Dynamic Matching Problems

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Examples of dynamic matching problems

- Assignment of jobs and transfers in a centralized organization
- Assignment of offices in a department
- Assignment of dormitory rooms
- Assignment of organs for transplants
- Assignment of social housing

Transfers of French high school teachers

- The assignment of teachers to high schools in France is done centrally by the Ministry of Education
- Every year in February, teachers can ask for a transfer.
- The procedure has two steps: (i) interregional transfers, (ii) intraregional assignments
- Teachers submit a list of preferences for regions in the first stage, for high schools inside a region in the second stage.
- At each stage, the assignment is made through a priority order given by the number of points of a teacher.
- Teachers collect points through seniority, family circumstances, career choices...

Points accumulation for high school teachers

Seniority	4 points per year + 49 points after 25 years of service
On the job seniority	10 points per year + 25 points every 4 years
Current job	50 points if first assignment 300 points if 4 years in violent high school
Family circumstances	150.2 points if spouse is transferred + 75 points per child

Thresholds for transfers 2008

Assignment of social housing in Paris

- 230 000 social housing units in Paris (20 % of housing units)
- Every year, 12000 units are allocated (turnover rate of 4.5 %)
- 130 000 households are listed in the queue (average waiting time higher than 5 years)
- The assignment of units to households is decided by special commissions, meeting regularly (every month)
- There are special priority rules for ”emergency cases”

Reforming the assignment of social housing in Paris

- Criteria for assignment, and role of order in the waiting list
- Quotas for emergency situation and optimal assignment
- Eliciting information about preferences and making households apply for specific units
- Merging queues between Paris and the surrounding municipalities.

Dynamic matching issues:

- **Taking time into account:**
  - Stochastic arrival of objects to assign
  - Waiting times, waiting costs, queuing priorities
- **Using dynamic elements to replace monetary transfers:**
  - Using dynamic sequences of assignments to elicit information about preferences or induce effort when monetary transfers are nor permissible
  - Using queuing priorities to elicit information about preferences or induce effort when monetary transfers are nor permissible
- **Reassigning the same object to overlapping generations of agents:**
  - The system is closed, and objects can only be reassigned when they become available (scheduling problem)
  - Tenants' rights: agents have temporary property rights over objects

Relevant literature

- **Tenants' rights:** Abdulkadiroglu and Sönmez (1999) (static), Kurino (2009) (dynamic, dormitories), Bloch and Cantala (2009) (dynamic)
- **Stochastic entry/exit:** Unver (2010) (dynamic kydney exchange), Leshno (2011) (social housing), Bloch and Cantala (2012) (social housing), Kennes, Monte and Tumennasan (2012) (daycare)
- **Dynamic sequences replacing monetary transfers:** Abdulkadiroglu and Loerscher (2007) (school choice) , Leshno (2011) (priority order in the queue)
- **Reassignment of objects and scheduling:** Bloch and Cantala (2009), Bloch and Houy (2011)

Assignments

- Set I of n agents, indexed by their age (or seniority) $i = 1, 2, ...n$
- Set J of n objects, indexed by their quality $j = 1, 2, ..n$
- An **assignment** μ is a mapping from I to J, $\mu(i)$ is the object held by agent i.

Dynamics of assignments

- Time is discrete, and runs as $t = 1, 2, ..$
- At each period in time, agent i becomes agent $i + 1$, agent n leaves society, a new agent i enters society.
- Object $\mu(n)$ left by the oldest player is reallocated to some agent i^1
- Then object $\mu(i^1)$ is reallocated to some agent i^2,...
- until agent 1 (the entering agent who does not have any object) receives an object.
- By convention, the null object held by the entering player is denoted 0.
- Assignments are done object by object rather than by a simultaneous reallocation of all objects.

Markovian assignments

- A state *s* is defined by an assignment μ.
- A **truncated assignment** ν given object j is a mapping from $I \setminus \{1\}$ to $J \setminus \{j\}$.
- A **Markovian assignment rule** is a collection of vectors $\alpha_j(\nu)$ in $\Re^n$ for $j = 1, 2, ...n$ satisfying: $\alpha_j(\nu, i) \geq 0$ for all i and $\sum_{i,\nu(i)<j} \alpha_j(\nu, i) = 1$.
- The number $\alpha_j(\nu, i)$ denotes the probability that agent i receives object j given the truncated assignment ν.

Four Markovian assignment rules

- The *seniority rule* assigns object j to the oldest agent with an object smaller than j, $\alpha_j(\nu, i) = 1$ if and only if $i = \max\{k|\nu(k) < j\}$.
- The *rank rule* assigns object j to the agent who currently owns object $j-1$, $\alpha_j(\nu, i) = 1$ if and only if $\nu(i) = j-1$.
- The *uniform rule* assigns object j to all agents who own objects smaller than j with equal probability, $\alpha_j(\nu, i) = \frac{1}{|\{k|\nu(k)<j\}|}$ for all i such that $\nu(i) < j$.
- The *replacement rule* assigns object j to the entering agent, $\alpha_j(\nu, i) = 1$ if and only if $i = 1$.

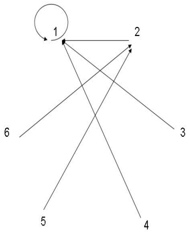
An example with three objects and three agents

$$\mu_1 : (1, 2, 3)$$
$$\mu_2 : (1, 3, 2)$$
$$\mu_3 : (2, 1, 3)$$
$$\mu_4 : (2, 3, 1)$$
$$\mu_5 : (3, 1, 2)$$
$$\mu_6 : (3, 2, 1) \quad .$$

The seniority rule

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

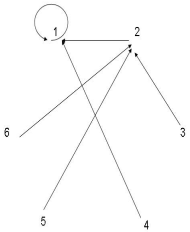
Transitions for the seniority rule



The rank rule

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

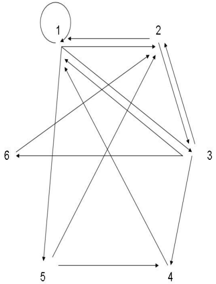
Transitions for the rank rule



The uniform rule

$$P = \begin{bmatrix} \frac{1}{6} & \frac{1}{3} & \frac{1}{6} & 0 & \frac{1}{3} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{6} & 0 & \frac{1}{6} & 0 & \frac{1}{3} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Transitions for the uniform rule



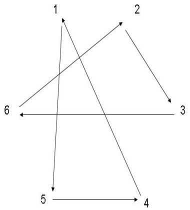
Invariant distribution for the uniform rule

$$p_1 = \frac{36}{127} \simeq 0.28, p_2 = \frac{28}{127} \simeq 0.22,$$
$$p_3 = \frac{30}{127} \simeq 0.24, p_4 = \frac{11}{127} \simeq 0.08,$$
$$p_5 = \frac{12}{127} \simeq 0.09; p_6 = \frac{10}{127} \simeq 0.07.$$

The replacement rule

$$P = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Transitions for the replacement rule



Independent assignment rules

- A Markovian assignment rule satisfies **independence** if the probability of assigning object j to agent i is independent of the objects held by the other players.
- Formally, for any j, for any i, for any ν, ν' such that $\nu(i) = \nu'(i)$, $\alpha_j(\nu, i) = \alpha_j(\nu', i)$.
- A Markovian assignment rule satisfies **strong independence** if the probability of assigning object j of agent i only depends on the object currently held by i and not his age.
- Formally, for any j, for any i, k, for any ν, ν' such that $\nu(i) = \nu'(k)$, $\alpha_j(\nu, i) = \alpha_j(\nu', k)$.
- The rank, uniform and replacement rules are strongly independent.
- The seniority rule does not satisfy independence.

Characterization of independent assignment rules

Lemma

If a Markovian rule α satisfies independence, then for any $j < n$, ν, ν' and i, k such that $\nu(i) = \nu'(k)$, $\alpha_j(\nu, i) = \alpha_j(\nu', k)$. Furthermore, for any ν, ν' such that
$\nu(i) = \nu'(j), \nu(j) = \nu'(i)$,$\alpha_n(\nu, i) + \alpha_n(\nu, j) = \alpha(\nu', i) + \alpha_n(\nu', j)$.

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- For objects $j < n$, the allocation to agent i only depends on i's current holding, not on his age.
- For object n, agents of different ages holding the same object can have different assignment probabilities.

Markov chains generated by assignment rules

- An assignment rule α generates a Markov chain over the set of assignments.
- From any μ to any μ', there exists a unique sequence of reassignments $i^0 = n + 1, i^1, ..., i^m, ..., i^M = 1$.
- The probability of reaching μ' from μ is:

$$p(\mu'|\mu) = \prod_{m=0}^{M-1} \alpha_{\mu(i^m-1)}(\nu^m, i^{m+1}) \tag{1}$$

where $\nu^m(i) = \mu(i-1)$ for $i \neq i^t$, $t = 1, 2, ..., m$ and $\nu^m(i) = \mu'(i)$ for $i = i^t$, $t = 1, 2, ..., m$.

Properties of finite Markov chains

- Two states i and j *intercommunicate* if there exists a path in the Markov chain from i to j and a path from j to i
- A set of states C is closed if, for any states $i \in C$, $k \notin C$, the transition probability between i and k is zero.
- A *recurrent set* is a closed set of states such that all states in the set intercommunicate. If the recurrent set is a singleton, it is called an *absorbing state*.

Convergent, irreducible and ergodic assignment rules

- A Markovian assignment rule α is **convergent** if the induced Markov chain is convergent (admits a unique absorbing state, and any initial assignment converges to the absorbing state).
- A Markovian assignment rule α is **irreducible** if the induced Markov chain is irreducible (the only recurrent set is the entire state set).
- A Markovian assignment rule α is **ergodic** if the induced Markov chain is ergodic (has a unique recurrent set).

Convergence and fairness

- An assignment rule is *fair* if for any two agents i and i' entering society at dates t and t', the assignment rule α generates a deterministic sequence of assignments such that $\mu^{t+\tau}(i) = \mu^{t'+\tau}(i')$ for $\tau = 0, 1, ..., n-1$.
- An assignment rule is *fair* if and only if it is convergent.

Convergent assignment rules

- Because $\mu^t(i) \geq \mu^{t-1}(i-1)$, the only possible absorbing state is the identity assignment $\iota(i) = i$.
- This will be an absorbing state if and only if

$$\prod_j \alpha_j(\tilde{\nu}^j, j) = 1, \tag{2}$$

where $\tilde{\nu}^j(i) = i - 1$ for $i \leq j$ and $\tilde{\nu}^j(i) = i$ for $i > j$.

- This condition is satisfied by the seniority and rank rules, where there is a path from any initial state to the identity assignment:

Theorem

Both the seniority and rank assignment rules are convergent.

Independent convergent assignment rules

Theorem

An assignment rule α is independent and convergent if and only if $\alpha_j(j-1) = 1$ for all $j < n$, $\alpha_n(\nu, n) = 1$ if $\nu(n) = n - 1$, and there exists $\lambda \in [0, 1]$ such that $\alpha_n(\nu, n) = \lambda$ and $\alpha_n(\nu, \nu^{-1}(n-1)) = 1 - \lambda$ if $\nu(n) \neq n - 1$.

Heterogeneous agents

- There is an ordered set $K = \{1, 2, ..., k, ..., m\}$ of types.
- Every entrant draws a type from K with independent probability, $q(k)$.
- The surplus σ is a strictly supermodular function of the object and the type: If $j' > j$ and k', k,

$$\sigma(j', k') + \sigma(j, k) > \sigma(j', k) + \sigma(j, k').$$

- Let θ be the type profile in society.
- The assignment rule now also depends on θ.

Independent assignment rules among heterogeneous agents

- An assignment rule α satisfies independence if: $\alpha_j(\nu, \theta, i) = \alpha_j(\nu', \theta', i)$ whenever $\nu(i) = \nu'(i)$ and $\theta(i) = \theta'(i)$.
- When agents are heterogeneous, independence with respect to the other agents' types is a very strong restriction:

Lemma

Let α be an independent assignment rule among heterogeneous agents. Then, for any θ, θ', any j, ν and i, $\alpha_j(\nu, \theta, i) = \alpha_j(\nu, \theta', i)$.

Efficient assignment rules

- We consider a dynamic notion of efficiency:
- An assignment rule α is efficient if it maximizes:

$$\mathbf{E} \sum_{t=0}^{\infty} \delta^t \sum_{i=1}^{n} u_i(\mu_t(i), \theta_t(i)).$$

- (We characterize dynamically efficient assignments in a companion paper.)

Quasi-convergent assignment rules

- There are two sources of randomness in the system: assignments and realizations of types.
- There cannot be an absorbing state, because $q(k) > 0$ for all k.
- Instead, we consider a weaker notion of quasi-convergence: A Markovian assignment rule α is **quasi convergent** if the induced Markov chain has a unique recurrent set of n^m states S such that, for any s, s' in S, $\theta(s) \neq \theta(s')$.
- This corresponds to a fairness principle. In the long run, two identical agents born at different dates in the same societies experience the same history.

The impossibility theorem

Theorem

Suppose that $|K| \geq 3$. There exist probability distributions over types q and/or discount factors δ such that no assignment rule can simultaneously satisfy efficiency and quasi-convergence.

The impossibility theorem: An illustration

- **There is no rule which is efficient and fair with more than three types.**
- $(M, M, M) \rightarrow (H, M, M) - > \rightarrow (H, H, M)$
- $(L, L, L) \rightarrow (M, L, L) - > \rightarrow (H, M, L) \rightarrow (H, H, M)$.

Efficient quasi-convergent rules for two types

- The *type-seniority* and *type-rank* rules use a lexicographic ordering: they first select the set of agents of highest type who may receive the object. If this set contains more than one type, the rule uses a tie-breaking rule (seniority or rank) to allocate the object.

Theorem

Suppose that $|K| = 2$. The type-rank and type-seniority rules are both efficient and quasi-convergent.

A characterization of dynamically efficient assignments

- Second paper ”Optimal assignment of durable goods to successive agents” (with Nicolas Houy)
- Considers a simpler framework with agents living two periods, one object to be assigned.

Agents

- A single durable object is allocated
- Agents live for two periods
- Agents have different values, θ for the object (measures the flow utility of the match)
- Agents draw their type from a continuous distribution F with compact support $\Theta \subset \Re_+$.

Social planner

- A benevolent social planner who maximizes discounted sum of assignment values
- The planner uses the same discount factor δ as the agents
- The planner chooses to allocate the object either to the young or old agent.
- If the young agent gets the object, he holds it for two periods.

The type-age tradeoff

- Giving the object to the old agent has an option value: you can always give it next period to the young, and maybe to someone with higher type.
- Hence, if young and old have the same type, the planner prefers to give the objet to the old
- Similar to the idea of ”transition popes” (John XXII (1316-1334) and John XXIII(1958-1965)).

The Markovian decision problem

- The planner's problem is:

$$V(\theta^y,\theta^o)=\max_{p^y,p^o} \quad p^y(\theta^y(1+\delta)+\delta^2\int_{\underline{\theta}}^{\overline{\theta}}\int_{\underline{\theta}}^{\overline{\theta}}V(t^y,t^o)f(t^y)f(t^o)dt^y dt^o$$
$$+p^o\theta^o+\delta(1-p^y)\int_{\underline{\theta}}^{\overline{\theta}}V(t^y,\theta^y)f(t^y)dt^y. \qquad (3$$

- Because types are positive, $p^o=1-p^y$
- The problem is linear in p^y: the optimal rule is a threshold rule.

The selectivity function

- The planner gives the good to the old agent if:

$$\theta^y(1+\delta)+\delta^2\int_{\underline{\theta}}^{\overline{\theta}}\int_{\underline{\theta}}^{\overline{\theta}} V(t^y,t^o)f(t^y)f(t^o)dt^y dt^o$$
$$\leq \quad \theta^o+\delta\int_{\underline{\theta}}^{\overline{\theta}} V(t^y,\theta^y)f(t^y)dt^y,$$

- We define the *selectivity function*

$$\phi(\theta)=\theta(1+\delta)-\delta\int_{\underline{\theta}}^{\overline{\theta}} V(t,\theta)f(t)dt+\delta^2\int_{\underline{\theta}}^{\overline{\theta}}\int_{\underline{\theta}}^{\overline{\theta}} V(t,z)f(t)f(z)dtdz.$$

The optimal assignment rule

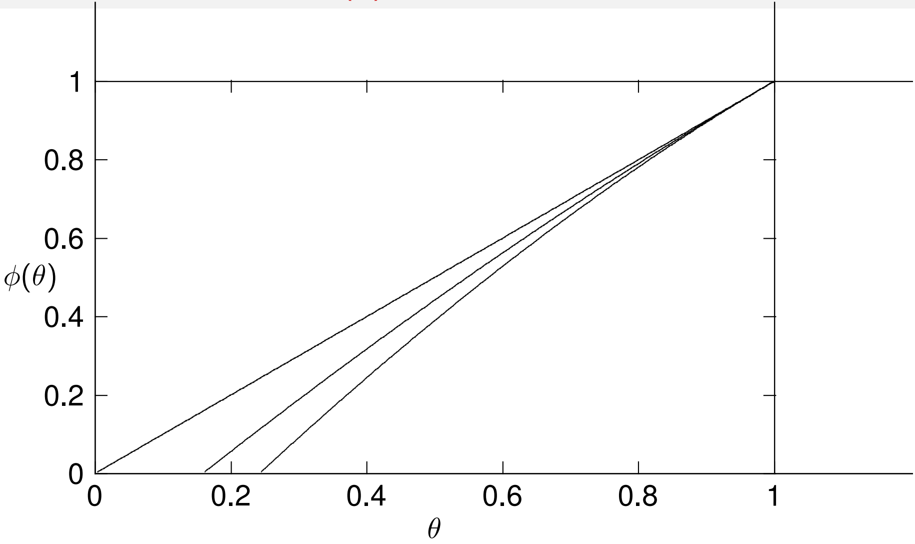
Theorem

There exists an optimal assignment policy characterized by a continuously differentiable selectivity function $\phi(\cdot)$ which is the unique solution of the iterative functional differential equation:

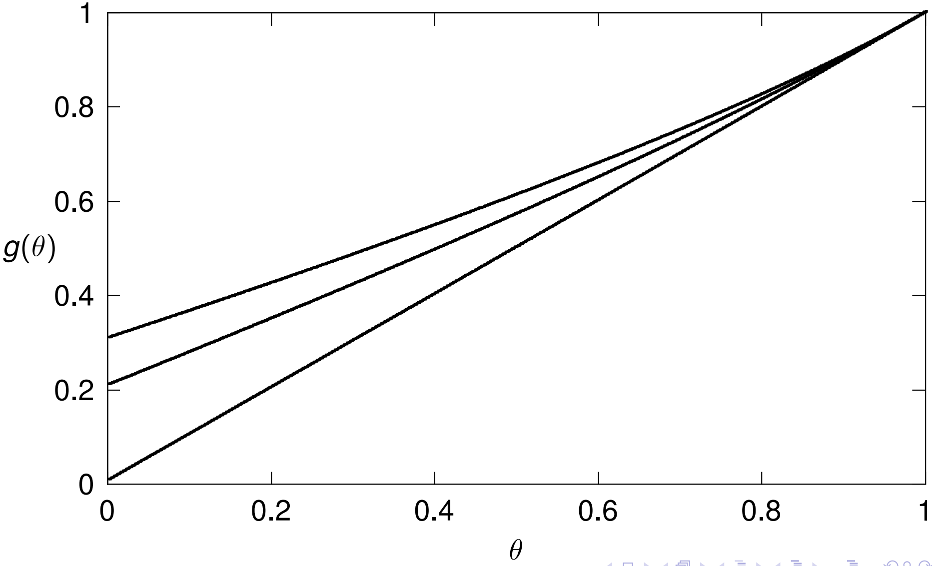
$$\phi'(\theta) = 1 + \delta - \delta F[\phi^{-1}(\theta)] \quad (4)$$

with initial condition $\phi(\overline{\theta}) = \overline{\theta}$.

The function ϕ for $F(\theta) = \theta$



The function g for $F(\theta) = \theta^2$



Comparative statics

Theorem

If $\delta > \delta'$, then $g(\delta, \theta) < g(\delta', \theta)$ for all θ.If $F'(\theta) \geq F(\theta)$ for all θ, then $g(F, \theta) \geq g(F', \theta)$ for all θ.

- If the discount factor is higher, then the threshold value $g(\theta)$ is lower.
- If the distribution F' puts less weight on higher values of θ than F, then the threshold value is lower.

Queuing and selection

- Agents are organized in a waiting list (of size 2+)
- At each period in time, a new object becomes available
- Agents in the waiting list (active agents) draw a value θ for the object according to a distribution $F(\cdot)$ with density $f(\cdot)$ over $[0, 1]$
- If an agent accepts the object, he leaves the waiting list and a new agent is drawn to enter the queue
- If the first agent takes the object, the second agent moves up in the waiting list.
- At each period in time, agents suffer an additive waiting cost c.
- Values are uncorrelated across agents and across time.

Efficiency measures

- There is no obvious measure of efficiency in the model, as the pool of active agents varies over time.
- *Static efficiency:* Assign the object to the agent (in the queue) who values it most.
- *Dynamic efficiency I:* Assign the object in order to maximize the sum of values of the agents
- *Dynamic efficiency II:* Assign the object in order to maximize the utility of an agent entering the queue (compatible with the absence of transfers)
- *Taking into account agents outside the queue:* The pool of agents is restricted. In order to take into account agents outside the queue, we may want to maximize the probability of assigning the object, in order to guarantee that new agents enter the pool.

Random serial dictatorship

- Suppose that all agents use the same threshold strategy characterized by θ^*.
- Let V be the value of the problem. With probability $\frac{1}{2}$, the agent is first to choose and obtains an expected payoff:

$$\int_{\theta^*}^{1} tf(t)dt + [1 - F(\theta^*)](V - c).$$

- With probability $\frac{1}{2}$, the agent chooses after the other agent and obtains an expected payoff:

$$F(\theta^*) \int_{\theta^*}^{1} tf(t)dt + [1 - F(\theta^*)](V - c)].$$

- Together with the stopping rule:

$$V = \theta^* - c,$$

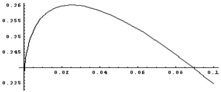
Optimal strategy

- We obtain a characterization of the optimal strategy:

$$H(\theta) \equiv [1 + F(\theta^*)][\int_{\theta^*}^{1} (t - \theta^*) f(t) dt].$$

Existence and uniqueness of the optimal stopping rule

- The optimal stopping rule always exists but *is not necessarily unique*.
- There are externalities across the two agents in the queue: the problem is a game rather than a decision problem.
- Suppose $F(\theta) = \sqrt{\theta}$. Then H is not monotonic:



Uniform distribution

- When the distribution F is uniform on $[0, 1]$, the optimal θ^* is the solution to a cubic equation:

$$-6c + 2 - 3\theta + \theta^3 = 0.$$

c	θ^*
0	1
0.05	0.652
0.1	0.481
0.15	0.328
0.2	0.175

Serial dictatorship

- The first agent is given the opportunity to choose first, and obtains an expected payoff of

$$\int_{\theta_1}^{1} tf(t)dt + F(\theta_1)(V_1 - c).$$

- The optimal strategy is given by:

$$G(\theta) \equiv \int_{\theta_1}^{1} (t - \theta_1)f(t)dt = c.$$

- This is identical to the optimal stopping rule in the traditional search literature.

Serial dictatorship II

- The second agent choose second. If the first rejects the good, he obtains an expected payoff:

$$F(\theta_1)[\int_{\theta_2}^{1} tf(t)dt + F(\theta_2)(V_2 - c)],$$

- If the first agent has accepted the good, the second agent moves in the seniority ranking and hence obtains a utility:

$$(1 - F(\theta_1))(V_1 - c).$$

- The optimal strategy is given by:

$$I(\theta_1, \theta) \equiv F(\theta_1)G(\theta) + (1 - F(\theta_1))(\theta_1 - \theta) = c.$$

Uniform distribution

- When the distribution F is uniform on $[0,1]$, the optimal reservation utilities $\theta^(1)$ and $\theta^*(2)$ are obtained as solutions to the system of cubic equations:

$$\begin{aligned} \theta(1) &= 1 - \sqrt{2c}, \\ 2\theta(2)(1-\theta(1)\theta(2)) &= \theta(1)(1-\theta(2)^2) + 2\theta(1)(1-\theta(1)) - 2c. \end{aligned}$$

Expected utilities with serial dictatorship

c	$\theta_(1)$	$\theta_(2)$
0	1	1
0.05	0.683	0.654
0.1	0.553	0.490
0.15	0.452	0.352
0.2	0.367	0.225

A comparison of the allocation rules

Proposition

The threshold (and utility) strategies satisfy:

$$\theta_1 > \theta_2, \theta_1 > \theta^*.$$

- The top agent in the seniority list obtains a higher value than the second agent (general result for any list)
- The top agent in the seniority list obtains a higher value than in a random serial dictatorship.

Comparison with fixed thresholds

Proposition

Suppose that the threshold θ is fixed. Then $V_1 > V^* > V_2$.

- Note that $G(x) + I(x, x) = 2H(x)$ and $G(x) < H(x)$. The result follows.
- This result shows that, when agents are nonstrategic, the top agent in the seniority list obtains the highest expected utility, followed by an agent in the random model, and the second agent in the priority list.

Main Result: Dynamic Efficiency I

Proposition

Suppose that $H(\theta)$ is decreasing and convex, then

$$\theta_1 + \theta_2 \geq 2\theta^*.$$

- We have:

$$2H(\theta^*) = 2c = G(\theta_1) + I(\theta_1, \theta_2).$$

- Suppose that $\theta^* > \frac{\theta_1+\theta_2}{2}$. Then,

$$H(\theta_1) + H(\theta_2) = \frac{(1 + F(\theta_1))G(\theta_1)}{2} + \frac{(1 + F(\theta_2)G(\theta_2)}{2} > 2H(\theta^*).$$

$$\begin{aligned} G(\theta_1) + I(\theta_1, \theta_2) &> G(\theta_1) + F(\theta_1)G(\theta_2), \\ &> \frac{(1 + F(\theta_1))G(\theta_1)}{2} + \frac{(1 + F(\theta_2)G(\theta_2)}{2}, \\ &= H(\theta_1) + H(\theta_2) > 2H(\theta^*), \text{ contradiction} \end{aligned}$$

Main Result: Dynamic efficiency II

Proposition

Suppose that $H(x) > G(x + \frac{G(x)}{2})$*, then* $\theta_2 > \theta^*$

- The condition is sufficient, not necessary. The necessary and sufficient condition is the more complex. Let

$$\phi(x) \equiv G^{-1}[\frac{(1 + F(x))G(x)}{2}],$$

$$F(\phi(x))G(x) + (1 - F(\phi(x)))(\phi(x) - x) - \frac{(1 + F(x))G(x)}{2} > 0.$$

- This (sufficient) condition is satisfied for $F(x) = x^\alpha$ with $\alpha \leq 1$.

Dynamic Efficiency II: Interpretation

- It results in the counterintuitive result that even the second agent prefers the serial dictatorship to the random serial dictatorship. this results from a trade-off: (i) in the random rule, both agents are treated symmetrically but (ii) in the serial dictatorship, the second agent is rewarded by moving to the top of the list if the other agent accepts the object.
- This also implies that the probability that the good is allocated (and a new agent enters the list) is higher under random serial dictatorship than under serial dictatorship.

Static efficiency

- Inefficiencies arise if the second agent who chooses draws a value higher than the first agent, but the value of the first agent is above the threshold.
- Given a threshold θ, this happens with probability

$$\begin{aligned}\pi &= \int_\theta^1 \int_\theta^u f(t)dtf(u)du, \\ &= \int_\theta^1 [F(u) - F(\theta)]f(u)du.\end{aligned}$$

- Hence π is decreasing in θ. The static inefficiency is higher in the random serial dictatorship than in the serial dictatorship model.

Inefficiency of punishment schemes

- Punishment schemes can either
  - limit the number of offers that the first agents can reject
  - evict the first agent from the queue with probability p after a rejection
  - place the first agents at the back of the queue with probability p after a rejection
- All three punishment schemes result in a loss of efficiency for the agents in the queue, as they amount to destroying part of the surplus
- These punishment schemes can only be valuable if one takes into account agents who are currently not in the queue and waiting to join the queue.

Queues of arbitrary length: serial dictatorship

- Consider a queue with n agents, but with only two possible values $\theta = 1$ (with probability π) and $\theta = 0$ (with probability $1 - \pi$).

Proposition

In the serial dictatorship model, there exists a rank k such that, for all $i > k$, agents immediately accept the object irrespective of its value whereas, for rank $i \leq k$, agents accept of the good if and only if it creates value 1. The rank k is the first integer such that:

$$V(k) = 1 + c - \frac{kc}{[1 - (1 - \pi)^k]} < 0.$$

The values of agents are given by:

$$\begin{aligned} V(i) &= 1 + c - \frac{ic}{[1 - (1 - \pi)^i]} \text{ if } i \leq k, \\ V(i) &= [1 - (1 - \pi)^k] - (i - 1)c \text{ if } i > k. \end{aligned}$$

Queues of arbitrary length: random serial dictatorship

Proposition

In the random serial dictatorship model, if

$$1 + c - \frac{nc}{[1-(1-\pi)^n]} > 0,$$

agents only accept the good if it creates value 1 *and obtain an expected utility*

$$V = 1 + c - \frac{nc}{[1-(1-\pi)^n]}.$$

Otherwise, all agents accept the good immediately irrespective of the value of the good, and obtain an expected utility:

$$V = -c(n-1).$$

Queues of arbitrary length: comparison of rules

- If $1 + c - \frac{nc}{[1-(1-\pi)^n]} > 0$, the two rules result in the same expected utility $V = V(n)$.
- If $1 + c - \frac{nc}{[1-(1-\pi)^n]} < 0$, the random serial dictatorship model results in a lower expected utility than the serial dictatorship model, $V < V(n)$.
- *The serial dictatorship model dominates the random serial dictatorship model for both agents.*