Summer School on Matching Problems, Markets and Mechanisms

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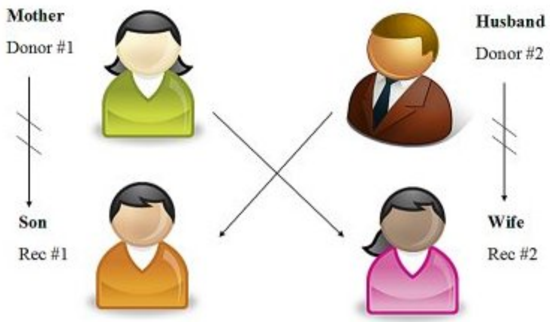
Outline

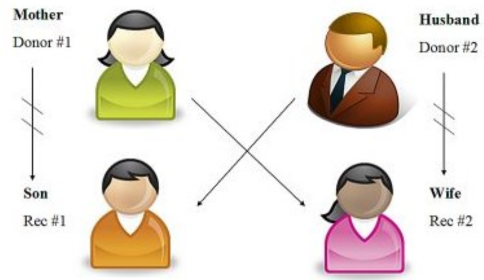
1. The Hospitals / Residents problem and its variants

2. The House Allocation problem

3. Kidney exchange



Kidney exchange



Primer: integer programming (1)

- Let $G=(V,E)$ be a graph
- A *colouring* of G is a function $f : V \rightarrow \{1,2,\ldots,k\}$, for some integer k, such that $f(u) \neq f(v)$ whenever $\{u,v\} \in E$
- The problem is to minimise k over all colourings of G
- Example:

University of Glasgow

Primer: integer programming (2)

- The graph colouring problem is NP-hard
- One possibility: solve the problem using *integer programming*
- Integer programming:
  - $\min \mathbf{c}^{\mathrm{T}}\mathbf{x}$ subject to $A\mathbf{x} \leq \mathbf{b}$

    Objective function · Constraints · Variables

  - where $\mathbf{c}=(c_1, c_2, \ldots, c_n)^{\mathrm{T}}$, $\mathbf{x}=(x_1, x_2, \ldots, x_n)^{\mathrm{T}}$, $\mathbf{b}=(b_1, b_2, \ldots, b_m)^{\mathrm{T}}$

    $A=(a_{ij})$ $(1 \leq i \leq m,\ 1 \leq j \leq n)$, the c_i, a_{ij} and b_j are real-valued known coefficients and the x_i are integer-valued variables
- Linear programming: relaxation in which x_i are real-valued
  - solvable in polynomial time
- General integer programming problem is NP-hard
  - but there are some powerful solvers

Primer: integer programming (3)

- Back to the graph colouring problem
- Suppose $|V|=n$. No colouring can use more than n colours.
- Define the following binary variables:

$$x_{v,c} = \begin{cases} 1 & \text{if vertex } v \text{ has colour } c \\ 0 & \text{otherwise} \end{cases} \quad \forall v \in V,\ \forall c\ (1 \leq c \leq n)$$

$$y_c = \begin{cases} 1 & \text{if colour } c \text{ is used} \\ 0 & \text{otherwise} \end{cases} \quad \forall c\ (1 \leq c \leq n)$$

- Define the following integer program:

$$\min \sum_{c=1}^{n} y_c \quad \text{subject to:} \quad \sum_{c=1}^{n} x_{v,c} = 1 \quad \forall v \in V$$

$$x_{u,c} + x_{v,c} \leq 1 \quad \forall c\ (1 \leq c \leq n) \quad \forall \{u,v\} \in E$$

$$n y_c \geq \sum_{v \in V} x_{v,c} \quad \forall c\ (1 \leq c \leq n)$$

$$x_{v,c} \in \{0,1\}\ \forall v \in V, \forall c\ (1 \leq c \leq n) \qquad y_c \in \{0,1\}\ \forall c\ (1 \leq c \leq n)$$

Each vertex must have one colour

Adjacent vertices have distinct colours

Minimise number of colours

If colour c is used then $y_c=1$

Variables are binary-valued

Kidney failure

- Treatment
  - Dialysis
  - Transplantation
- Need for donors
  - 6325 on active transplant list as of 31 March 2013
  - Median waiting time: 1168 days (adults), 354 days (children) [based on patient registrations during 1 April 2005 – 31 March 2009]
  - Deceased donors
    - 1916 transplants from deceased donors between 1 April 2012 and 31 March 2013
  - Living donors
    - 1068 transplants from living donors between 1 April 2012 and 31 March 2013
    - 36% of all donations from living donors
    - But: blood type incompatibility (e.g. A →B)
    - Positive crossmatch (tissue-type incompatibility)
- Source of figures: NHS Blood and Transplant (NHSBT)

NHS Blood and Transplant

Human Tissue Act

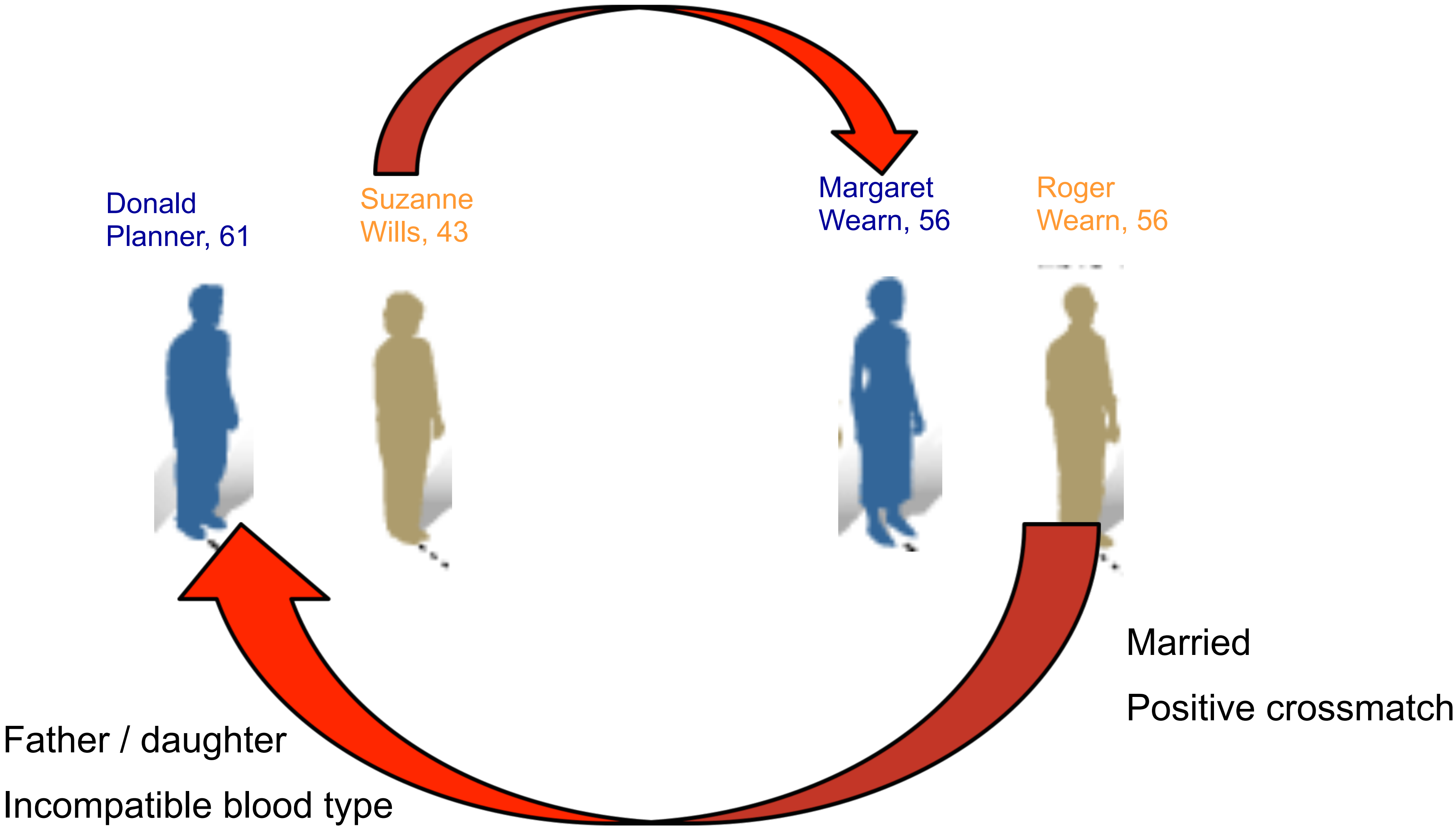
- Prior to 1 September 2006, transplants could only take place between those with a genetic or emotional connection

- Human Tissue Act 2004 and Human Tissue (Scotland) Act 2006:
  - legal framework created to allow transplants between strangers

- New possibilities for live-donor transplants:
  - *Paired kidney donation*: a patient with a willing but incompatible donor can swap their donor with that of another similar patient
  - *Altruistic* (non-directed) donors
    - they can donate directly to the *deceased donor waiting list* (DDWL)
    - they can trigger *domino paired donation (DPD) chains*

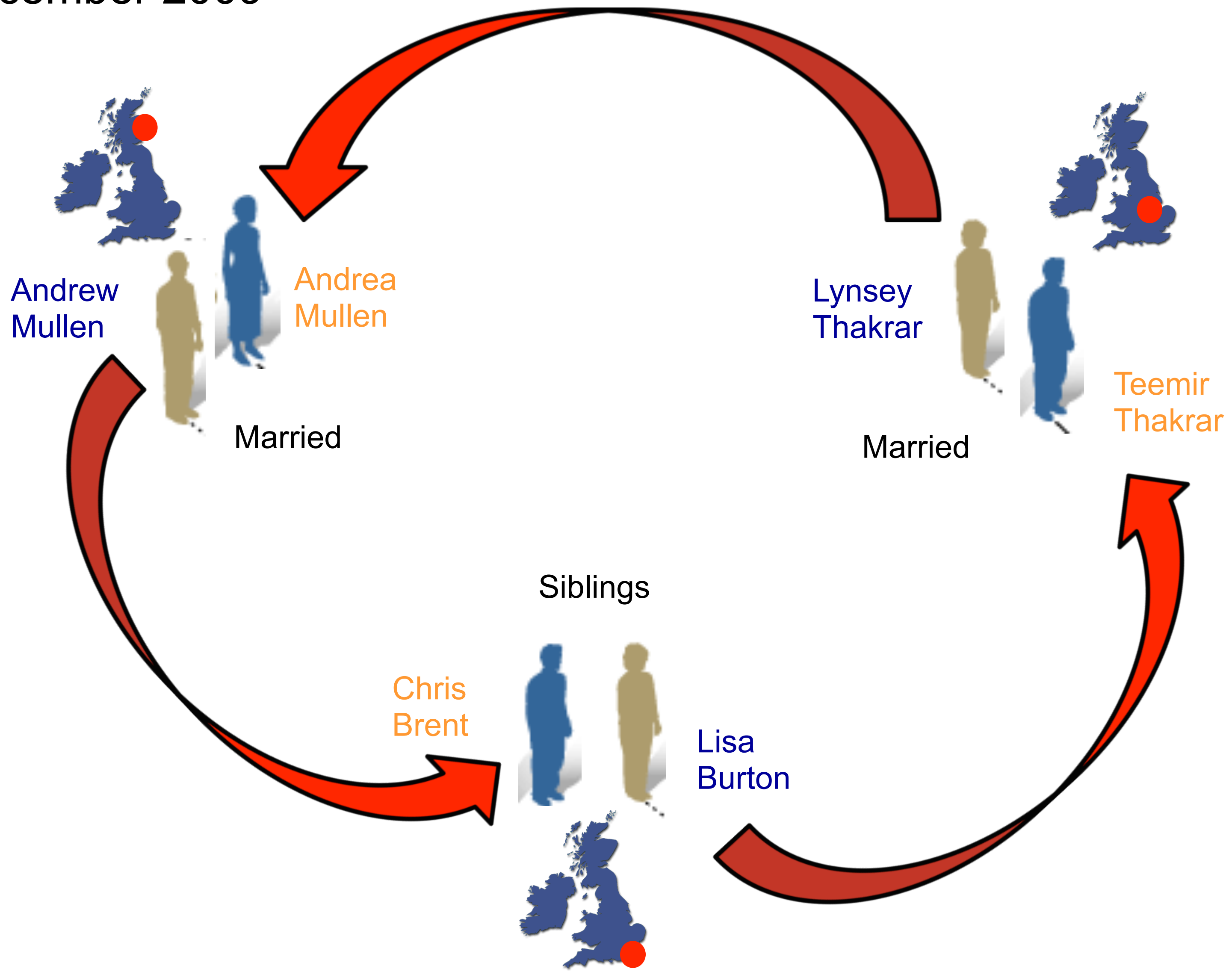
Pairwise exchange

- Portsmouth / Plymouth 2007



3-way exchange

- 4 December 2009

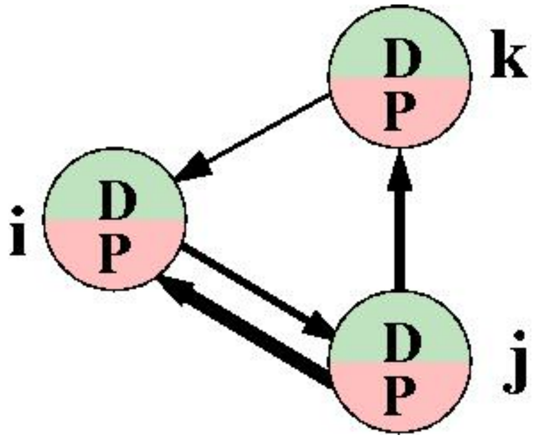


Kidney exchange programs around the world

- US Programs:
  - New England Program for Kidney Exchange since 2004
  - Alliance for Paired Donation
    - [Roth, Sönmez and Ünver, 2004, 2005]
  - Dec 2010: first exchanges performed as part of a national pilot program by the Organ Procurement and Transplantation Network
  - Mostly involving pairwise and 3-way exchanges, but sometimes even longer (a 6-way exchange was performed in April 2008)
- Other countries:
  - The Netherlands
    - [Keizer, de Klerk, Haase-Kromwijk and Weimar, 2005; Glorie, Wagelmans and van de Klundert, 2012]
  - South Korea
  - Romania
  - UK
    - National Living Donor Kidney Sharing Schemes (NHS Blood and Transplant)
    - [M and O'Malley, 2012]
- Cycles should be as short as possible

Modelling the problem

- We consider patient-donor pairs as single vertices of a directed graph ***D*=(*V*,*A*)**



- **(*i*,*j*)∈*A*** if and only if donor ***i*** is compatible with patient ***j***

- 2-cycles and 3-cycles in ***D*** correspond to pairwise and 3-way exchanges (no cycles of length >3 permitted)

- Arc weights can likelihood of success of corresponding transplants, patient priorities etc.

Interlude: The Stable Roommates problem

- Input: n agents; each agent ranks a subset of the others in strict order
- Output: a *stable matching*

Definitions

- A *matching* is a set of disjoint pairs of acceptable pairs of agents
- A *blocking pair* of a matching M is an acceptable pair of agents $\{a_i,a_j\}\notin M$ such that:
  - a_i is unmatched or prefers a_j to his partner in M, and
  - a_j is unmatched or prefers a_i to his partner in M
- A matching is *stable* if it admits no blocking pair

Connection with kidney exchange

- Here agent a_i corresponds to donor-patient pair (d_i, p_i)
- a_i finds a_j acceptable if and only if d_i is compatible with p_j
- Preference lists can reflect varying level of compatibility
- A matching is then a set of pairwise exchanges
- Example SR instance I_1:

a_1: a_3 a_2 a_4
a_2: a_4 a_3 a_1
a_3: a_2 a_1 a_4
a_4: a_1 a_3 a_2

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- The matching is not stable as $\{a_1,a_3\}$ blocks

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- Example SR instance I_1:

a_1: (a_3) a_2 a_4
a_2: (a_4) a_3 a_1
a_3: a_2 (a_1) a_4
a_4: a_1 a_3 (a_2)

- Stable matching

Non-existence of a stable matching

- Example SR instance I_2:

$$\begin{array}{llll} a_1: & a_3 & a_2 & a_4 \\ a_2: & a_1 & a_3 & a_4 \\ a_3: & a_2 & a_1 & a_4 \\ a_4: & a_1 & a_2 & a_3 \end{array}$$

Non-existence of a stable matching

- Example SR instance I_2:

a_1: a_3 a_2 a_4
a_2: a_1 a_3 a_4
a_3: a_2 a_1 a_4
a_4: a_1 a_2 a_3

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Non-existence of a stable matching

- Example SR instance I_2:

a_1: a_3 a_2 a_4
a_2: a_1 a_3 a_4
a_3: a_2 a_1 a_4
a_4: a_1 a_2 a_3

- The matching is not stable as $\{a_2,a_3\}$ blocks

Non-existence of a stable matching

- Example SR instance I_2:

a_1: a_3 a_2 a_4
a_2: a_1 a_3 a_4
a_3: a_2 a_1 a_4
a_4: a_1 a_2 a_3

- The matching is not stable as $\{a_1,a_2\}$ blocks

Non-existence of a stable matching

- Example SR instance I_2:

$$\begin{array}{l} a_1\text{: } a_3 \ \ a_2 \ \ a_4 \\ a_2\text{: } a_1 \ \ a_3 \ \ a_4 \\ a_3\text{: } a_2 \ \ a_1 \ \ a_4 \\ a_4\text{: } a_1 \ \ a_2 \ \ a_3 \end{array}$$

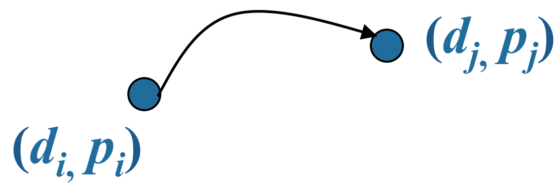
- So no stable matching exists

- [Irving, 1985]: $O(m)$ algorithm to find a stable matching or report that none exists, where m is the total length of the preference lists

- Drawbacks of the model:
  - Ordinal preferences
  - Pairwise exchanges only
  - Potential non-existence of a solution

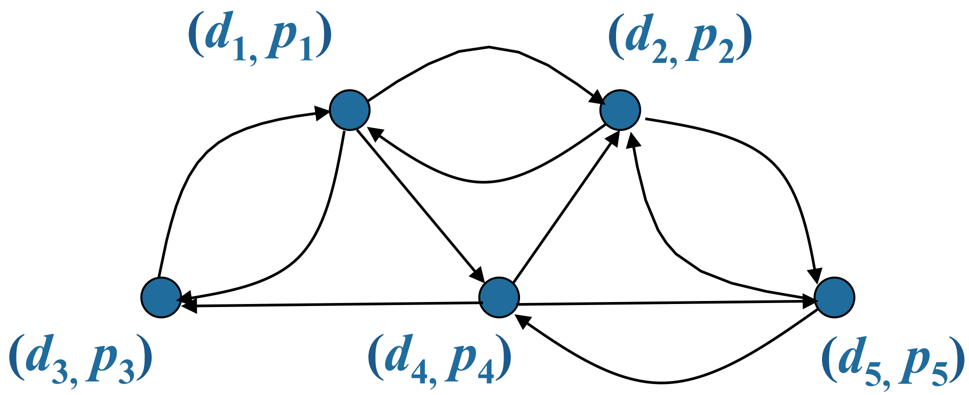
NHS Blood and Transplant's scoring system

A score ≥ 0 is given to each arc (i,j):

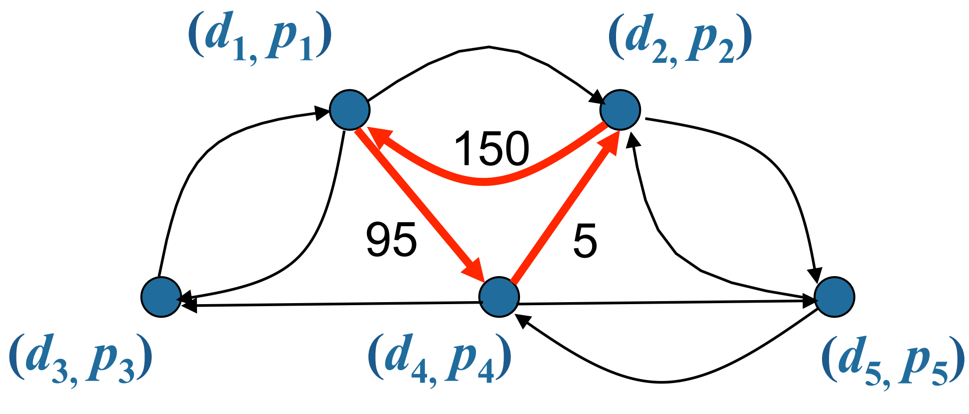


- Waiting time
  - 50 × number of previous matching runs that p_j has been involved in
- Sensitisation points (0-50)
  - Based on calculated sensitisation ("panel reactive antibody") test % for p_j divided by 2
- HLA mismatch points (0, 5, 10 or 15)
  - HLA ("Human Leukocyte Antigen") mismatch levels determine tissue-type incompatibility between d_i and p_j
- Donor-donor age difference (0 or 3)
  - 3 points if $|\text{age}(d_i) - \text{age}(d_j)| \leq 20$ years, 0 otherwise
- "Final discriminator" involving *actual* donor-donor age difference

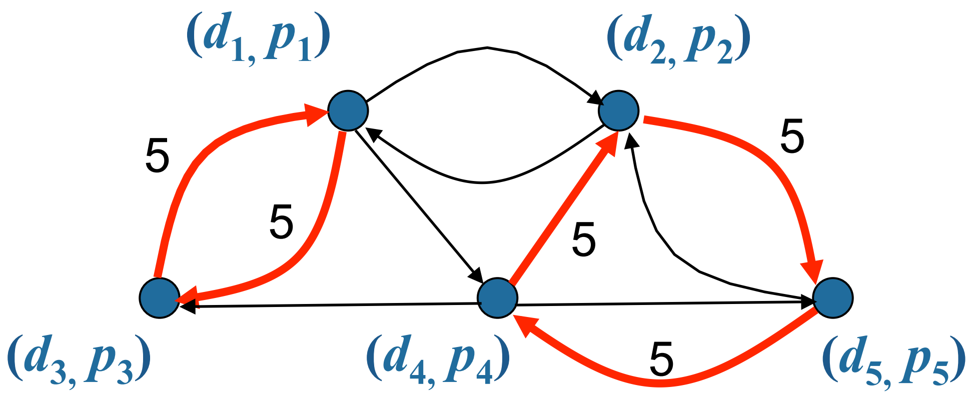
What to optimise?



What to optimise?



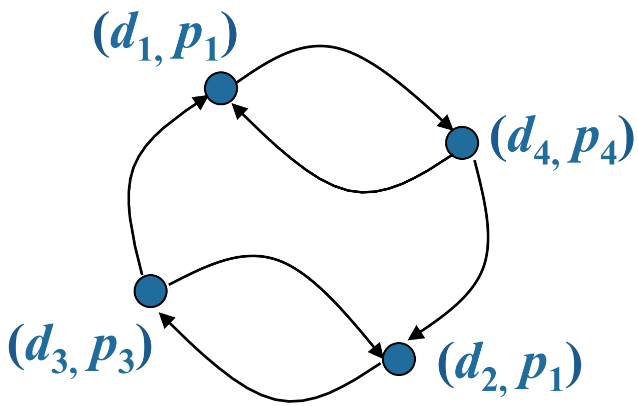
3 transplants
Total weight 250



5 transplants
Total weight 25

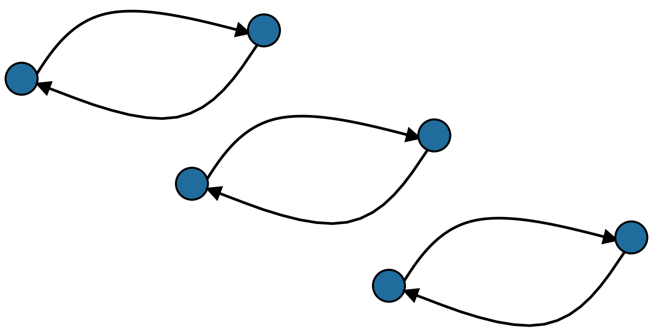
Some extensions

- Patients with multiple donors
  - e.g., both parents (d_1 and d_2) are willing donors for their child (p_1)

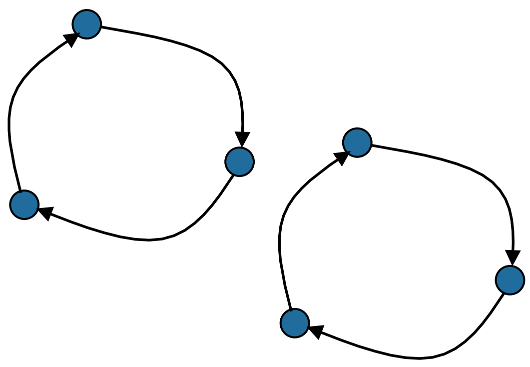


  - at most one of d_1 and d_2 should be used!

- Minimising the number of 3-way exchanges

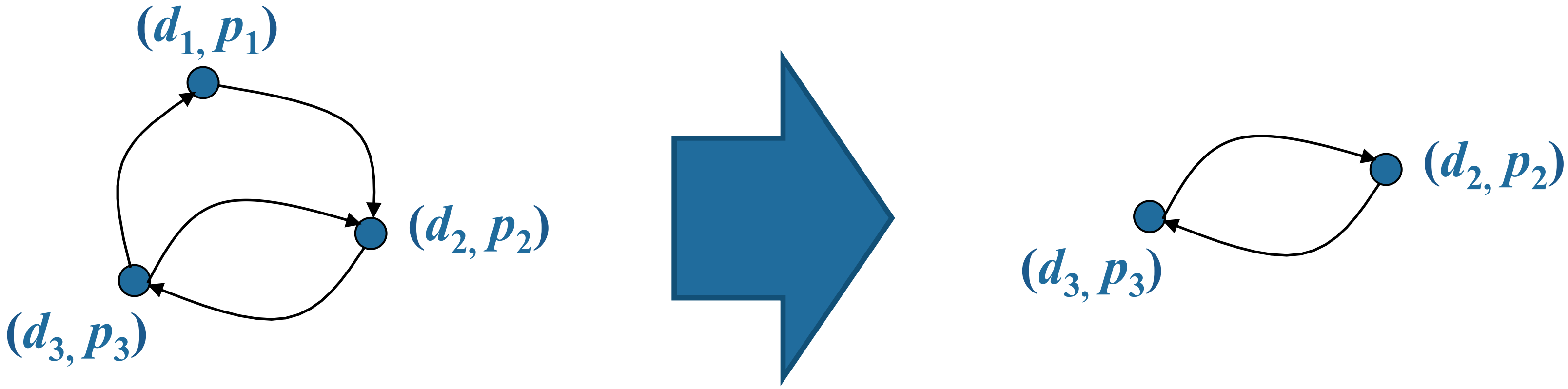


is less risky than



3-way exchanges with "back-arcs"

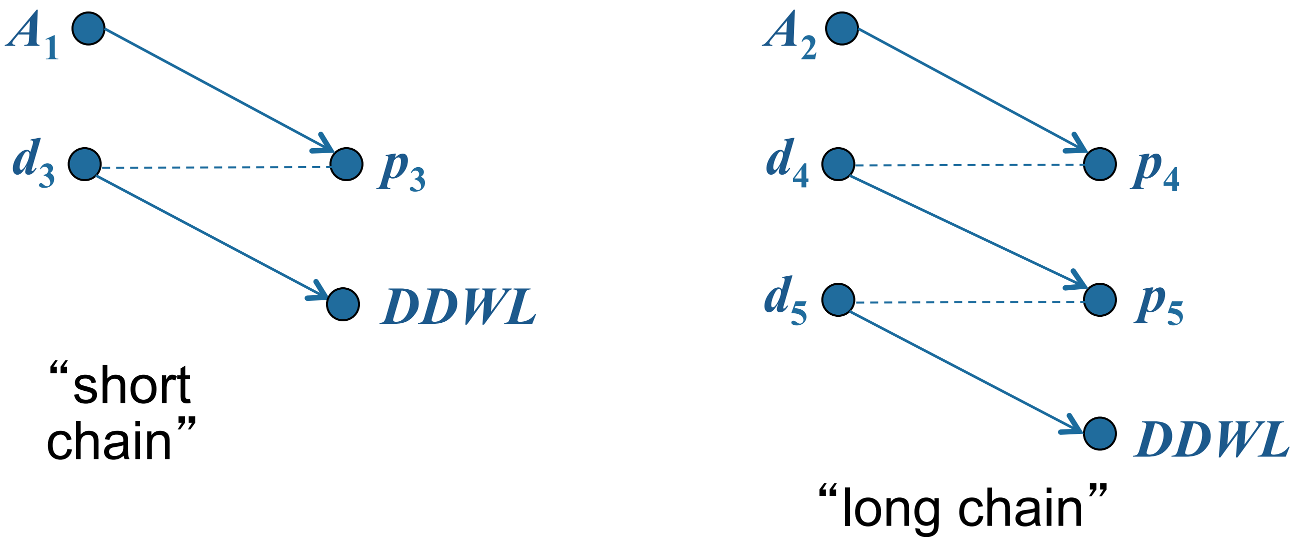
- A 3-way exchange with a *back-arc* has an embedded pairwise exchange



  - If (d_1,p_1) drops out then the embedded pairwise exchange could still proceed
  - So the pairwise exchange involving (d_2,p_2) and (d_3,p_3) could be "extended" to a 3-way exchange involving (d_1,p_1) too, with relatively little additional risk
  - If either (d_2,p_2) or (d_3,p_3) drops out then drops out then the pairwise exchange would have failed in any case

Domino paired donation chains

- Altruistic donors can trigger "domino paired donation chains" (DPD chains)



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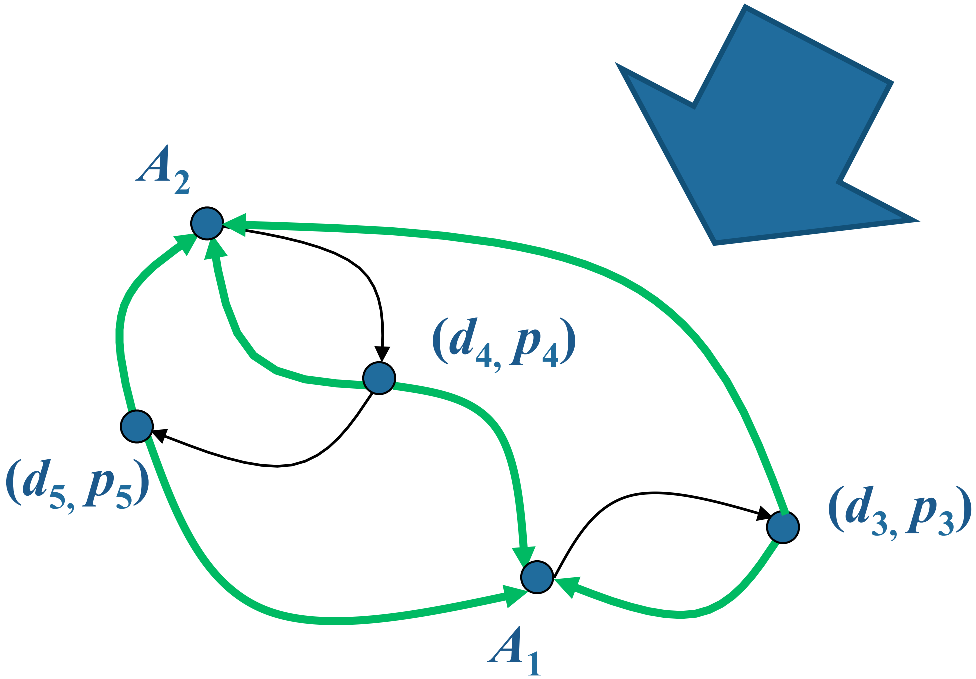


- At most one altruistic donor per cycle!

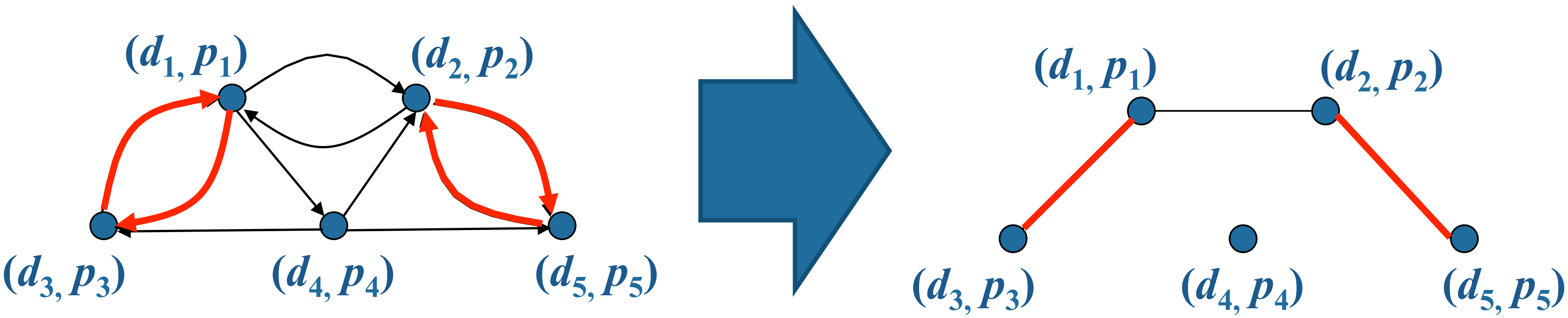
The optimisation problem

- A *set of exchanges* is a permutation π of V into cycles of length ≤ 3 such that $i \neq \pi(i)$ implies $(i,\pi(i)) \in A(D)$
- A vertex $i \in V$ is *covered* by π if $i \neq \pi(i)$
- A set of exchanges is *optimal* if

1. the number of *effective pairwise exchanges* (i.e., no. pairwise exchanges plus no. 3-way exchanges with a back-arc) is maximised
2. subject to (1), the number of vertices covered by π (i.e., the total number of transplants) is maximised
3. subject to (1)-(2), the number of 3-way exchanges is minimised
4. subject to (1)-(3), the number of back-arcs in the 3-way exchanges is maximised
5. subject to (1)-(4), the overall weight is maximised.

1: Maximising pairwise exchanges

- We transform the directed graph ***D*** to an undirected graph ***G***



- A maximum (cardinality) matching in ***G*** corresponds to a maximum set of pairwise exchanges in ***D***

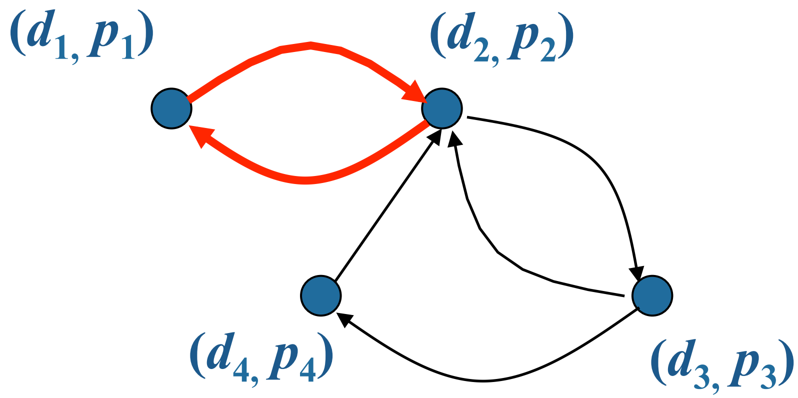
- The problem of finding a maximum matching in ***G*** can be solved in polynomial time by Edmonds' algorithm
  - [Micali and Vazirani, 1980]

- Let N_2 be the size of a maximum matching M in ***G***

2: Maximising overall number of transplants

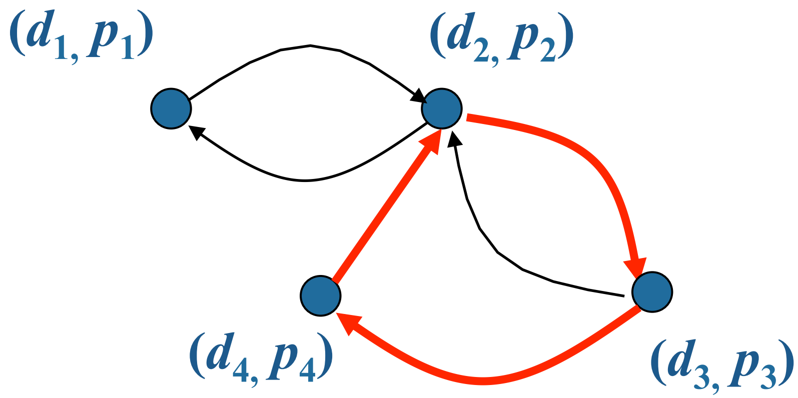
- Maximising the overall number of vertices covered by π

- Finding a maximum cycle cover in D involving only 2- and 3-cycles is:
  - NP-hard
    - [Abraham, Blum and Sandholm, 2007]
  - APX-hard
    - [Biró, M and Rizzi, 2009]



2 transplants

- Heuristics are not acceptable
  - must find an optimal solution



3 transplants

- Exponential-time exact algorithm
  - avoid trying out all possibilities
  - use integer programming
  - [M and O'Malley, 2012]

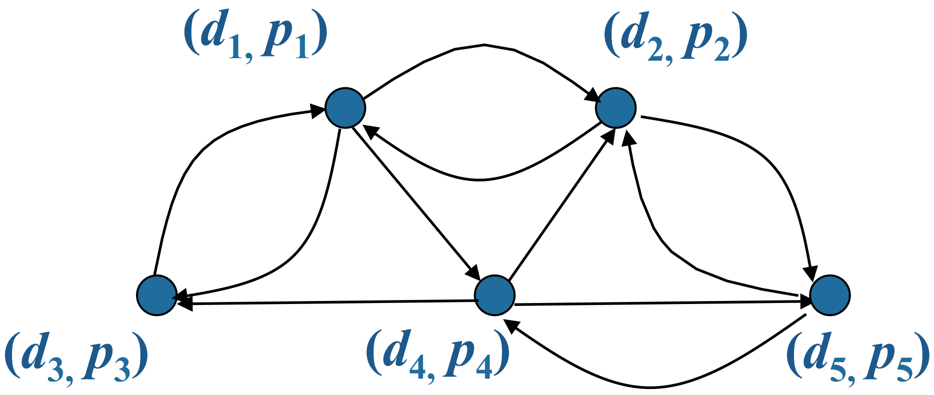
Integer linear program

- We create an integer program as follows:

  - list all the possible cycles (exchanges) of lengths 2 and 3 in the directed graph as $C_1, C_2, \ldots, C_m$
  - use binary variables $x_1, x_2, \ldots, x_m$
  - where $x_i = 1$ if and only if C_i belongs to an optimal solution
  - build an $n \times m$ matrix A where $n = |V|$ and $A_{i,j} = 1$ if and only if v_i is incident to C_j
  - let b be an $n \times 1$ vector of 1s
  - let c be a $1 \times m$ vector of values corresponding to the optimisation criterion, e.g., c_j could be the length of C_j
  - Then solve $\max cx$ such that $Ax \leq b$, subject to $x \in \{0,1\}^m$
  - [Roth, Sönmez and Ünver, 2007]

Integer linear program: example

max cx
s.t. $Ax \leq b$
and $x_i \in \{0, 1\}$



where

$$A = \left[\begin{array}{cccc|ccc} 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right], \quad b = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} \text{ and}$$

$$c_s = \left[\begin{array}{cccc|ccc} 2 & 2 & 2 & 2 & 3 & 3 & 3 \end{array}\right]$$

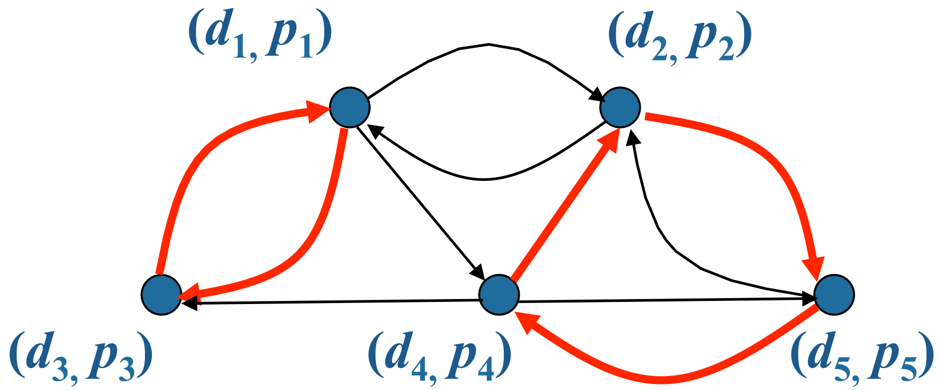
Integer linear program: example

max cx

s.t. $Ax \leq b$

and $x_i \in \{0, 1\}$

where



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$$c_s = \left[\begin{array}{cccc|ccc} 2 & 2 & 2 & 2 & 3 & 3 & 3 \end{array}\right] \qquad \max c_s x = 5$$

Amended ILP

- Suppose that, in D:
  - the 2-cycles are $C_1, \ldots, C_{n_2}$
  - the 3-cycles are $C_{n_2+1}, \ldots, C_{n_2+n_3}$
  - the 3-cycles with back-arcs are $C_{n_2+1}, \ldots, C_{n_2+n_b}$ $(n_b \leq n_3)$
- Add the following constraint to the ILP and solve:

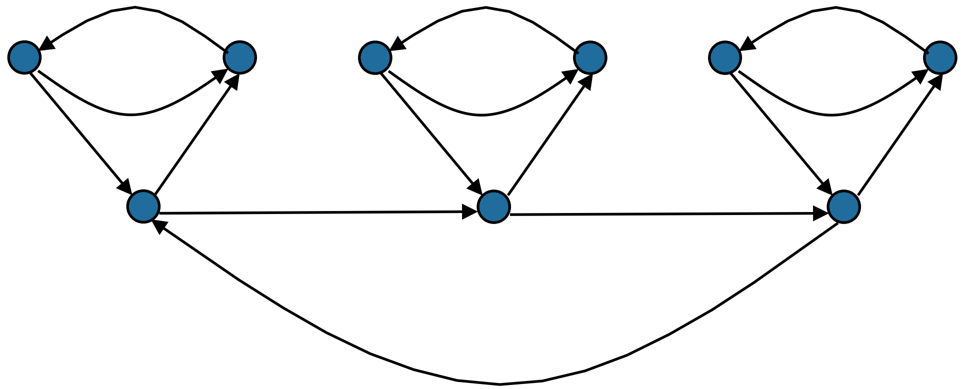
$$x_1 + \ldots + x_{n_2+n_b} \geq N_2$$

- Let $N_{2,3}$ be the maximum number of vertices covered by π (i.e., the number of transplants given by an optimal solution)
- Add the following constraint to the ILP:

$$2x_1 + \ldots + 2x_{n_2} + 3x_{n_2+1} + \ldots + 3x_{n_2+n_3} \geq N_{2,3}$$

3: Minimising number of 3-way exchanges

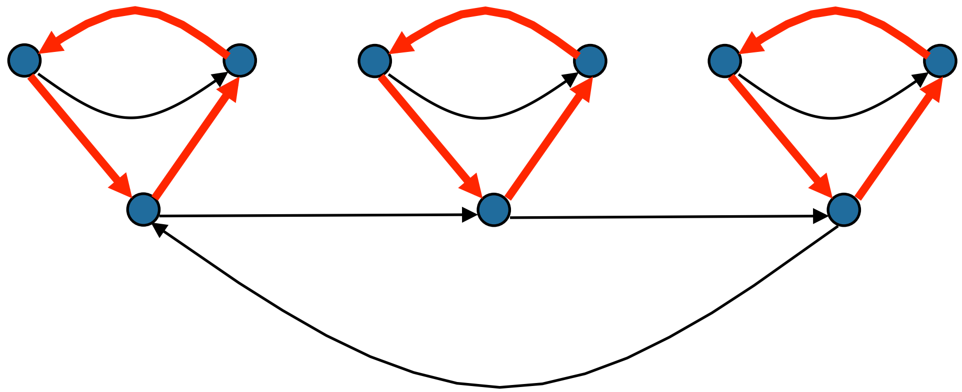
- Example

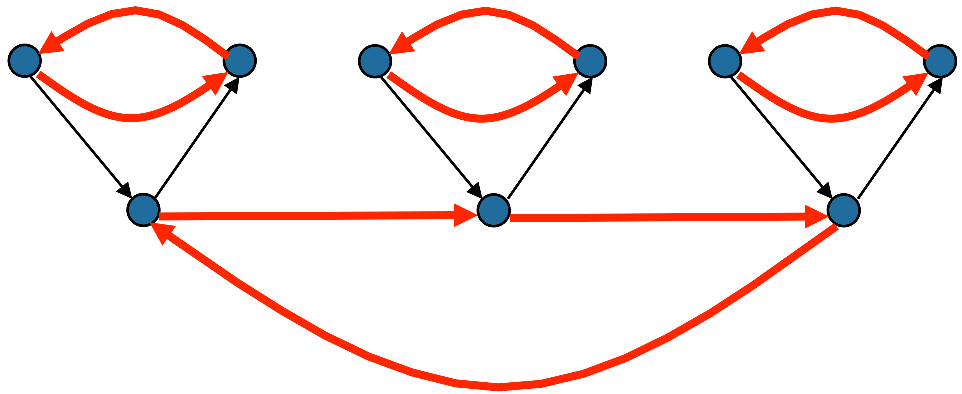


3: Minimising number of 3-way exchanges

- An optimal solution involves **9** transplants

achievable by **3**
three-way exchanges





or by **3** pairwise and **1**
three-way exchange

- Both solutions have **3** effective pairwise exchanges

3: Minimising number of 3-way exchanges

- Let $c_i = 0$ $(1 \le i \le n_2)$, let $c_i = 1$ $(n_2+1 \le i \le n_2+n_3)$ and solve the ILP (objective is to minimise)

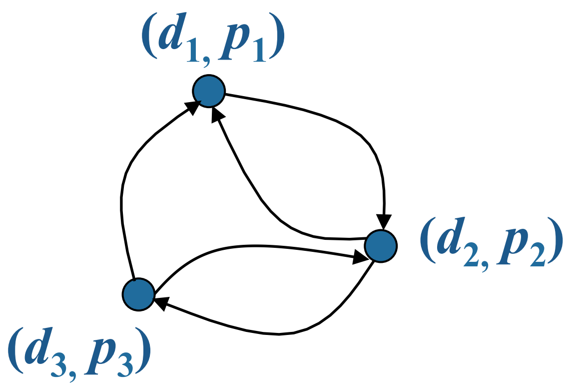
- Let N_3 be the number of 3-way exchanges used in an optimal solution

- Add the following constraint to the ILP:

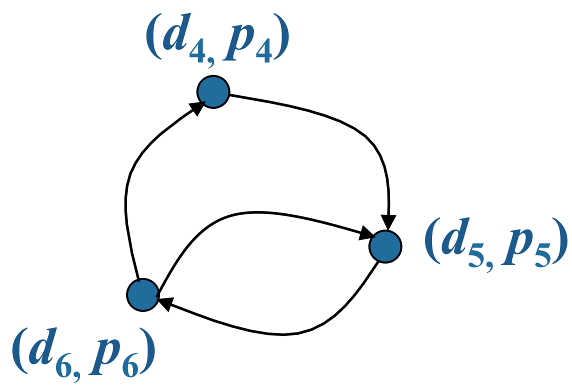
$$x_{n_2+1} + \ldots + x_{n_2+n_3} \le N_3$$

4: Maximising number of back-arcs

- E.g.



beats



- Let $c_i = 0$ $(1 \le i \le n_2)$ and let $c_i = k_i$ $(n_2+1 \le i \le n_2+n_3)$ where k_i is the number of back-arcs in C_i and solve the ILP (objective is to maximise)

- Let N_B be the number of back-arcs in an optimal solution

- Add the following constraint to the ILP:

$$k_{n_2+1} x_{n_2+1} + \dots + k_{n_2+n_3} x_{n_2+n_3} \ge N_B$$

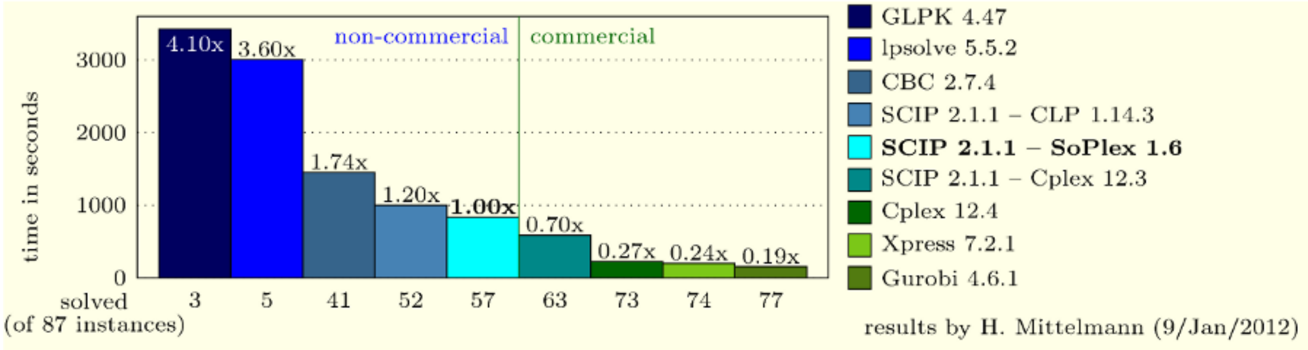
5: Maximising overall weight

- Let c_i be the weight of C_i (sum of the weights of the arcs in C_i)

- Solve the ILP (objective is to maximise)

Choosing an IP solver

- Many free and proprietary solvers on the market
- Difference in performance can be significant



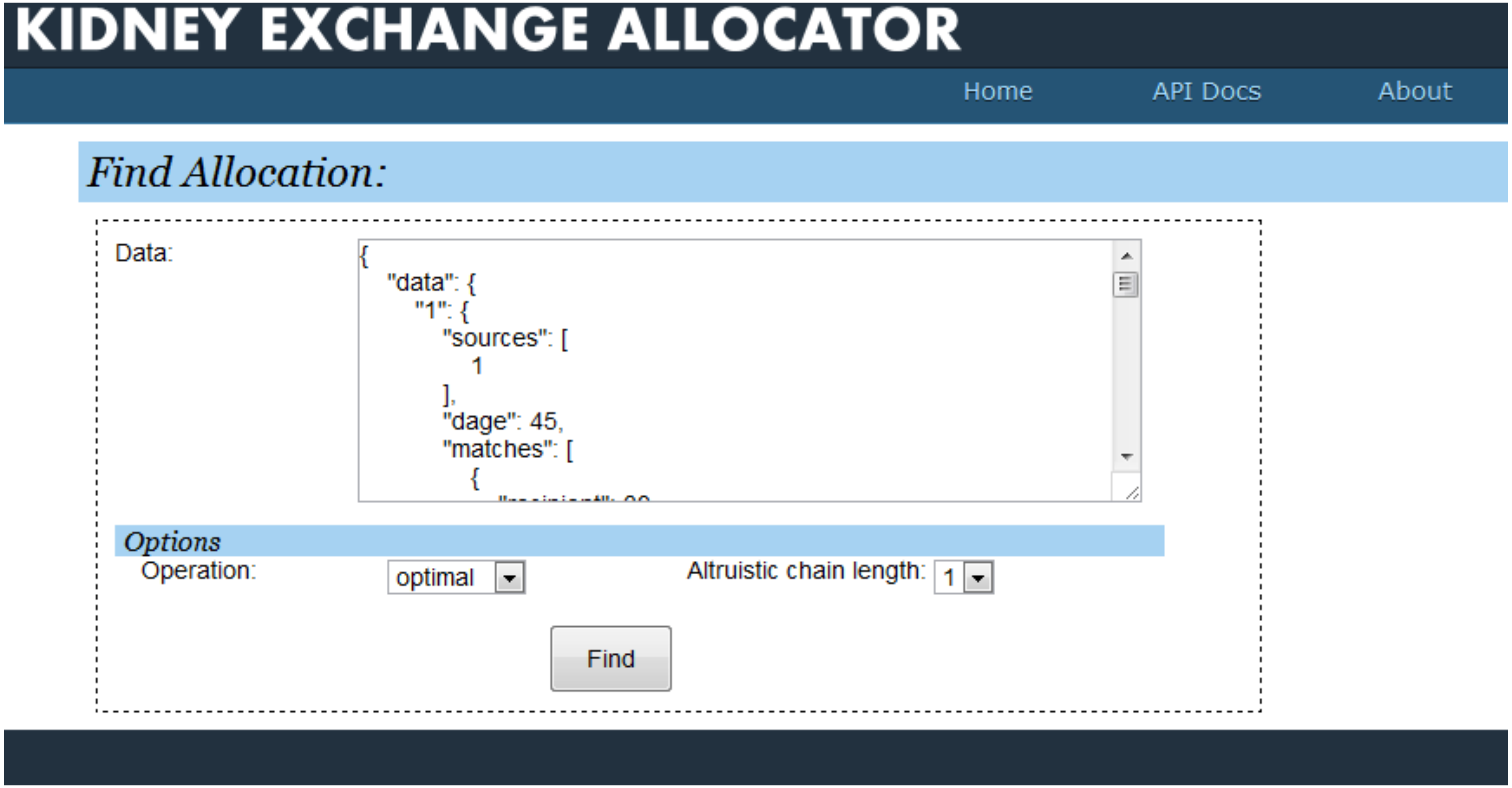
- Cost of many commercial solvers can easily reach >€100k depending on the deployment environment
- We opted for COIN-Cbc
  - Open-source solver library written in C++

Implementation

- Software implemented in C++ using the following packages:
  - COIN-Cbc (ILP solver)
  - LEMON (graph matching library for maximum matching)
  - Ruby on Rails framework for web service
  - Google Test (testing framework)
- Data formats for input / output:
  - XML or JSON
  - Called via the SOAP or REST protocols
- Software can be deployed on Windows, Linux or Solaris
- Demonstration version hosted at kidney.optimalmatching.com
- Running time under 1 second for all real data sets to date

The application UI

- **`http://kidney.optimalmatching.com`**



The application results page

Summary

Operation: optimal *Time taken:* 0.282417s *Total number of 2-cycles:* 113
Total number of 3-cycles: 26

Detailed output

Exchanges

(COIN) Optimal set of exchanges

Exchanges:	[124, 53] [122, 28] [125, 115] [123, 104] [121, 97] [6, 61] [10, 31] [90, 26] [66, 40] [72, 79] [32, 13] [39, 55] [1, 30, 14] [17, 93, 49] [2, 114, 33] [22, 5, 25] [7, 94, 18] [102, 45, 3] [16, 44, 20] [27, 60, 46] [48, 82, 117] [63, 58, 120]
Weight:	2301.62
Total Transplants:	54
Three-Ways:	10
Two-Ways:	12 (15)

All Cycles

2 Cycles

[122, 1] [124, 28] [124, 7] [124, 25] [124, 102] [124, 3] [124, 44] [124, 87] [124, 49] [124, 15] [124, 21] [124, 72] [124, 109] [124, 13] [124, 46] [124, 62] [124, 53] [124, 56] [124, 85] [124, 78] [124, 89] [124, 88] [124, 107] [124, 91] [124, 99] [124, 119] [122, 28] [122, 24] [122, 4] [122, 114] [122, 33] [122, 49] [122, 15] [122, 32] [122, 54] [122, 83] [122, 58] [122, 81] [122, 117] [125, 11] [125, 24] [125, 31] [125, 102] [125, 26] [125, 35] [125, 18] [125, 40] [125, 62] [125, 105] [125, 112] [125, 106] [125, 99] [125, 108] [125, 115] [125, 119] [123, 24] [123, 6] [123, 17] [123, 10] [123, 31] [123, 2] [123, 94] [123, 90] [123, 20] [123, 66] [123, 49] [123, 41] [123, 12] [123, 111] [123, 21] [123, 39] [123, 43] [123, 75] [123, 62] [123,

Results from NHSBT matching runs (1)

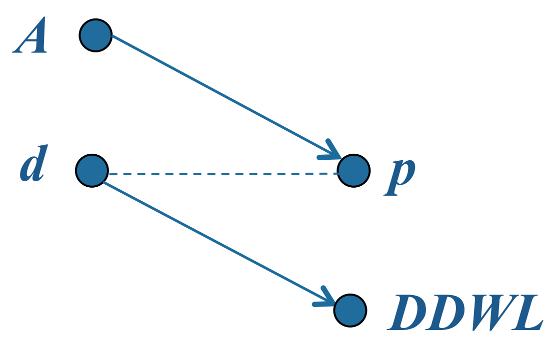
Matching run		2008		2009				2010			
		Jul	Oct	Jan	Apr	Jul	Oct	Jan	Apr	Jun	Oct
Number of pairs		83	123	126	128	141	147	150	158	152	191
Number of arcs		628	1406	1256	1413	1926	1715	1527	1635	1310	1943
Number of 2-cycles		2	14	17	20	55	4	17	23	4	20
Number of 3-cycles		0	116	72	71	166	4	33	77	1	39
Optimal solution	#2 cycles	1	6	5	5	4	0	3	2	3	3
	#3 cycles	0	3	1	2	7	2	1	6	0	2
	size	2	21	13	16	29	6	9	22	6	12
	weight	6	930	422	618	1168	300	135	782	261	473
Actual transplants	#pairwise	1	4	5	2	3	0	2	4	0	3
	#3-way	0	0	0	0	2	2	0	3	0	1
	Total	2	8	10	4	12	6	4	17	0	9

+ 4 pairwise exchanges identified between Apr 07 – Apr 08

Results from NHSBT matching runs (2)

Matching run		2011			
		Jan	Apr	Jun	Oct
Number of pairs		202	176	189	197
Number of arcs		2366	1701	2130	2007
Number of 2-cycles		19	9	34	18
Number of 3-cycles		145	27	101	73
Optimal solution	#2 cycles	3	0	5	7
	#3 cycles	10	4	4	5
	size	36	12	22	29
	weight	1328	464	794	1094
Actual transplants	#pairwise	2	0	2	6
	#3-way	5	2	4	3
	Total	19	6	16	21

- Altruistic donors were introduced into the scheme in January 2012
  - at present they can trigger only short chains or donate directly to the DDWL



"short chain"

Results from NHSBT matching runs (3)

Matching run		2012				2013	
		Jan	Apr	Jun	Oct	Jan	Apr
Number of pairs		195	190	187	215	233	223
Number of altruistic donors		2	3	1	4	9	11
Number of arcs		2902	2494	2190	3315	3905	3720
Number of 2-cycles		115	21	22	35	201	218
Number of 3-cycles		87	46	33	77	46	50
Optimal solution	#2 cycles	1	0	2	6	4	5
	#short chains	2	2	0	4	6	8
	#3 cycles	6	5	2	5	3	5
	size	24	20	11	35	29	41
	weight	2882	1872	1175	3599	2968	4745
Actual transplants	#pairwise	1	1	0	6	5	?
	#short chains	0	1	0	3	3	?
	#3-way	2	4	1	1	1	?
	Total	10	18	4	22	25	?

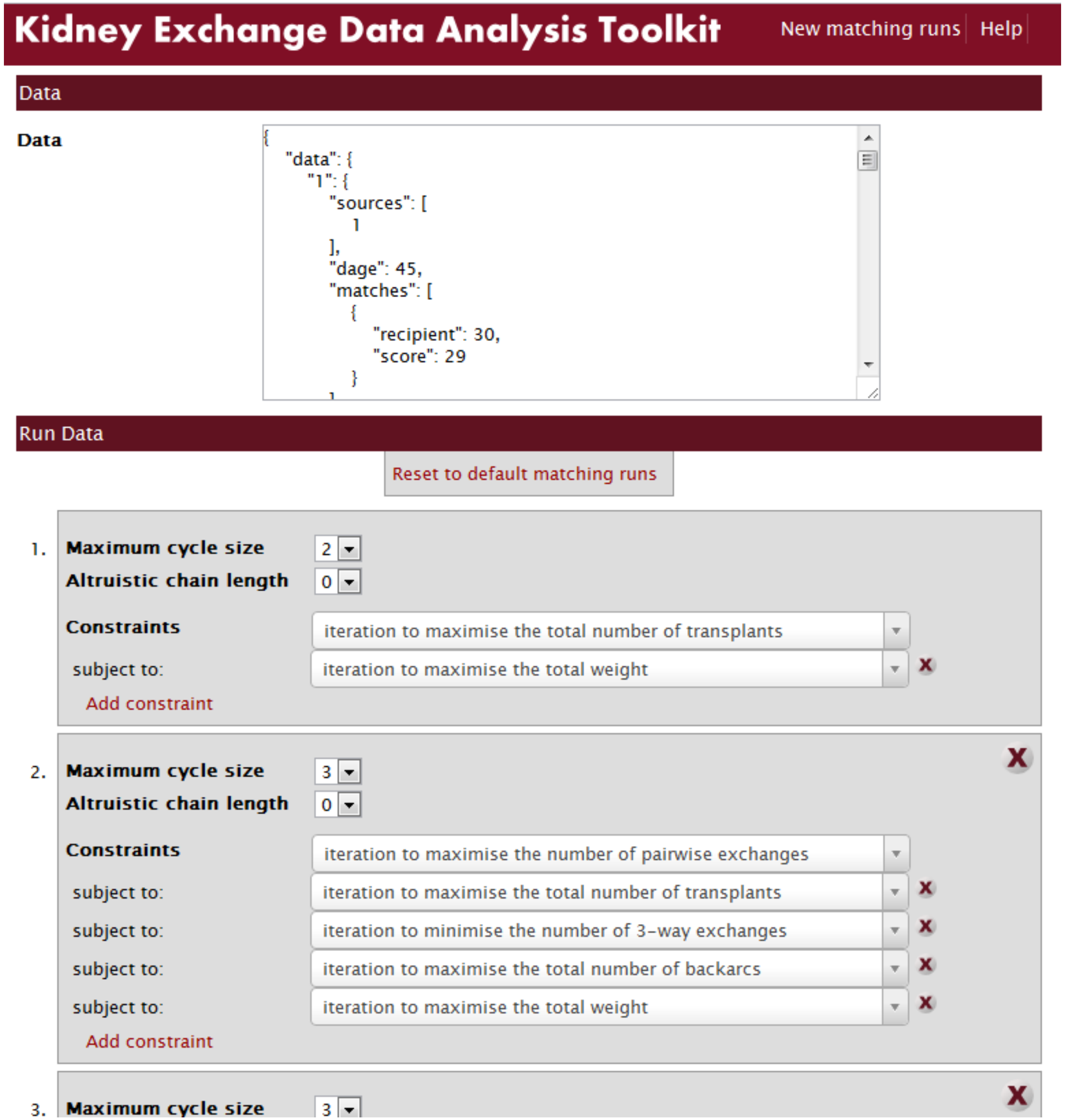
Summary of results

- <u>Identified transplants</u> (over 20 matching runs):
  - Pairwise exchanges: 65
  - 3-way exchanges: 73
  - Short chains: 22
  - Unused altruistic donors: 8
  - Total transplants: **401**
- <u>Actual transplants</u> (over 19 matching runs):
  - Pairwise exchanges: 47
  - 3-way exchanges: 31
  - Short chains: 7
  - Unused altruistic donors: 8
  - Total transplants: **209**

Data Analysis Toolkit

- Due to its complex nature NHSBT were interested in analysing the effect of each constraint on the optimality criteria
- Changing the optimality criteria involved changing code in the C++ library
- It would be easier if there was an application that allowed us to specify the constraints on the matching and the order to apply these constraints
- Even better would be to allow the dynamic creation of new constraints as well
- `http://toolkit.optimalmatching.com`

Data Analysis Toolkit UI



Data Analysis Toolkit results page

Results

SAVE TO DISK

<<Back

Run number	Total transplants	Total paired transplants	Total transplants from altruistic donor chains	Number of unused altruistic donors	Number of 2-ways	Number of 3-ways	Number of 3-ways with embedded	Number of 4-ways	Number of short ADCs	Number of long ADCs	Effective pairwise	Effectiv 3-way
1	25	20	0	5	10	0	0	0	0	0	10	0
2	46	41	0	5	10	7	0	0	0	0	10	7
3	49	44	0	5	7	10	3	0	0	0	10	10

Incentive compatibility

- Hospitals may withhold their easiest-to-match pairs, reporting only their hardest-to-match pairs to the matching scheme

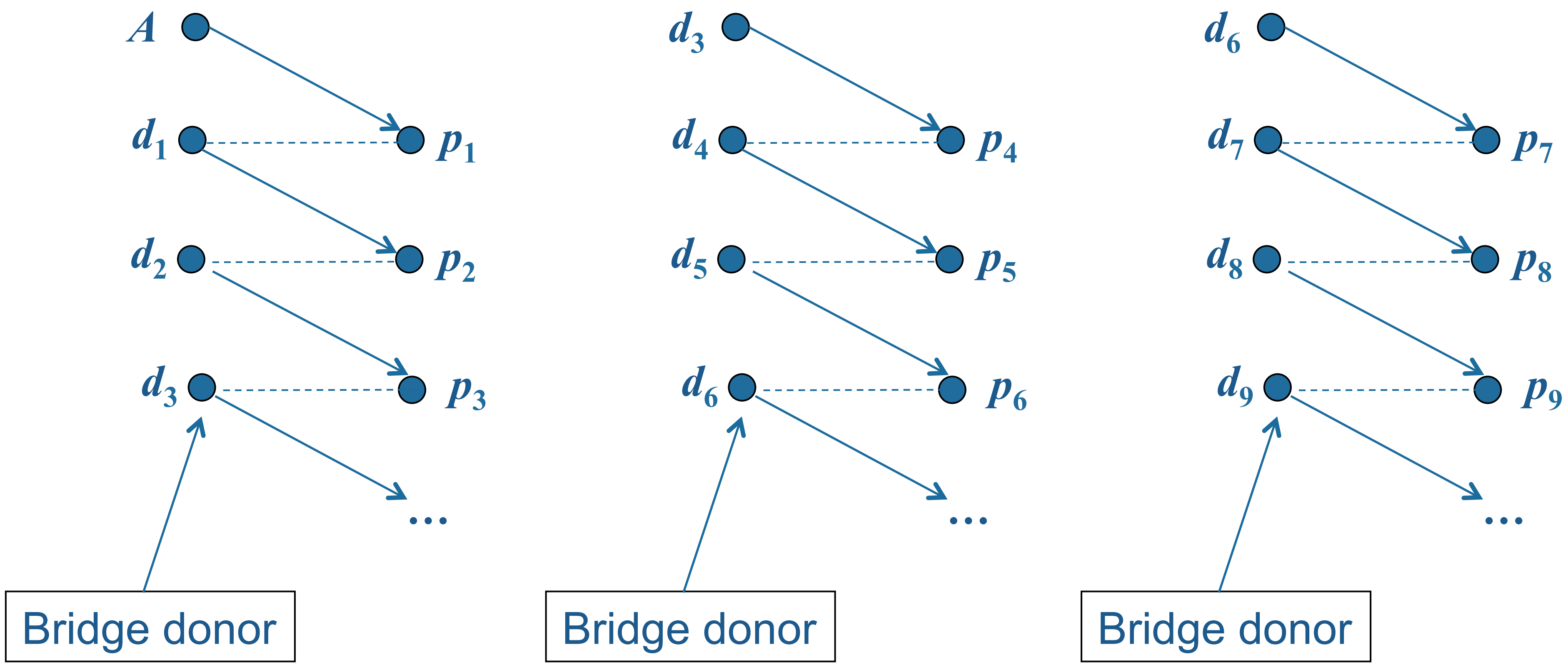
- Patients at other hospitals may lose out on a transplant they may otherwise have received

- Need to incentivise hospitals to behave truthfully
  - [Ashlagi et al., 2010; Ashlagi and Roth, 2011; Caragiannis et al., 2011; Toulis and Parkes, 2011; Ashlagi and Roth, 2012]

- Not an issue in the UK
  - no legal framework allowing a hospital to undertake exchanges outside of the NLDKSS due to tight regulation by the HTA

Future Work (1)

- NEAD (Non-simultaneous Extended Altruistic Donor) chains
  - [Rees MA, Kopke JE, Pelletier, R.P. et al., 2009]



  - Chain segments need not be of the same length

Future Work (2)

- Larger size of datasets
- Further empirical investigation
  - Require artificial dataset generator
- Allow "compatible couples"
  - E.g., d_1 is a willing and compatible donor for p_1, but p_1 could obtain a better match d_2 via a pairwise or 3-way exchange

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    - Joanne Allen (Senior Statistician)

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