Computational complexity of competitive equilibria in exchange markets

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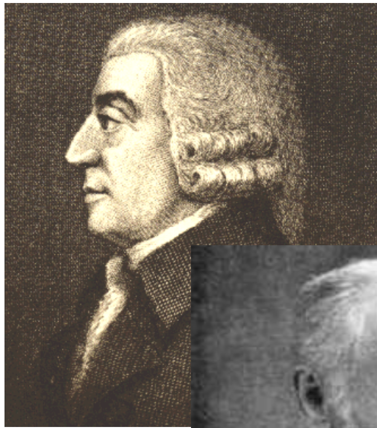
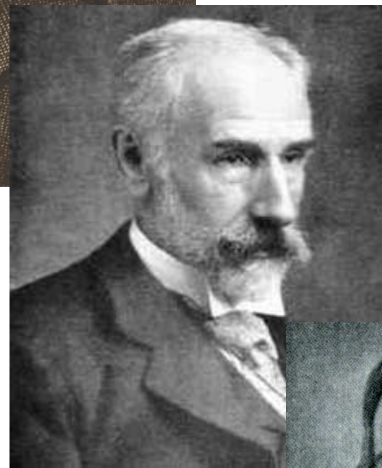
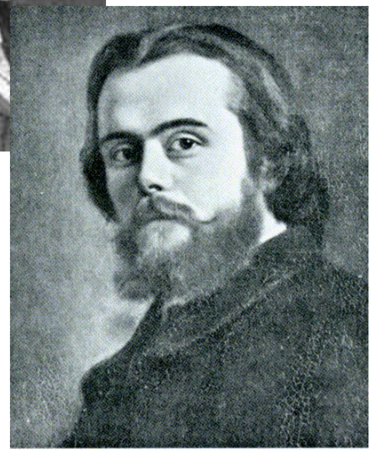
Budapest, Summer school, 2013

Outline of the talk

- brief history of the notion of competitive equilibrium
- model computation for divisible goods
- indivisible goods – housing market
- Top trading cycles algorithm
- housing market with duplicated houses
  - algorithm and complexity
  - approximate equilibrium and its complexity

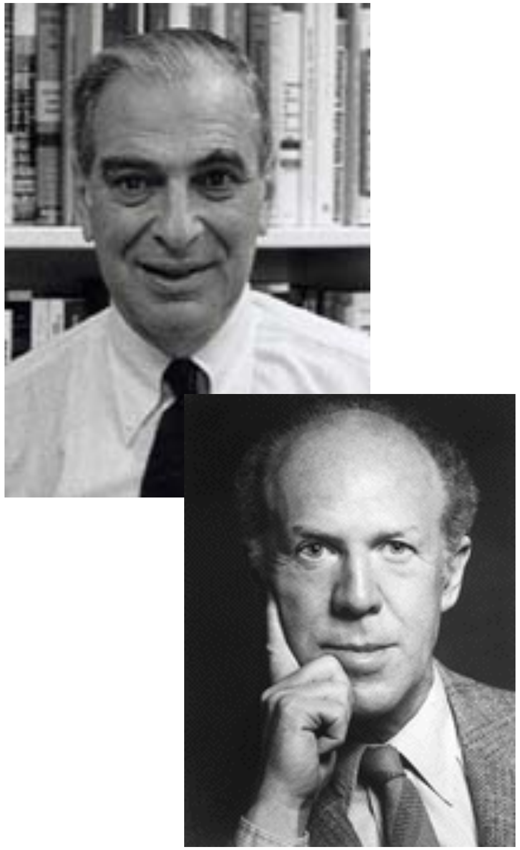
First ideas

- **Adam Smith:** An Inquiry into the Nature and Causes of the **Wealth of Nations (1776)**
- **Francis Ysidro Edgeworth:** Mathematical Psychics: An Essay on the Application of Mathematics to the Moral Sciences (1881)
- **Marie-Ésprit Léon Walras:** Elements of Pure Economics (1874)
- **Vilfredo Pareto:** Manual of Political Economy (1906)


Exchange economy

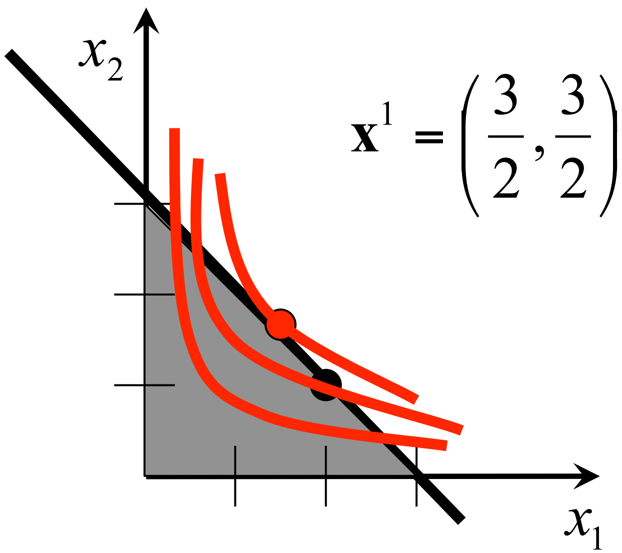
- set of agents, set of commodities
- each agent owns a commodity bundle and has preferences over bundles
- economic equilibrium: pair
  (prices, redistribution) such that:
  - each agent owns the best bundle he can afford given his budget
  - demand equals supply
- if commodities are infinitely divisible and preferences of agents strictly monotone and strictly convex, equilibrium always exists

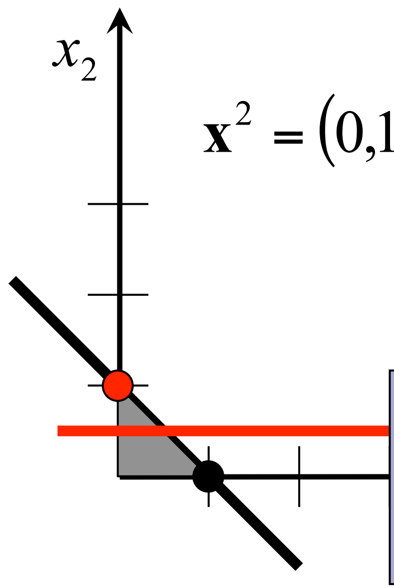


Kenneth Arrow & Gérard Debreu (1954)

Example: two agents, two goods

- agent 1: $\boldsymbol{\omega}^1 = (2,1);\ \ u^1(x_1, x_2) = x_1 x_2$
- agent 2: $\boldsymbol{\omega}^2 = (1,0);\ \ u^2(x_1, x_2) = x_2$
- prices (1,1)





$$\sum \mathbf{x}^i = \left(\frac{3}{2}, \frac{5}{2}\right)$$

$$\sum \omega^i = (3,1)$$

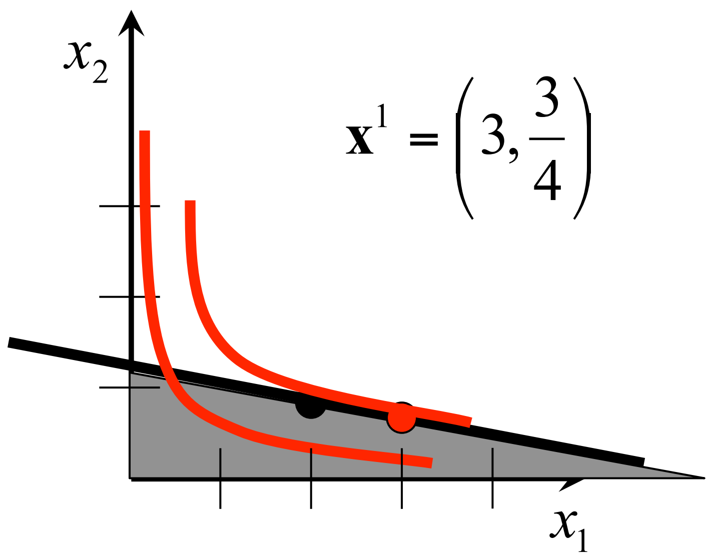
prices (1,1) are not equilibrium, as **supply ≠demand**

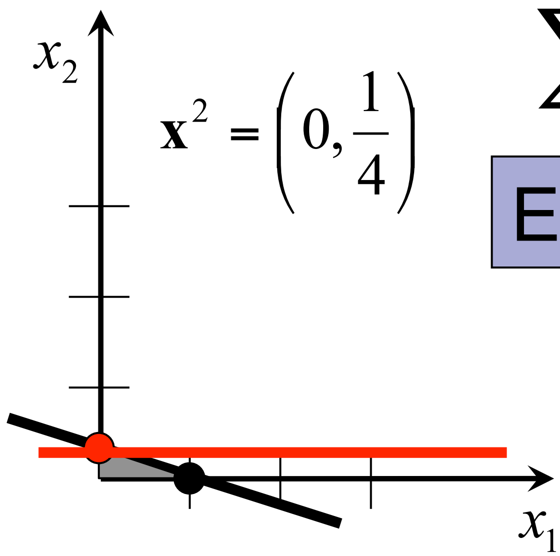
Example - continued

- agent 1: $\boldsymbol{\omega}^1 = (2,1);\ \ u^1(x_1, x_2) = x_1 x_2$
- agent 2: $\boldsymbol{\omega}^2 = (1,0);\ \ u^2(x_1, x_2) = x_2$
- prices (1,4)

$$\sum \mathbf{x}^i = (3,1)$$

$$\sum \boldsymbol{\omega}^i = (3,1)$$

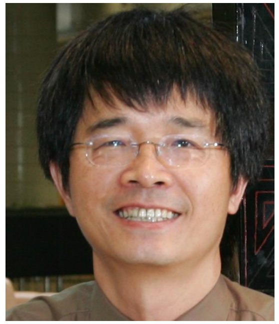




Equilibrium!

Economy with indivisible goods

Equlibrium might not exists!



X. Deng, Ch. Papadimitriou, S. Safra (2002):

Decision problem:

Does an economic equilibrium exist in exchange economy with indivisible commodities and linear utility functions?

NP-complete, already for two agents

Housing market

- **n agents,** each owns one unit of a unique indivisible good – **house**
- **preferences of agent**: linear ordering on a subset of houses
- **Shapley-Scarf economy (1974)**
- **housing market is a model of:**
  - kidney exchange
  - several Internet based markets

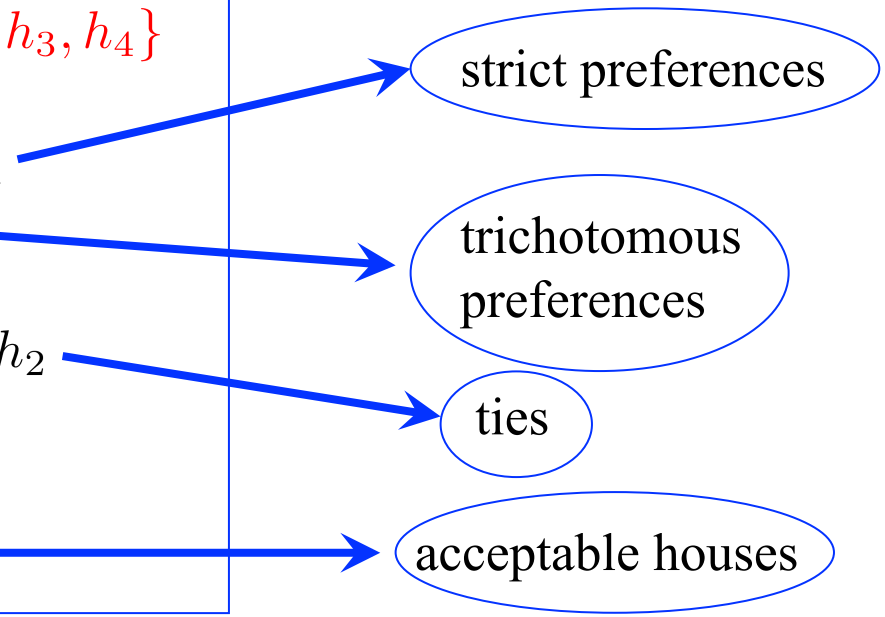
Definition.

A housing market is a quadruple $\mathcal{M} = (A, H, \omega, \mathcal{P})$ where

- A is a set of n agents, H is a set of m house types
- $\omega : A \to H$ is the endowment function
- preference profile $\mathcal{P}$ is an n-tuple of agents' preferences,

$A = \{a_1, a_2, \ldots, a_7\}; \quad H = \{h_1, h_2, h_3, h_4\}$

$$
\begin{array}{ll}
\omega(a_1) = h_1; & P(a_1) : h_4, h_3, h_2, h_1 \\
\omega(a_2) = h_4; & P(a_2) : (h_1, h_3), h_4 \\
\omega(a_3) = h_1; & P(a_3) : h_2, h_4, h_1 \\
\omega(a_4) = h_2; & P(a_4) : (h_1, h_3), h_4, h_2 \\
\omega(a_5) = h_2; & P(a_5) : h_4, h_1, h_2 \\
\omega(a_6) = h_3; & P(a_6) : h_4, h_2 \\
\omega(a_7) = h_4; & P(a_7) : h_1, h_3, h_4
\end{array}
$$



Definition.

A function $x : A \to H$ is an allocation if there exists a bijection π on A such that $x(a) = \omega(\pi(a))$ for each $a \in A$.

Each allocation consists of trading cycles.

$A = \{a_1, a_2, \ldots, a_7\}; \quad H = \{h_1, h_2, h_3, h_4\}$

$\omega(a_1) = h_1;$	$P(a_1) : h_4, h_3, h_2, h_1$
$\omega(a_2) = h_4;$	$P(a_2) : (h_1\ h_3), h_4$
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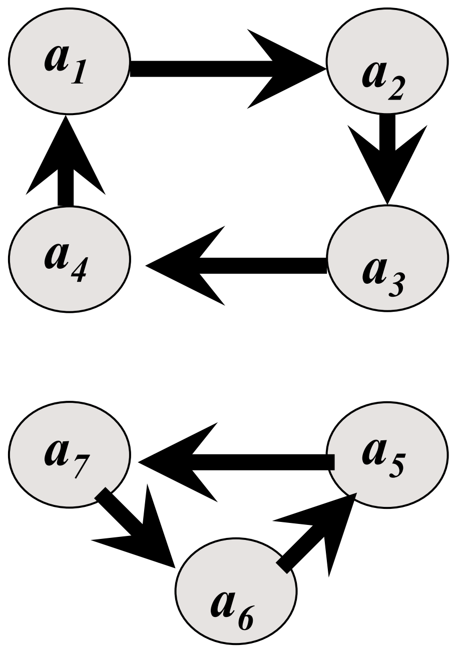

Definition.

A pair (p, x), where $p : H \to R$ is a price function and x is an allocation on A is an economic equilibrium for market $\mathcal{M}$ if for each $a \in A$, house $x(a)$ is among the most preferred house types in his budget set.

Lemma.

If (p, x) f (x, p) is an economic equilibrium for market $\mathcal{M}$ then $p(x(a)) = p(\omega(a))$ for each $a \in A$.

$A = \{a_1, a_2, \ldots, a_7\}; \quad H = \{h_1, h_2, h_3, h_4\}$

$\omega(a_1) = h_1;$	$P(a_1) : h_4, h_3, h_2, h_1$
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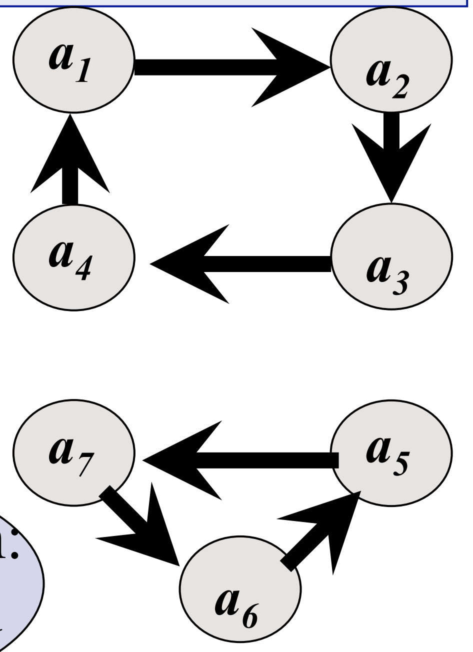

not equilibrium: a_6 not satisfied

Top Trading Cycles algorithm for Shapley-Scarf model (m=n,ω identity)

Step 0. N:=A, round r:=0, p_r=n.

Step 1. Take an arbitrary agent a_0.

Step 2. a_0 points to a most preferred house, in N, its owner is a_1 . Agent a_1 points to the most preferred house a_2 in N etc. A cycle C arises.

Step 3. r:=r+1, p_r= p_r-1; C_r:=C, all houses on C receive price p_r, N:=N-C.

Step 4. If N $\neq\emptyset$, go to Step 1, else end.

- **Shapley & Scarf** (1974): author D. Gale
- **Abraham, KC, Manlove, Mehlhorn** (2004): implementation linear in the size of the market

Top Trading Cycles algorithm for Shapley-Scarf model (m=n,ω identity)

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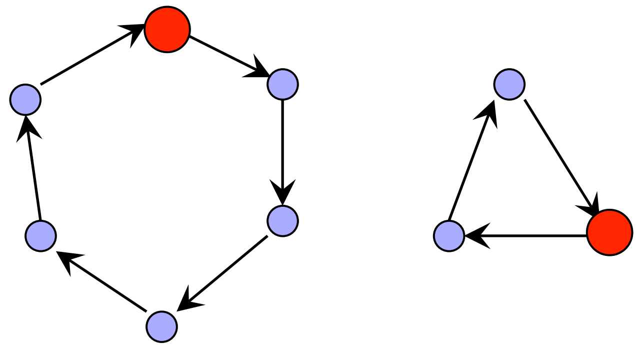
Step 4. If N $\neq \emptyset$, go to Step 1, else end.

Theorem (Gale 1974).

For each Shapley-Scarf housing market, the TTC algorithm outputs an equilibrium pair.

Theorem (Fekete, Skutella , Woeginger 2003).

If the houses of several agents in the market are equivalent, it is NP-complete to decide whether an economic equilibrium exists.



Theorem (KC & Fleiner 2008).

If a trichotomous housing market it is NP-complete to decide whether an economic equilibrium exists. If a market with strict preferences, the existence of equilibrium can be decided in polynomial time.

$$A = \{a_1, a_2, \ldots, a_7\}; \quad H = \{h_1, h_2, h_3, h_4\}$$

$$\begin{array}{ll}
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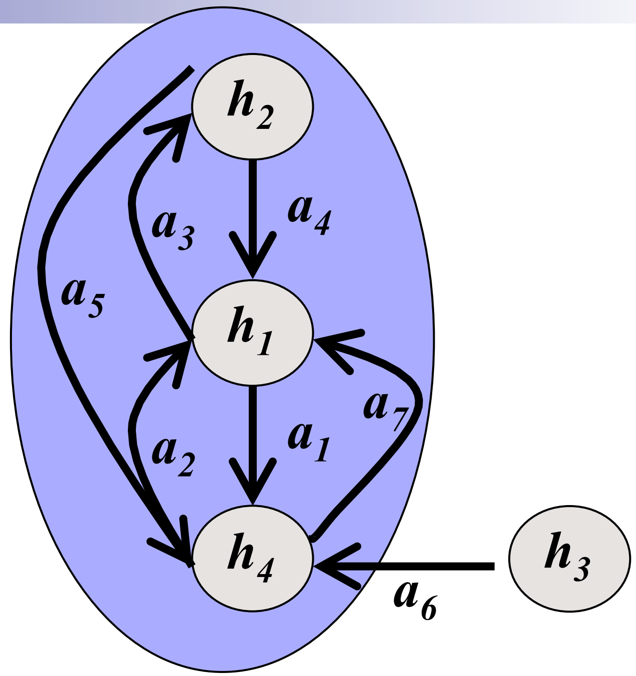
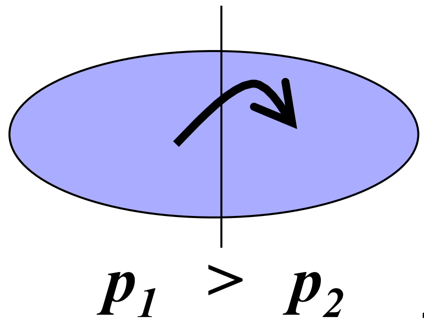
\end{array}$$

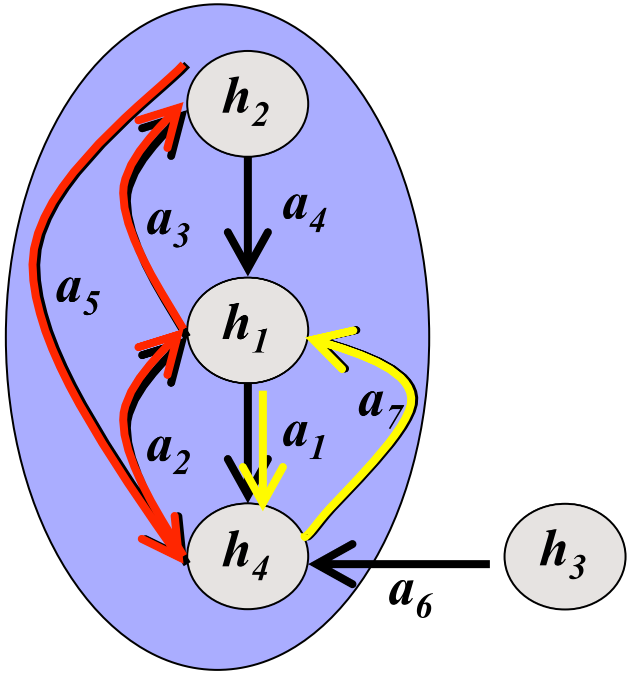


- directed graph $G = (H, E)$; vertices correspond to house types
- arcs correspond to agents: $e(a) = (h_i, h_j) \in E$ if $\omega(a) = h_i$ and h_j is the most preferred house type of a
- strongly connected digraph, strongly connected component (SCC)
- construct condensation of G
- all houses in a top SSC have equal price
- all agents in a top SCC trade with each other
- the top SCC must be Eulerian



Definition.

The deficiency $\mathcal{D}(\mathcal{M})$ of a housing market $\mathcal{M}$ is the minimum number of unhappy agents in a price-allocation pair.



Theorem (KC & Schlotter 2010).

DEFICIENCY is NP-complete even if each agent prefers only one house type to his endowment and there are at most two houses of each type.

Theorem (KC & Schlotter 2010).

- If the number of house types m is fixed then $\mathcal{D}(\mathcal{M})$ can be computed in any housing market in $O(m^m\sqrt{n}L)$ time.
- If preferences are strict, then a brute force algorithm with complexity $O(L^{\alpha+1})$ decides whether $\mathcal{D}(\mathcal{M}) \leq \alpha$.

Approximating the number of satisfied agents

Let (p, x) be a price-allocation pair. Agent a is

- unhappy if $x(a)$ is not among the most preferred houses is his budget set under p
- happy otherwice

$$\mathcal{D}(\mathcal{M}) = n - opt(\mathcal{M})$$

$$\text{mininization of } \mathcal{D}(\mathcal{M}) \Longleftrightarrow \text{maximization of } opt(\mathcal{M})$$

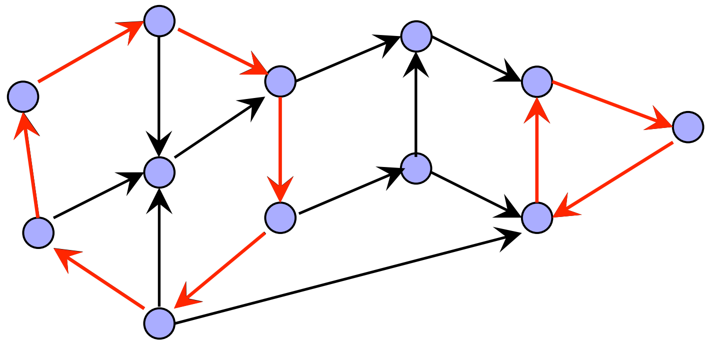
Definition.

An algorithm $\mathcal{A}$ for a maximization problem $\mathcal{S}$ is said to have

approximation factor r if $\dfrac{opt(I)}{\mathcal{A}(I)} \leq r$ for each instance I of $\mathcal{S}$.

Theorem (KC & Jelínková 2011).

There is a 2-approximation algorithm for deficiency in trichotomous markets. Moreover, this guarantee is tight.



- let C be a maximum cycle packing of $\mathcal{M}$, covering agents A_C
- If $|A_C| \geq n/2$: give all houses the same price and let the cycles of C be trading cycles

Theorem (KC & Jelínková 2011).

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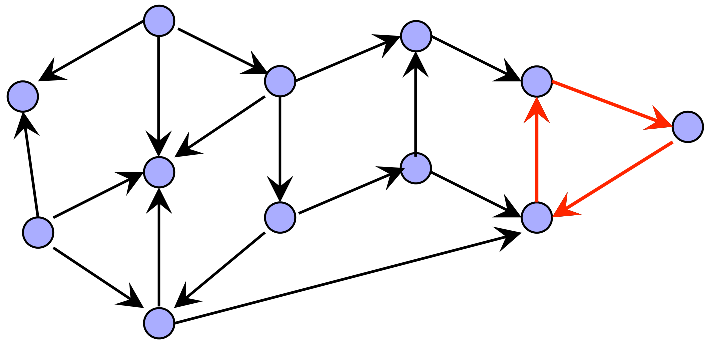

- let C be a maximum cycle packing of $\mathcal{M}$, covering agents A_C
- If $|A_C| < n/2$: delete the agents of A_C

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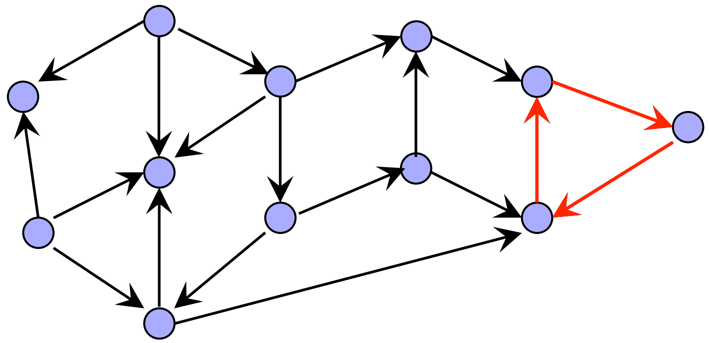

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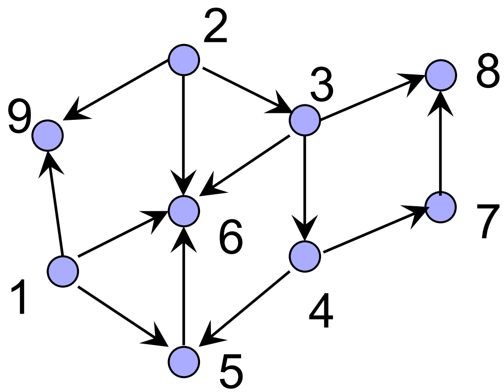
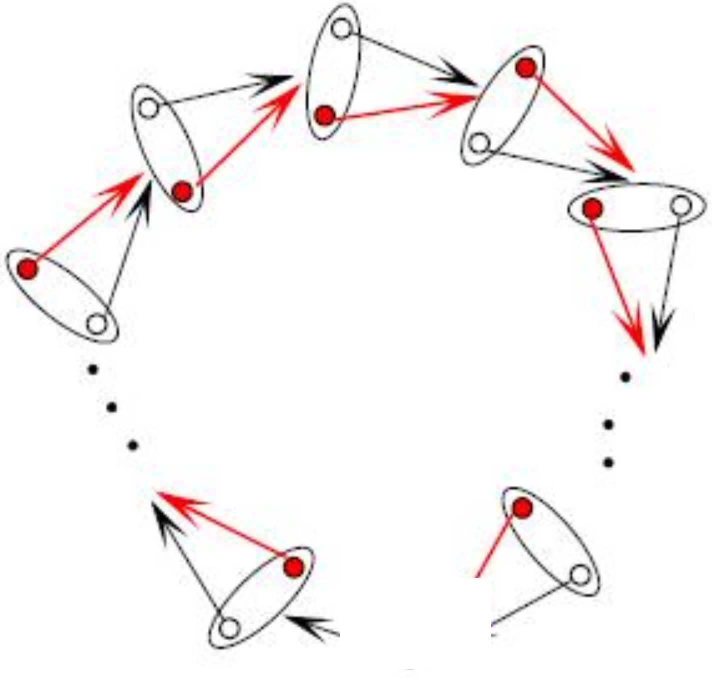


- let C be a maximum cycle packing of $\mathcal{M}$, covering agents A_C
- If $|A_C| < n/2$: delete the agents of A_C
- remaining market is acyclic $\Longrightarrow$ agents own different houses
- give prices to houses according to a topological ordering
- nobody is trading but all agents are happy

Theorem (KC & Jelínková 2011).

There is a 2-approximation algorithm for deficiency in trichotomous markets. Moreover, this guarantee is tight.



$2q + 1$ agents

Each cycle packing makes $q + 1$ agents happy

But it is possible to make $2q$ agents happy

Theorem (KC & Jelínková 2011).

It is NP-hard to approximate $opt(\mathcal{M})$ for trichotomous markets with an approximation factor smaller than $21/19$.

Thank you for your attention!