Stable allocations and flows

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Summer School on
Matching Problems, Markets, and Mechanisms
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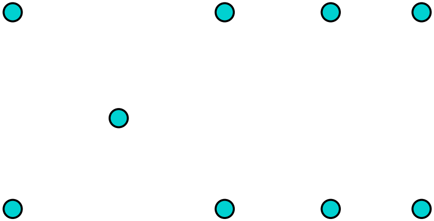
[1]Budapest University of Technology and Economics

Stable matchings

Model:

Stable matchings



Model: **Boys**

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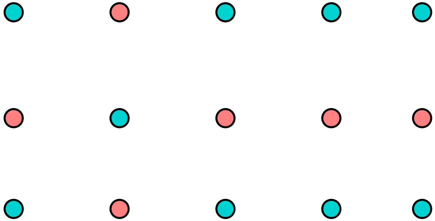

Model: **Boys** and **girls**

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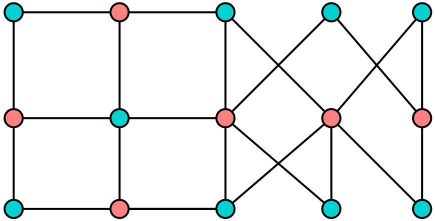

Model: **Boys** and **girls** with possible marriages are given.

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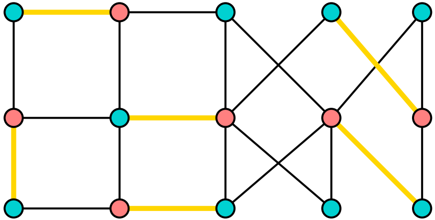

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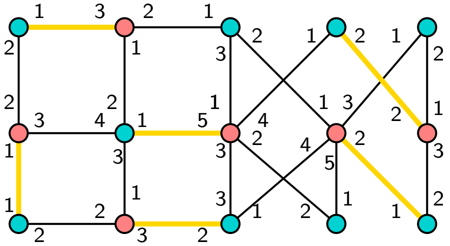
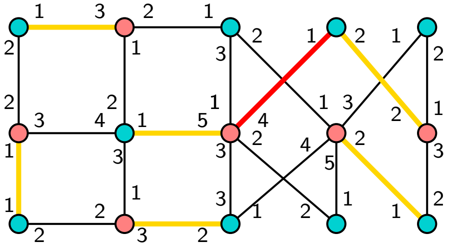
Marriage scheme: **matching**.

Stable matchings



Model: Boys and girls with possible marriages are given.
Marriage scheme: matching. **Preferences** on possible partners.

Stable matchings



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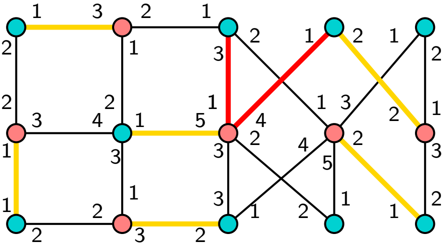

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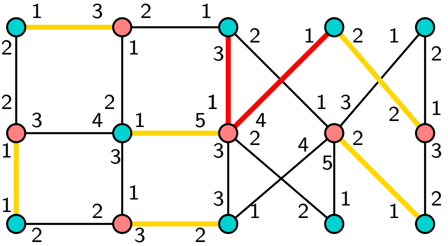

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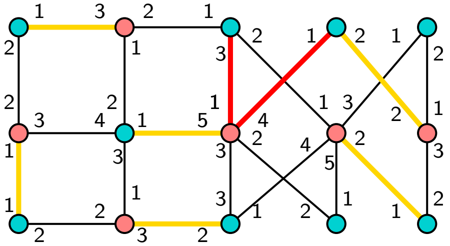

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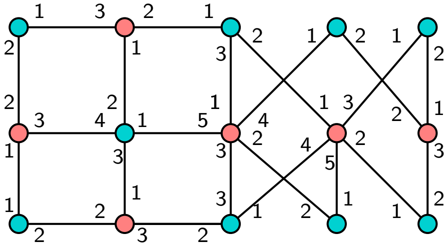

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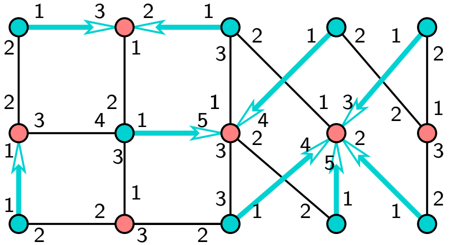

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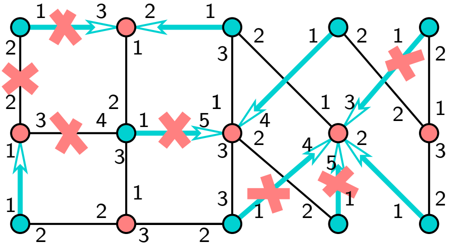

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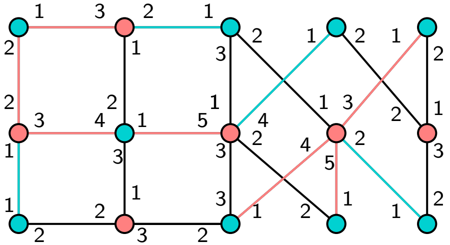

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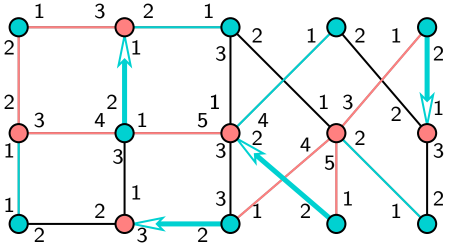

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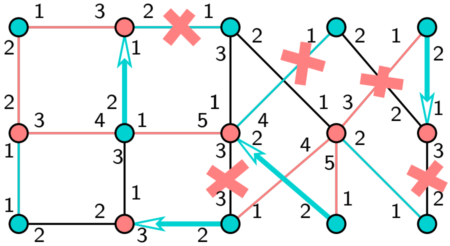

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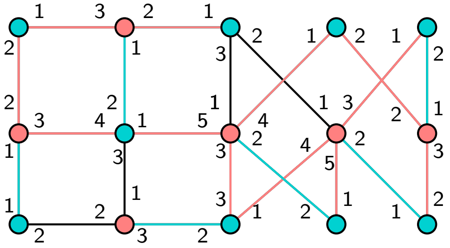

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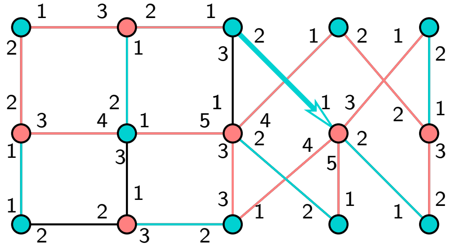

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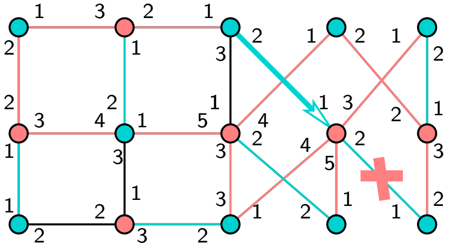

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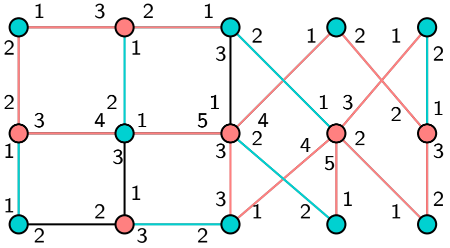

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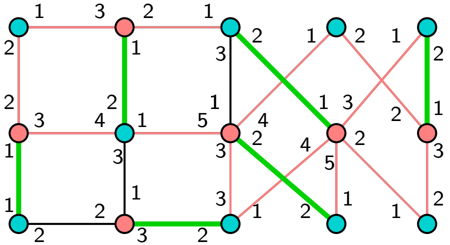

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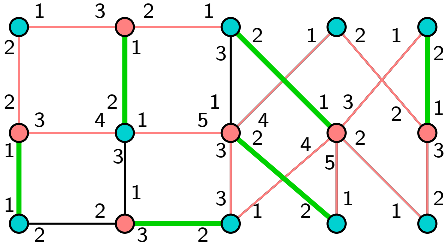

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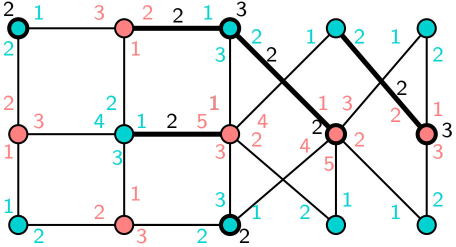
Man-optimality: each boy gets the best stable partner.

Stable allocations and properties

Extension of the model: capacities for vxs and edges (partnerships).

Stable allocations and properties



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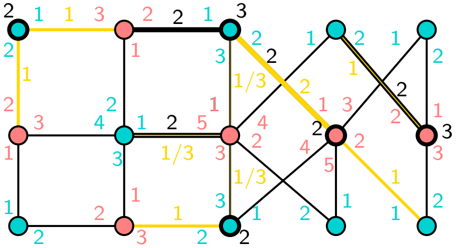

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An **allocation** an assignment of intensities to edges st capacities of edges and vxs are observed.

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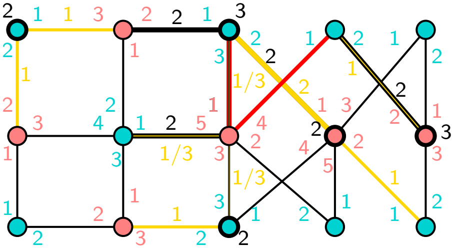

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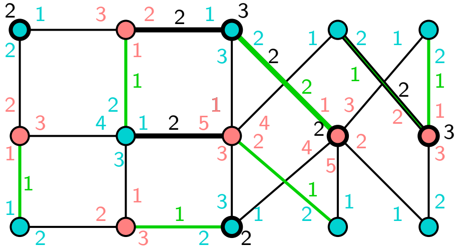

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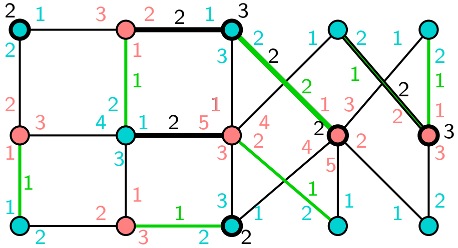

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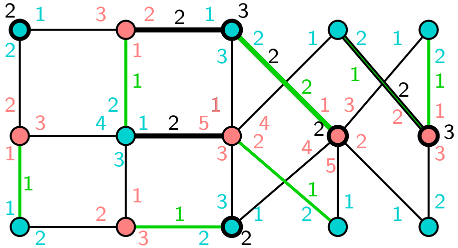

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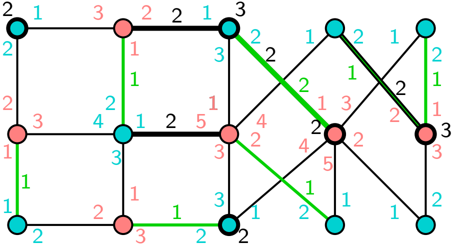

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Lattice property: if boys freely select from two stable alloc's then a stable alloc is created where girls get their worse choice.

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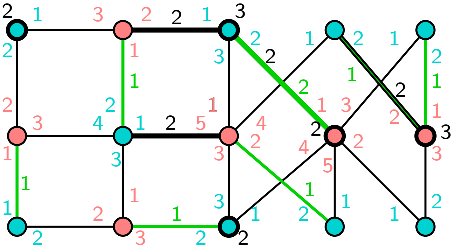

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Stable flows

Network flows: generalization of bipartite matching.

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Stability for network flows??

Stable flows

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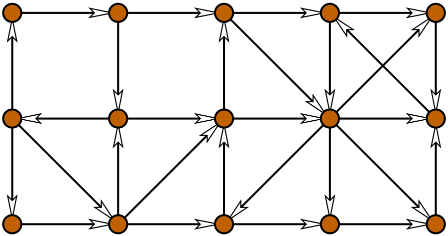
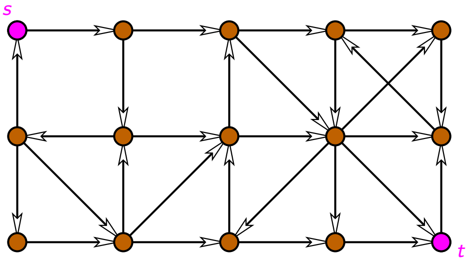


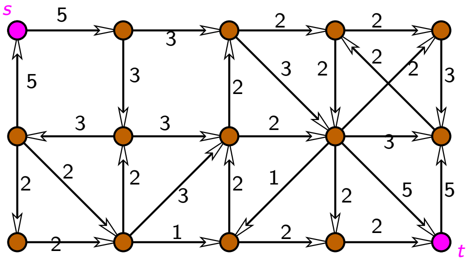
Model: Digraph

Stable flows



Model: Digraph, terminals s, t

Stable flows



Model: Digraph, terminals s, t, capacities on the arcs

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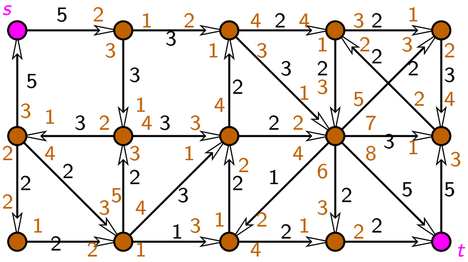

Model: Digraph, terminals s, t, capacities on the arcs and **preferences** on the arcs of the nonterminals.

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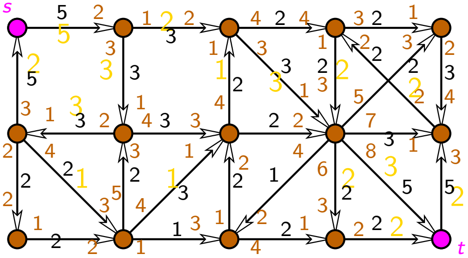

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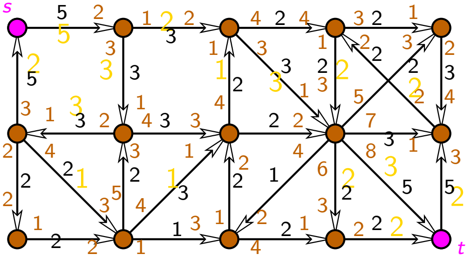

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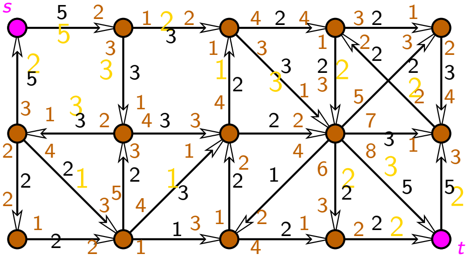

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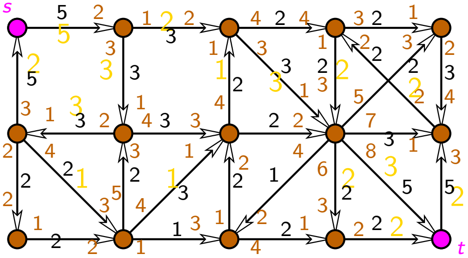

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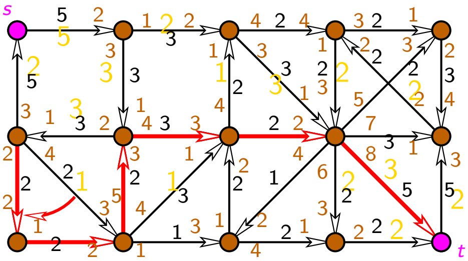

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Formally: a flow is stable if no **blocking walk** exists, i.e. a directed walk on unsaturated arcs such that both ends of the walk is either a terminal or can improve its position by moving some flow from a worse arc onto the walk.

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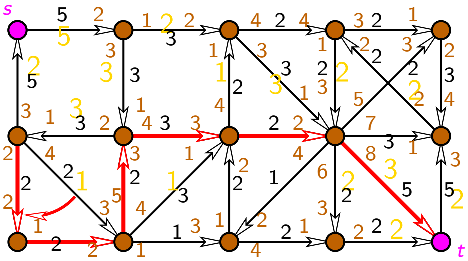

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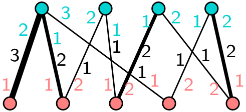
Stable allocations as stable flows



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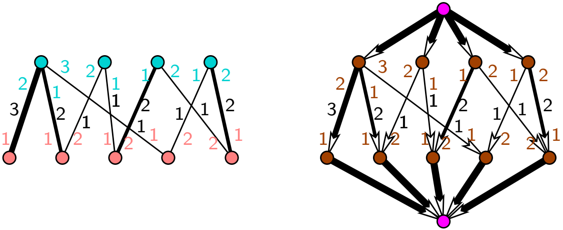

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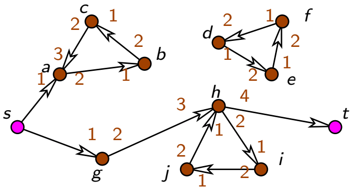
Introduce new terminals s and t and high capacity arcs from s to one color class, and to t from the other color class. Orient all edges from one color class to the other one and keep preferences. (...)

This way any stable allocation can be naturally transformed into a stable flow and any stable flow induces a stable allocation on the original instance.

An example



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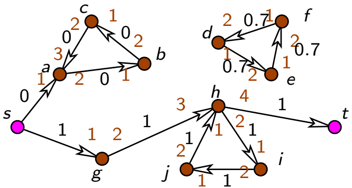

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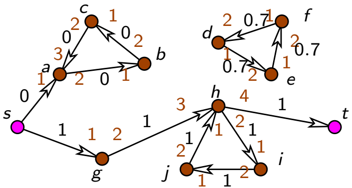

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Directed cycle *abc* cannot carry any flow as otherwise *sa* would be a blocking path.

Directed cycle *def* can carry any flow between 0 and 1.

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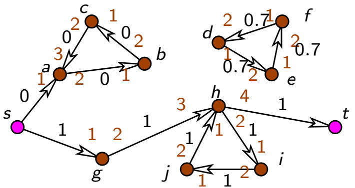
An example



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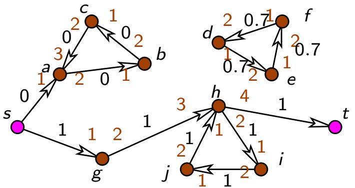

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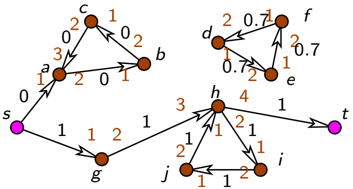

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Theorem: Deciding the existence of a fully stable flow is NP-complete.

Stable flows and stable allocations

Possible proof: extension of the Gale-Shapley algorithm.

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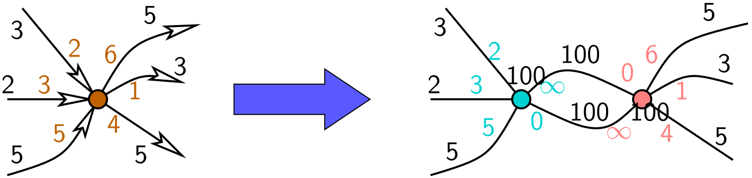

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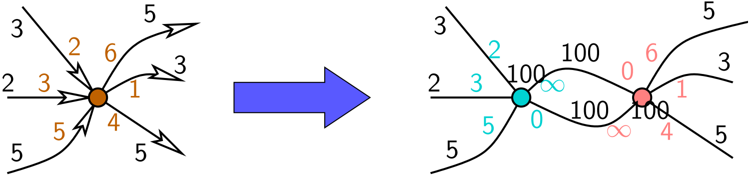

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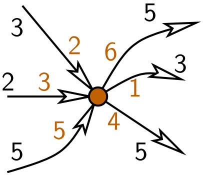

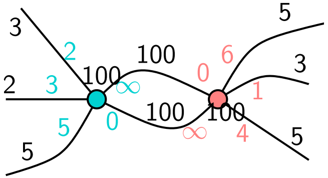

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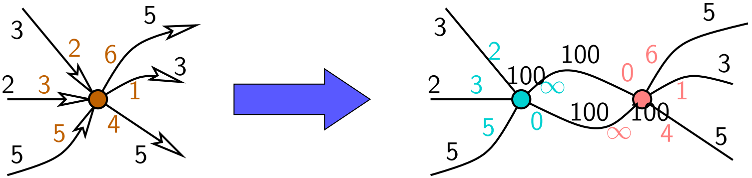

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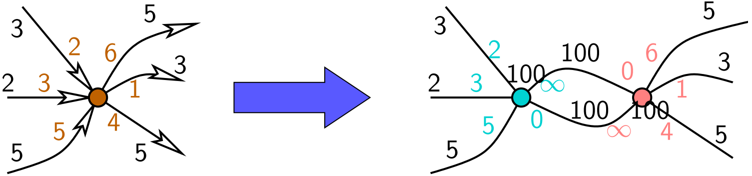

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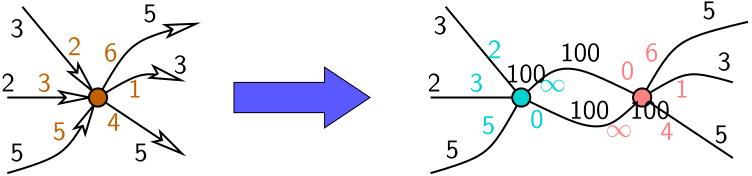
Stable flows and stable allocations



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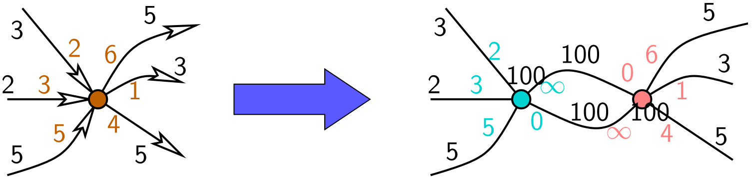
Facts: (1) Any two stable flows have the same value.

Stable flows and stable allocations



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Facts: (1) Any two stable flows have the same value.
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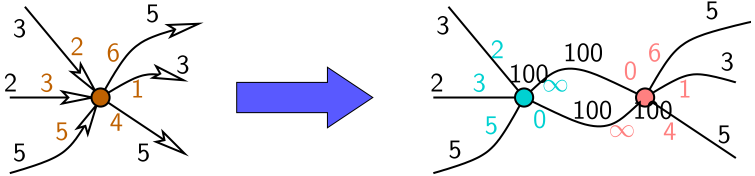

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Facts: (1) Any two stable flows have the same value.
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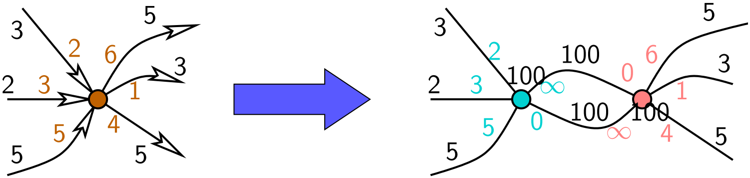
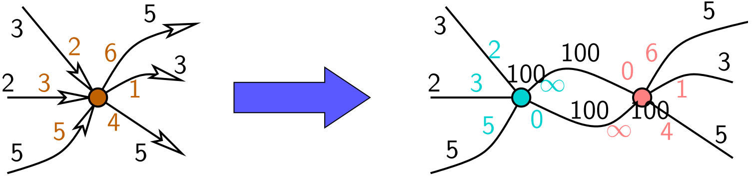
(3) The lattice structure of stable allocations can be generalized.

Lattice structure of stable flows



If f is a stable flow, then each nonterminal vertex is either a "customer" or a "vendor" determined by the canonical stable allocation of f.

Lattice structure of stable flows



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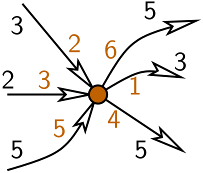

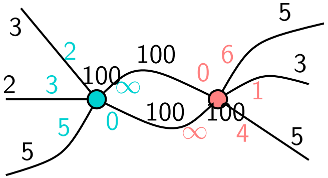

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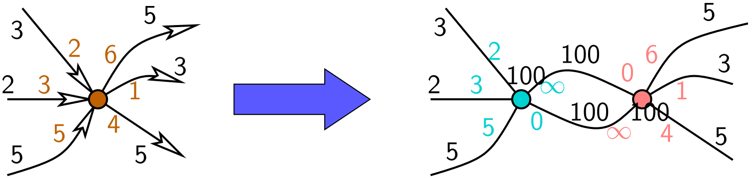

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A vendor prefers to transmit more flow.

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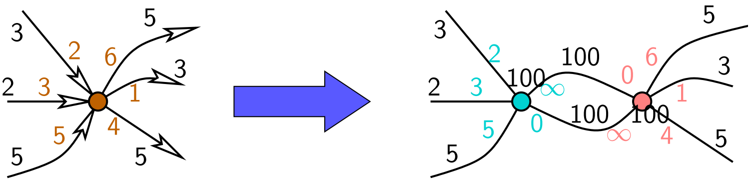

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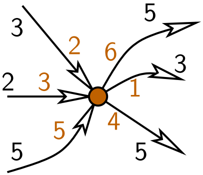

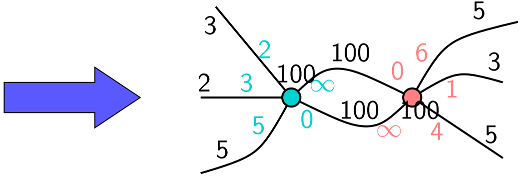

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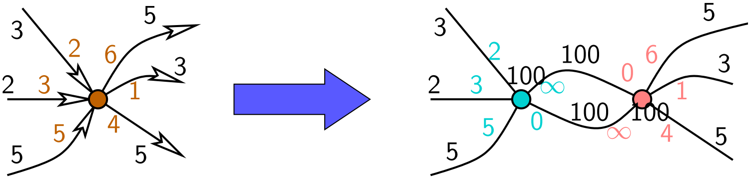

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Otherwise the better selling (worst buying) position is preferred.
Lattice property of stable flows: If two stable flows are given and each nonterminal picks the better (worse) position from the two flows then another stable flow is constructed.

Conclusion

Closely related: Ostrovsky has an earlier result on supply chains.
On one hand, he assumed that the network is **acyclic**.
On the other hand, he could considerably relax the Kirchhoff rule to so called same side substitutability and cross side complementarity. His requirement is that each "agent" transmits goods in a certain monotone manner: buying more means selling more and vice versa.

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Thank you