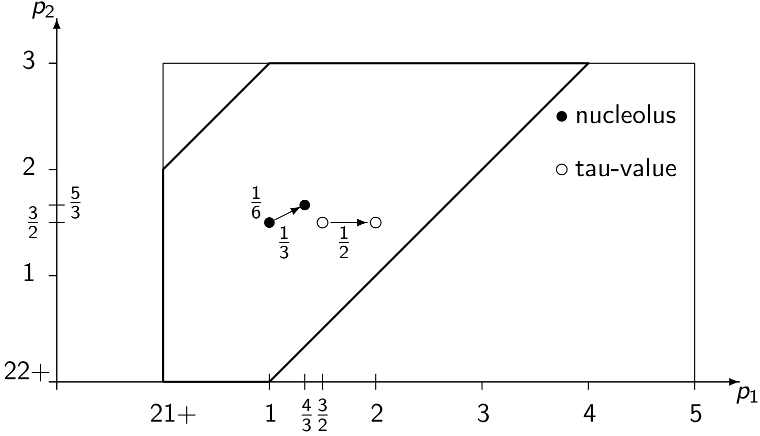


Example 2×2 continued

0	6	2
0	4	3

$\longrightarrow$

$1+0$	5	1
0	4	3

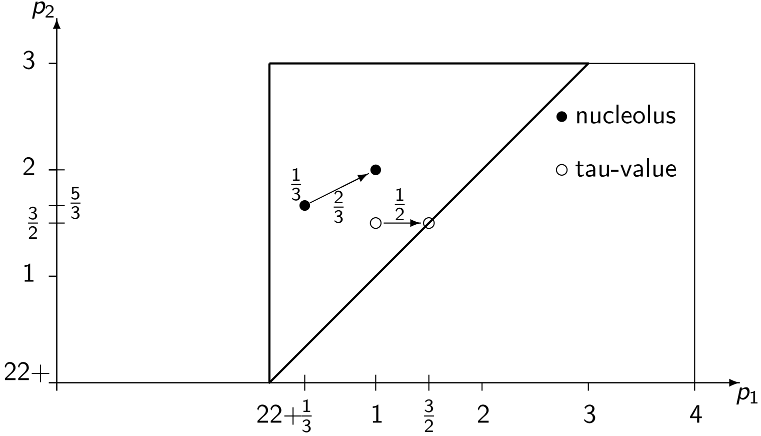


Example 2×2 continued

$1+0$	$\boxed{5}$	1
0	4	$\boxed{3}$

$\longrightarrow$

$2+0$	$\boxed{4}$	0
0	4	$\boxed{3}$



Manipulability results for sellers

Let seller k increase his reported reservation price by $0 \leq c$ small enough not to change the optimal assignment. In new input matrix

$$a'_{kj} = (a_{kj} - c)^+ \text{ for all } j \quad \text{and} \quad a'_{ij} = a_{ij} \text{ for all } j \text{ and } (i \neq k).$$

Then

- for the **buyer-optimal corner** $(\underline{u}, \overline{v})$, $\quad 0 + \underline{u}_k \leq c + \underline{u}'_k \leq c + \underline{u}_k$
  Seller k can increase the minimum eq. price, but by at most c.
- for the **seller-optimal corner** $(\overline{u}, \underline{v})$, $\quad 0 + \overline{u}_k = c + \overline{u}'_k$
  Seller k cannot influence the maximum eq. price.
- Consequently, for the **tau-value** τ, $\quad 0 + \tau_k \leq c + \tau'_k \leq \tau_k + \frac{c}{2}$.
  Seller k can increase the fair equilibrium price, but by at most $\frac{c}{2}$.
- for the **nucleolus** η, $\quad 0 + \eta_k \leq c + \eta'_k \leq \eta_k + \frac{d}{d+1}c$
  Seller k can increase the nucleolus price, but by at most $\frac{d}{d+1}c$, where d is the number of houses sold.

Moreover, all bounds are sharp.

Summary of first part

We considered

- assignment markets à la Shapley and Shubik (1972)
- equilibrium prices induced by core solutions of associated games (special corners, tau-value, nucleolus)
- sensitivity of these equilibrium prices w.r.t. market data changes

We found that each player could influence any of these prices to his benefit, and established sharp upper bounds for its extent.

Further questions:

[?] Manipulability of other stable assignment mechanisms?

[?] Is there a 'least manipulable' stable assignment mechanism?

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Core – minimum matrix covers

Recall

the **core** of the assignment game induced by the matrix $A = \left[a_{ij} \geq 0\right]_{i \in M, j \in M'}$
$=$ the (non-empty) set of **minimum covers** of matrix A
$=$ the (non-empty) set of optimal solutions to the dual assignment LP:

$$\begin{array}{rcll} \sum_{i \in M} u_i + \sum_{j \in M'} v_j & \to & \min & = w_A(M \cup M') \\ u_i + v_j & \geq & a_{ij} & \text{for all } i \in M, j \in M' \\ u_i, v_j & \geq & 0 & \text{for all } i \in M, j \in M' \end{array}$$

By complementary slackness between primal and dual optimal solutions, for any maximum value assignment μ_X, i.e. $w_A(M \cup M') = \sum_{(i,j) \in \mu_X} a_{ij}$ and for any core vector (u, v)

$$\begin{array}{rcll} u_i + v_j & = & a_{ij} & \text{for all matched } \quad (i, j) \in \mu_X \\ u_i & = & 0 & \text{for all unmatched } i \in M \setminus \mu_X \\ v_j & = & 0 & \text{for all unmatched } j \in M' \setminus \mu_X \end{array}$$

Core description – unified notation

Assume w.l.o.g.

- $|M| = |M'|$ (0 rows/columns $\to$ dummy player)
- no player is unmatched in an optimal matching
- the diagonal matching is optimal (rearrange rows/columns)

Introduce a fictitious seller 0 and a fictitious buyer $0'$

Extend any optimal matching with the pair of 0 and $0'$.

Denote dual slacks $$f_{ij}(u,v) = \begin{cases} u_i + v_j - a_{ij} & \text{for} \quad i \in M, j \in M' \\ u_i & \text{for} \quad i \in M, j = 0' \\ v_j & \text{for} \quad i = 0, j \in M' \\ 0 & \text{for} \quad i = 0, j = 0' \end{cases}$$

Then, under any optimal matching,

$$(u;v) \in Co \Leftrightarrow \begin{cases} f_{i0'}(u,v) \geq 0 & \text{for each seller } i \\ f_{0j}(u,v) \geq 0 & \text{for each buyer } j \\ f_{ij}(u,v) \geq 0 & \text{for each unmatched pair } i,j \\ f_{ij}(u,v) = 0 & \text{for each matched pair } i,j \end{cases}$$

Dual slacks in Example 3×3

square matrix

7	1	4
8	**6**	7
5	0	**3**

with an optimal matching in the diagonal

core constraints

	$v_1 \geq 0$	$v_2 \geq 0$	$v_3 \geq 0$
$0 \leq u_1$	$u_1 + v_1 = 7$	$u_1 + v_2 \geq 1$	$u_1 + v_3 \geq 4$
$0 \leq u_1$	$u_2 + v_1 \geq 8$	$u_2 + v_2 = 6$	$u_2 + v_3 \geq 7$
$0 \leq u_2$	$u_3 + v_1 \geq 5$	$u_3 + v_2 \geq 0$	$u_3 + v_3 = 3$

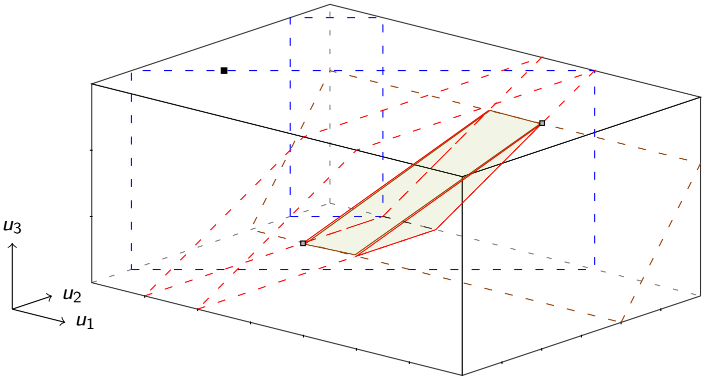
in extended dual slacks $f_{ij} = f_{ij}(u, v)$

$f_{00} = 0$	$v_1 = f_{01} \geq 0$	$v_2 = f_{02} \geq 0$	$v_3 = f_{03} \geq 0$
$u_1 = f_{10} \geq 0$	$f_{11} = 0$	$f_{12} \geq 0$	$f_{13} \geq 0$
$u_2 = f_{20} \geq 0$	$f_{21} \geq 0$	$f_{22} = 0$	$f_{23} \geq 0$
$u_3 = f_{30} \geq 0$	$f_{31} \geq 0$	$f_{32} \geq 0$	$f_{33} = 0$

Example 3 × 3: box allocations /1

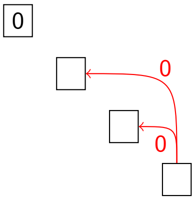
0	6	4	0
1	0	4	−3
2	0	0	−5
3	4	7	0

box allocation $(1, 2, 3; 6, 4, 0)$ not in core

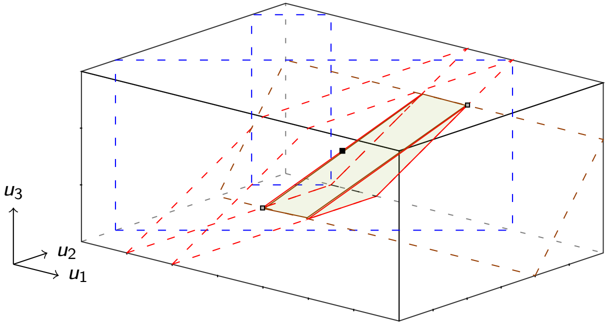


Example 3×3: box allocations /2

0	5	1	2
2	0	2	0
5	2	0	0
1	1	2	0

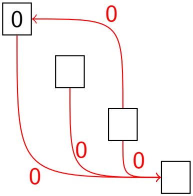


allocation $(2, 5, 1; 5, 1, 2)$ is in the core, but not an extreme point

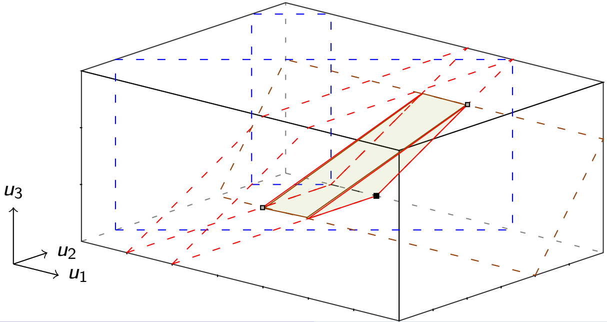


Example 3×3: box allocations /3

0	5	0	3
2	0	1	1
6	3	0	2
0	0	0	0



allocation $(2,6,0;5,0,3)$ is a core extreme point



Nucleolus = lexicographic center

The nucleolus was introduced by Schmeidler (1969). Alternatively, Maschler, Peleg, Shapley (1979) describe a finite process that iteratively reduces the set of payoffs to a singleton, called the lexicographic center. It is shown to be exactly the nucleolus.

The lexicographic center procedure can be easily implemented. Each iteration can be carried out by solving LPs with $n+1$ variables and 2^n constraints including only -1, 0, or 1 coefficients. E.g. Sankaran (1991) proposed such a formulation with at most $\mathcal{O}(2^n)$ LPs to be solved for an n-person game.

For assignment games, Solymosi and Raghavan (1994) applied graph-related techniques instead of LPs. Their algorithm generates a finite sequence of payoffs leading to the nucleolus. For an (m,n)-person game with $m = \min(m,n)$, the algorithm determines the nucleolus in at most $m(m+3)/2$ steps, each one requiring at most $\mathcal{O}(m \cdot n)$ elementary operations.

Lexicographic center for assignment games

Let σ be a fixed optimal matching. Construct a sequence $(\Delta^0, \Sigma^0), \ldots, (\Delta^\rho, \Sigma^\rho)$ of partitions of (M_0, M_0') with $\Sigma^0 \supset \ldots \supset \Sigma^\rho$, and a nested sequence $X^0 \supset \ldots \supset X^\rho$ of sets of payoff vectors.

Initially, let $\Delta^0 = \sigma$, $\Sigma^0 = (M_0, M_0') \setminus \sigma$,

$$(u^0, v^0) \text{ with } u_i^0 = 0 \quad \forall\, i \in M,\ v_j^0 = a_{jj} \quad \forall\, j \in M'$$

$$\alpha^0 = \min\{f_{ij}(u^0, v^0) : (i,j) \in \Sigma^0\}$$

$$X^0 = \{(u,v) \geq (0,0),\ f_{ij}(u,v) = 0 \quad \forall\, (i,j) \in \Delta^0,\ f_{ij}(u,v) \geq \alpha^0 \quad \forall\, (i,j) \in \Sigma^0\}.$$

For $r = 0, 1, \ldots, \rho$ define recursively

1. $\alpha^{r+1} = \max_{(u,v)\in X^r} \min_{(i,j)\in\Sigma^r} f_{ij}(u,v)$
2. $X^{r+1} = \left\{(u,v) \in X^r : \min_{(i,j)\in\Sigma^r} f_{ij}(u,v) = \alpha^{r+1}\right\}$
3. $\Sigma_{r+1} = \left\{(i,j) \in \Sigma^r : f_{ij}(u,v) = \text{ constant on } X^{r+1}\right\}$
4. $\Sigma^{r+1} = \Sigma^r \setminus \Sigma_{r+1},\ \Delta^{r+1} = \Delta^r \cup \Sigma_{r+1},$

where ρ is the last value of index r for which $\Sigma^r \neq \emptyset$.

The set $X^{\rho+1}$ is the **lexicographic center** of $A(M, M')$.

Algorithm (Solymosi, Raghavan, 1994)

Given: square matrix $(a_{ij} \geq 0)_{(M_0, M_0')}$ with diagonal optimal matching σ

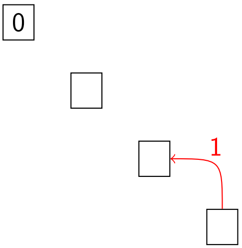
Initially let $r = 0$, $\Delta = \sigma$, $\Sigma = (M_0, M_0') \setminus \sigma$, $u_i = 0 \quad \forall\, i \in M_0$, $v_j = a_{jj} \quad \forall\, j \in M_0'$, $f_{ij} = u_i + v_j - a_{ij} \quad \forall\, (i,j) \in (M_0, M_0')$, $\alpha = \min\{f_{ij} : (i,j) \in \Sigma\}$.

While $\Sigma \neq \emptyset$ do (**iteration** r)

1. Build the graph $G := G(r, \alpha)$
2. While G is proper do (**step** (r, α))
    1. Find direction (s, t)
    2. Find step size $\beta := \beta(r, \alpha)$
    3. Update arcs in graph, $G := G(r, \alpha + \beta)$
    4. Update payoff, $(u, v) := (u, v) + \beta \cdot (s, t)$
    5. Update $f_{ij} := f_{ij} + \beta \cdot (s_i + t_j) \quad \forall\, (i,j) \in \Sigma$
    6. Update guaranteed level, $\alpha := \alpha + \beta$
3. Find to-be-settled coalitions $\overline{\Sigma} := \Sigma_{r+1}$
4. Update partition, $\Sigma := \Sigma \setminus \overline{\Sigma}$, $\Delta := \Delta \cup \overline{\Sigma}$
5. Set $r := r + 1$.

Iteration r=0, step $\alpha = -4$

$\boxed{0}$	7	6	3
0	$\boxed{0}$	5	-1
0	-1	$\boxed{0}$	-4
0	2	6	$\boxed{0}$

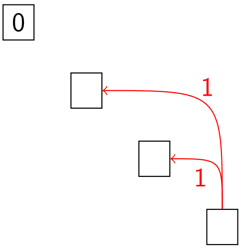


$\boxed{0}$	7_0	6_{-1}	3_0	0
0_0	$\boxed{0}$	5_{-1}	-1_0	0
0_{+1}	-1_{+1}	$\boxed{0}$	-4_{+1}	$+1$
0_0	2_0	6_{-1}	$\boxed{0}$	0
0	0	-1	0	

$\beta = 3$

Iteration r=0, step $\alpha = -1$

[0]	7	3	3
0	[0]	2	-1
3	2	[0]	-1
0	2	3	[0]

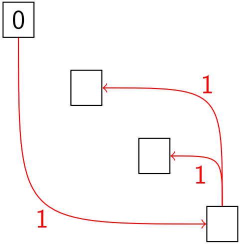


[0]	7_{-1}	3_{-1}	3_0	0
0_{+1}	[0]	2_0	-1_{+1}	$+1$
3_{+1}	2_0	[0]	-1_{+1}	$+1$
0_0	2_{-1}	3_{-1}	[0]	0
0	-1	-1	0	

$\beta = 1$

Iteration r=0, step $\alpha = 0$

$\boxed{0}$	6	2	3
1	$\boxed{0}$	2	0
4	2	$\boxed{0}$	0
0	1	2	$\boxed{0}$

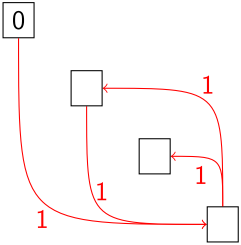


$\boxed{0}$	6_{-2}	2_{-2}	3_{-1}	0
1_{+2}	$\boxed{0}$	2_0	0_{+1}	$+2$
4_{+2}	2_0	$\boxed{0}$	0_{+1}	$+2$
0_{+1}	1_{-1}	2_{-1}	$\boxed{0}$	$+1$
0	-2	-2	-1	

$\boxed{\beta = 1/2}$

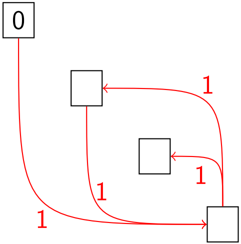
Iteration r=0, step $\alpha = 1/2$

$\boxed{0}$	5	1	$\frac{5}{2}$
2	$\boxed{0}$	2	$\frac{1}{2}$
5	2	$\boxed{0}$	$\frac{1}{2}$
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\boxed{0}$



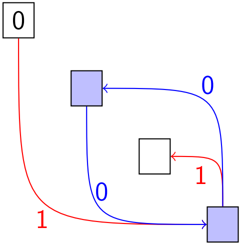
Iteration r=0, step $\alpha = 1/2$

0	5	1	$\frac{5}{2}$
2	0	2	$\frac{1}{2}$
5	2	0	$\frac{1}{2}$
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	0



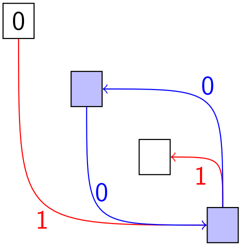
Settling step

0	5	1	$\frac{5}{2}$
2	0	2	$\frac{1}{2}$
5	2	0	$\frac{1}{2}$
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	0



Iteration r=1, step $\alpha = 1/2$

0	5	1	$\frac{5}{2}$
2	0	2	$\frac{1}{2}$
5	2	0	$\frac{1}{2}$
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	0

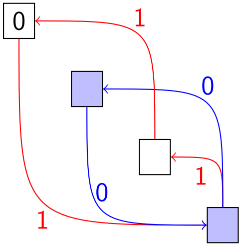


0	5_{-1}	1_{-2}	$\frac{5}{2}_{-1}$	0
2_{+1}	0	2_{-1}	$\frac{1}{2}$	+1
5_{+2}	2_{+1}	0	$\frac{1}{2}_{+1}$	+2
$\frac{1}{2}_{+1}$	$\frac{1}{2}$	$\frac{3}{2}_{-1}$	0	+1
0	−1	−2	−1	

$\beta = 1/6$

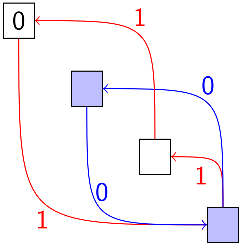
Iteration r=1, step $\alpha = 2/3$

0	$\frac{29}{6}$	$\frac{2}{3}$	$\frac{7}{3}$
$\frac{13}{6}$	0	$\frac{11}{6}$	$\frac{1}{2}$
$\frac{16}{3}$	$\frac{13}{6}$	0	$\frac{2}{3}$
$\frac{2}{3}$	$\frac{1}{2}$	$\frac{4}{3}$	0



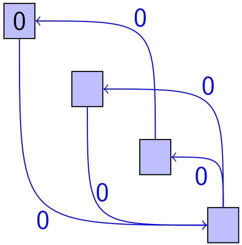
Iteration r=1, step $\alpha = 2/3$

0	$\frac{29}{6}$	$\frac{2}{3}$	$\frac{7}{3}$
$\frac{13}{6}$	0	$\frac{11}{6}$	$\frac{1}{2}$
$\frac{16}{3}$	$\frac{13}{6}$	0	$\frac{2}{3}$
$\frac{2}{3}$	$\frac{1}{2}$	$\frac{4}{3}$	0



Settling step → STOP → the nucleolus $=(\frac{13}{6}, \frac{16}{3}, \frac{2}{3}; \frac{29}{6}, \frac{2}{3}, \frac{7}{3})$

0	$\frac{29}{6}$	$\frac{2}{3}$	$\frac{7}{3}$
$\frac{13}{6}$	0	$\frac{11}{6}$	$\frac{1}{2}$
$\frac{16}{3}$	$\frac{13}{6}$	0	$\frac{2}{3}$
$\frac{2}{3}$	$\frac{1}{2}$	$\frac{4}{3}$	0



References /2

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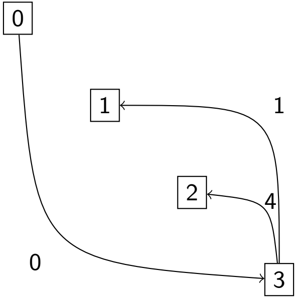
Example 3×3: Trees of longest paths to special corners

extended matrix

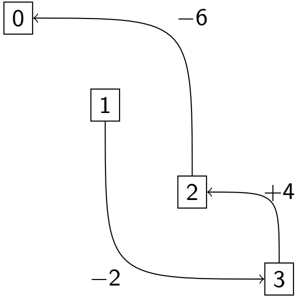
[0]	0	0	0
0	[7]	1	4
0	8	[6]	7
0	5	0	[3]

arc lengths

.	-7	-6	-3
0	.	-5	$+1$
0	$+1$	.	$+4$
0	-2	-6	.



The minimal $\underline{u} = (1, 4, 0)$



The maximal $\overline{u} = (4, 6, 2)$

THANK YOU
FOR YOUR ATTENTION