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CHAPTER 5

Group Activity Selection Problems

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5.1 Introduction

Group activity selection consists in selecting one or several activities for a set of participants (agents) and of assigning these agents to one of the different selected activities (or possibly to no activity at all), according to the agents' preferences. The specificity of the group activity selection problem is that agents' preferences bear both on activities and on the number of participants for a given activity. As a concrete example of a group activity selection problem consider the organisers of a workshop[1] who are planning to have a set of group activities taking place during a free afternoon:

- Activities are held in parallel, so that each participant can take part in at most one activity.
- The possible activities include a hike, a bus trip to a nearby historic city, and a table tennis competition.
- There can be several hiking groups, and similarly, several buses can be rented for the trip; however, there can be only one group for the table tennis competition, as there is only one table.
- The cost of renting a bus (or several buses) has to be shared between the participants of the trip, therefore the participants generally prefer a bus trip with more participants over one with few participants.
- As for the table tennis competition, a plausible preference about the number of participants would be that the number should neither be too small nor too large; typically, the players will neither want to wait to long for their turn nor wish to play permanently (without reasonable breaks).

There are several natural variations over this problem. For instance, there may be only one activity (say, a dinner or another social event) and we look for a set of invitees, where potential invitees have preferences about the number of invitees, and possibly also about the other invitees: this variant is called the

[1]This example is adapted from a real scenario that took place at a Dagstuhl Seminar on Computation and Incentives in Social Choice in 2012. The problem is apparently known by the Dagstuhl staff as the 'Dagstuhl group activity selection problem'!

stable invitation problem (Lee and Shoham, 2015). This problem can also be extended to a setting in which the preferences of the invitees depend also on the date the event takes place (Lee and Shoham, 2014).

If activities are not typed (or equivalently, if it is known beforehand that every group will be assigned to the same activity), then we have an *anonymous hedonic game*, where agents' preferences bear only on the size in their group; if, on the other hand, agents can have preferences about the identity of the other participants in their group (and not only about its size) then we have more generally a *hedonic game*.

The goal of this chapter is to describe these problems formally, to show how solution concepts that have been well-studied for hedonic games also apply to group activity selection problems, and to consider additional solution concepts from social choice theory to group activity selection. As there is a recent and detailed survey chapter about hedonic games (Aziz and Savani, 2016), we do not cover general hedonic games here.

In this chapter we hence consider the model for group activity selection problems in which the agents' preferences are over pairs "(activity, group size)", and the above mentioned related problems/variations. In Section 5.2 we describe the different problems formally, and we clarify the relationships between them. In Section 5.3 we discuss natural assumptions that one might make about agents or activities (domain restrictions). In Section 5.4 we review several solution concepts from the literature on hedonic games, show how they specialize or adapt to group activity selection along with some concrete examples, and apply solution concepts from social choice theory to group activity selection. In Section 5.5 we address the computational issues for these solution concepts and in Section 5.6 we briefly consider strategic issues. Section 5.7 gives a short conclusion and some further links to other fields of research.

5.2 Models

In this section, we present models of different classes of group activity selection problems. We begin with some basic definitions.

5.2.1 Basic Definitions

For the various models below we describe only the input, with various possible assumptions about the nature of preferences. In the whole chapter, we will consider a set of *agents* $N = \{1, \ldots, n\}$. We now consider in sequence activities, assignments, alternatives, and preference profiles.

Activities and Assignments. We consider a set of *activities* $A = A^* \cup \{a_\emptyset\}$, where $A^* = \{a_1, \ldots, a_m\}$. Activity $a_\emptyset$ is called the *void activity*; an agent being assigned to $a_\emptyset$ means that the agent will not participate in any concrete activity.

An *assignment* for (N, A) is a mapping $\pi : N \to A$. We denote by π^0 the set of agents i such that $\pi(i) = a_\emptyset$ and for each $j \leqslant m$, π^j the set of agents i such that

$\pi(i) = a_j$. Finally, the coalition structure induced by π is defined as $CS_\pi = \{\{i\} \mid i \in \pi^0\} \cup \{\pi^j \mid 1 \leqslant j \leqslant m, \pi^j \neq \emptyset\}$.

In addition, we may consider a more general version of the problem with constraints (especially, cardinality constraints) restricting the set of possible sets of activities that can be jointly organised, as well as constraints concerning the number of participants to an activity. In this case, we denote the set of all constraints by Γ, and an assignment is *feasible* if it satisfies the constraints in Γ.

Alternatives and Preference Profiles. Agents have preferences that bear both on the activity they will be assigned to, and on the set of agents who will participate in the same activity. An *alternative for agent* i is either $a_\emptyset$ or a pair $(a, S) \in A^* \times N_i$, where N_i is the set of all subsets of N containing i. The set of *alternatives* for i is X_i.

Each agent i has some preferences over X_i. A *preference relation for agent* i $\succsim_i$ is a reflexive and transitive order over X_i; the strict part and the indifference parts of $\succsim_i$ are denoted respectively by $\succ_i$ and $\sim_i$. A *preference profile* is an n-tuple $P = (\succsim_1, \ldots, \succsim_n)$ where $\succsim_i$ is a preference relation for i.

Recall that agent i is assigned to $a_\emptyset$ if she is not assigned to any activity (and will stay alone). If $(a, S) \succsim_i a_\emptyset$ then i likes being assigned to a with coalition S at least as much as staying alone: in this case we say that alternative (a, S) is *admissible* for i. When (a, S) is not admissible for i, i.e., $a_\emptyset \succ_i (a, S)$, then i would prefer to 'leave the game' and stay alone. An assignment is *individually rational* if it admissible for all agents. Individual rationality is the most basic and most important stability criterion for assignments (more complex stability notions will be defined in Section 5.4.)

A simple preference restriction is one where agents simply approve alternatives that are admissible for them, without ranking them. If, in addition, no agent is indifferent between being alone and some other activity, we will say that the agent has *trichotomous* preferences: in this case, agent i specifies a set of alternatives S_i, which induces a partition of the set of alternatives in three clusters, ranked in this order: S_i, then $\{a_\emptyset\}$, and last, $X_i \setminus (S_i \cup \{a_\emptyset\})$. Another preference restriction consists in requiring that each agent expresses a *strict order* over X_i.

Finally, a *group activity selection problem* is a triple (N, A, P). If, in addition, we have feasibility constraints, then we have a *constrained group activity selection problem*. We now consider several classes of group activity selection problems, depending on the nature of the preference relations in P.

5.2.2 Classes of Group Activity Selection Problems

Hedonic Games. We say that $\succsim_i$ is activity-independent if $(a, S) \sim_i (a', S)$ for all activities a, a' and coalitions S: i's preference relation $\succsim_i$ depends only on the set of agents in i's coalition, and not on the activity to which i is assigned. When $P = (\succsim_1, \ldots, \succsim_n)$ is such that every $\succsim_i$ is activity-independent, (N, A, P) degenerates into a *hedonic game*, where agents only care about which agents are in their coalition (Drèze and Greenberg, 1980; Banerjee et al., 2001; Bogomolnaia and Jackson, 2002).

If, in addition to being activity-independent, $\succsim_i$ depends only on the *cardinality* of i's coalition, $\succsim$ is said to be *anonymous*; if every $\succsim_i$ in P is activity-independent and anonymous, then (N, A, P) is an *anonymous hedonic game*.

The input of an anonymous hedonic game thus consists, without loss of generality, of a preference relation $\succsim_i^*$ over $\{1, \ldots, n\}$ for each agent, where $k \succsim_i^* k'$ if and only if $S \succsim_i S'$ for some (and, equivalently, for all) S and S' containing i such that $|S| = k$ and $|S'| = k'$, meaning that agent i likes being in a coalition of k agents at least as much as in a coalition of k' agents. Anonymous hedonic games have been studied by Ballester (2004).

While the input of an anonymous hedonic game can be represented succinctly in $O(n^2)$, the input of a hedonic game (and *a fortiori* of a group activity selection problem) needs in general an exponentially large space. Because of this, some restrictions of hedonic games have been considered: apart of anonymous hedonic games, we find *additive* hedonic games (each agent specifies a utility value for each other agent, and the utility of a coalition for i is the sum of the values of agents in it), *fractional* hedonic games (similar to additive hedonic games, except that the utility of a coalition for i is the *average* of the values of agents in it), *friends-and-enemies* hedonic games (an agent partitions N between two sets, corresponding to friends and enemies, and his preference relation depends on the number of friends and the number of enemies in his coalition), *optimistic* (respectively, *pessimistic*) hedonic games (each agent ranks other agents and values a coalition according to the best (respectively, worst) agent in his coalition), or *Boolean* hedonic games (each agent specifies the set of coalitions he approves, possibly succinctly, using a propositional formula). A recent survey on hedonic games, with a focus on how they can be represented succinctly and how various forms of equilibria can be computed, is that of Aziz and Savani (2016).

The extension to hedonic games studied by Spradling et al. (2013) considers agents who have preferences over pairs consisting of the role they will play in their coalition and of the composition of roles in their coalition. See the work of Darmann et al. (2012) for a discussion on the relation of this model to group activity selection.

Anonymous Group Activity Selection. Anonymous group activity selection is to non-anonymous (or general) activity selection what anonymous hedonic games are to hedonic games: agents care only about the activity they belong to and the number of participants to that activity (this number, of course, depends on the activity). To keep the terminology consistent with existing papers, the general model (where agents care about the assigned activity and the identity of the agents in their coalitions) will be called *generalised group activity selection* and the anonymous model will be called, simply, *group activity selection*. Therefore, without loss of generality, a group activity selection problem (GASP) is a triple (N, A, P), where P consists of n preference relations over $\{a_\emptyset\} \cup (A \times \{1, \ldots, n\})$. If, furthermore, all agents have approval-based preferences, then we will say that P is an *approval-based group activity selection problem*, for short a-GASP (Darmann et al., 2012): in such a problem, P consists in n subsets of $A \times \{1, \ldots, n\}$, namely the *approval sets* S_i, $i \in N$; and if all agents have strict preferences (without any ties between alternatives), then we will say that P is an *ordinal group activity selection problem*, for short o-GASP (Darmann, 2015).

Example 5.1. *We have two non-void activities: $A^* = \{a, b\}$, and 5 agents with the following preferences (which we truncate at $a_\emptyset$, because, as we will be interested in individually rational assignments, preferences below $a_\emptyset$ will not play any role).*

$$\begin{array}{ll} 1: & (a,1) \succ (a,2) \succ (a,3) \succ a_\emptyset \succ \dots \\ 2: & (b,5) \succ (a,1) \succ (a,2) \succ (b,4) \succ (b,3) \succ (b,2) \succ a_\emptyset \succ \dots \\ 3: & (b,5) \succ (b,4) \succ (b,3) \succ (b,2) \succ (a,1) \succ a_\emptyset \succ \dots \\ 4: & (b,5) \succ (b,4) \succ (b,3) \succ (b,2) \succ (a,1) \succ (a,2) \succ a_\emptyset \succ \dots \\ 5: & (a,1) \succ (a,2) \succ (b,5) \succ (b,4) \succ a_\emptyset \succ \dots \end{array}$$

There is no feasibility constraint so we have here a (non-constrained) anonymous group activity selection problem, and more precisely an instance of o-GASP*. Two individually rational assignments are π and λ with*

$$\begin{array}{llll} \pi: & 1,2 \mapsto a; & 3,4 \mapsto b; & 5 \mapsto a_\emptyset \\ \lambda: & 1 \mapsto a; & 2,3,4,5 \mapsto b & \end{array}$$

An analogue of the above example in the approval-based setting of a-GASP would be the following.

Example 5.2. *In the setting of* a-GASP*, note that we only need to take into account the approval sets of the agents. These are given by*

$$\begin{array}{rl} S_1 = & \{(a,1),(a,2),(a,3)\} = \{a\} \times [1,3] \\ S_2 = & \{b\} \times [2,5] \cup \{a\} \times [1,2] \\ S_3 = & \{b\} \times [2,5] \cup \{(a,1)\} \\ S_4 = & \{b\} \times [2,5] \cup \{a\} \times [1,2] \\ S_5 = & \{b\} \times [4,5] \cup \{a\} \times [1,2]. \end{array}$$

Clearly, both assignments π and λ defined in Example 5.1 are individually rational also in this example.

Lu and Boutilier (2012) discuss a model of cooperative group buying, which can be embedded into GASP (more specifically, GASP with decreasing preferences, see Section 5.3). See Darmann et al. (2012) for further discussion.

Stable Invitations. If there is only one non-void activity, i.e., $A^* = \{a\}$, where a is called the *event*, then we have a *stable invitation* problem: in an invitation problem, agent i's preferences bear on $a_\emptyset \cup \{(a, S) \mid S \in N_i\}$. If moreover, preferences are anonymous, then we have an *anonymous stable invitation problem*, for short ASIP (Lee and Shoham, 2015): in this case, each agent has a preference over $\{0, \dots, n\}$, where 0 means that the agent is not attending the event, while if $t > 0$ then t means that the agent attends together with $t - 1$ other agents. Lee and Shoham (2015) also define a *general stable invitation problem*, for short GSIP, in which agents preferences are restricted and given as follows: each agent i has a preference relation over $\{0, \dots, n\}$ as before, together with an *acceptance set* $F_i \subseteq N \setminus \{i\}$ and a *rejection sets* $R_i \subseteq N \setminus \{i\}$; i is willing to attend only if all agents of F_i attend, no agent in R_i attends, and the number of attendees is preferred to 0 (that is, to not attending); among such acceptable coalitions, i prefers (a, S) to (a, S') if she prefers $|S|$ to $|S'|$.

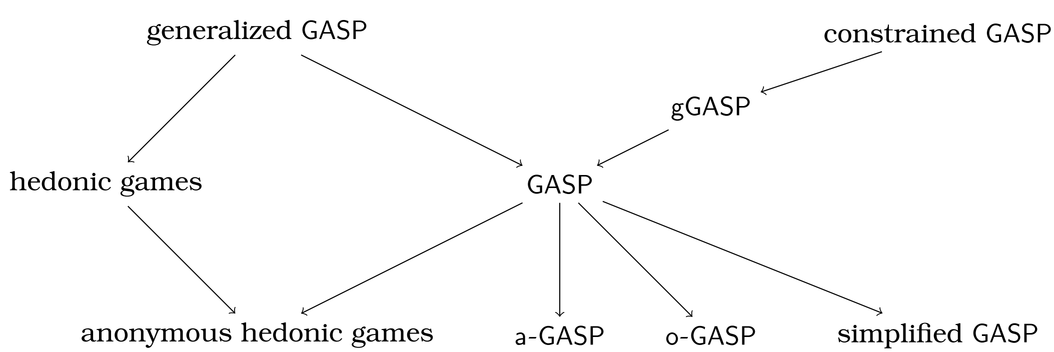
Example 5.3. *Here is an anonymous stable invitation problem, with 6 agents and the following preferences (which we truncate at* 0*, because again we will be interested only in individually rational assignments (invitations)).*

$$\begin{array}{ll} 1: & 6 \succ 5 \succ 4 \succ 0 \\ 2: & 6 \succ 5 \succ 4 \succ 3 \succ 2 \succ 0 \\ 3: & 3 \succ 4 \succ 0 \\ 4: & 2 \succ 3 \succ 0 \\ 5: & 6 \succ 5 \succ 4 \succ 0 \\ 6: & 1 \succ 2 \succ 3 \succ 4 \succ 5 \succ 6 \succ 0 \end{array}$$

Some individually rational invitations are $\{1, 2, 3, 5\}$, $\{1, 2, 5, 6\}$ *and* $\{2, 3, 4\}$.

Group Activity Selection on Social Networks. Igarashi et al. (2017) and Igarashi et al. (2017) consider a constrained group activity selection problem where the agents are linked through an undirected graph G (representing social interactions) and assigning a coalition of agents to an activity is feasible only if this coalition is connected with respect to this graph. Although the global feasibility constraint Γ_G pays attention to the identity of agents, their preferences are anonymous. We denote by gGASP such a game. Note that GASP is obtained as a special case where G is a complete graph.

Relation between the Classes of Problems. We end the section by a diagram showing the inclusion relationship between the different problems. In the diagram we have added one more class: *simplified* GASP, in which the agents' preferences depend only of the activity they are assigned to (this problem, of course, is nontrivial only if there are feasibility constraints).



5.3 Domain Restrictions

Now that we have defined group activity selection and some interesting subproblems, it is worth going further and to consider restrictions of the problem obtained either by assuming specific assumptions on the agents' preferences or on the available activities.

5.3.1 Restrictions on Preferences

For any activity $a \in A^*$ and any agent i with preference relation $\succsim_i$, let $\succsim_i^{\downarrow a}$ denote the projection of $\succsim_i$ on a defined by: for all subsets of agents S, S' containing i, $S \succsim_i^{\downarrow a} S'$ if $(S, a) \succsim_i^{\downarrow a} (S', a)$. Now, we say that $\succsim_i$ is *monotonic* with respect to a if $\succsim_i^{\downarrow a}$ is monotonic, that is, if $S \supseteq S'$ implies $S \succsim_i^{\downarrow a} S'$. Likewise, $\succsim_i$ is *antimonotonic* with respect to a if $\succsim_i^{\downarrow a}$ is antimonotonic, that is, if $S \supseteq S'$ implies $S' \succsim_i^{\downarrow a} S$.

When group activity selection is anonymous, then monotonicity and antimonotonicity – called increasingness and decreasingness in the setting of GASP (see Darmann et al., 2012; Darmann, 2015) – simplifies into the following:

The preferences of agent i are

- *increasing* with respect to activity $a \in A^*$ if $(a, k) \succsim_i (a, k-1)$ for any $k \in \{2, \dots, n\}$
- *decreasing* with respect to activity $a \in A^*$ if $(a, k-1) \succsim_i (a, k)$ for any $k \in \{2, \dots, n\}$.

An instance $\mathcal{I} = (N, A, P)$ of GASP is called increasing (respectively, decreasing) if for each $a \in A^*$ and each agent $i \in N$ the preferences of agent i are increasing (respectively, decreasing) with respect to a. An instance is called *mixed increasing/decreasing*, if there is a set $A' \subseteq A^*$ such that for each agent $i \in N$ her preferences are increasing with respect to each $a \in A'$ and decreasing with respect to each $a \in A^* \setminus A'$.

Clearly, for the setting of o-GASP these definitions translate in a straightforward way. For a-GASP, however, these definitions can be simplified by considering only the set of approved alternatives. Informally, in a-GASP an agent i has increasing preferences with respect to activity a, if the set of group sizes k for which agent i approves (a, k) forms an interval with upper bound n; formally, in a-GASP agent i has *increasing preferences* with respect to activity a, if there is a threshold ℓ_i^a such that $\{k \mid (a, k) \in S_i\} = [\ell_i^a, n]$. Analogously, in a-GASP agent i has *decreasing preferences* with respect to activity a, if there is a threshold u_i^a such that $\{k \mid (a, k) \in S_i\} = [1, u_i^a]$.

Observe that the instances in Example 5.1 and Example 5.2, respectively, are mixed increasing/decreasing.

5.3.2 Restrictions on Activities

Among the non-void activities, some of them might be organised in several 'copies': for instance, if there are three guides in the local museum, up to three guided tours can be organised concurrently. Formally, we say that two activities

a and a' are *copies* of each other if for each agent i and subset S containing i, i is indifferent between (a, S) and (a', S). We say that A^* contains k copies of a (or, a is k-copyable) if there exists $k - 1$ other activities in A^* that are copies of a.

Sometimes, an activity can exist in as many copies as we like (for instance, a hike). In this case it will be said to be *infinitely copyable*; since there are only n agents and considering more than n copies would be irrelevant, an activity is infinitely copyable just if it is n-copyable. An activity is said to be *simple* if it is not k-copyable for any $k > 1$. Unless stated otherwise, an activity is assumed to be simple.

5.4 Solution Concepts

We define solution concepts for hedonic games first and show how these specialize to the model of GASP and the stable invitation problem respectively. To simplify notation, in hedonic games we consider assignment π as a partition of the set of agents and identify $\pi(i)$ with the coalition to which i is assigned under π.

5.4.1 Maximum Individual Rationality and Pareto Optimality

A *maximum individually rational* assignment is one among the individually rational ones that maximizes the number of agents assigned to a non-void activity. E.g., in the instance considered in Example 5.1 assignment λ is a maximum individually rational assignment; in particular, λ assigns each agent to an activity different from $a_\emptyset$.

The next concept considered is the one of *Pareto optimality*. In a hedonic game, a partition π is called Pareto optimal, if there is no partition π' such that $\pi'(i) \succsim \pi(i)$ holds for each $i \in N$ and for at least one agent $j \in N$ we have $\pi'(i) \succ \pi(i)$. Translating the concept to GASP, an individually rational assignment π is *Pareto optimal* if there is no assignment π' such that for at least one $i \in N$ $(\pi'(i), |\pi'_i|) \succ_i (\pi(i), |\pi_i|)$ holds and there is no $i \in N$ with $(\pi'(i), |\pi'_i|) \prec_i (\pi(i), |\pi_i|)$. Note that in the approval-scenario of a-GASP, a maximum individually rational assignment is also Pareto optimal.

As particular examples for o-GASP, both assignments π and λ are Pareto optimal with respect to Example 5.1.

5.4.2 Stability Notions

In this subsection, we consider the aspect of stability with respect to agents' incentive to deviate from the considered assignment. In this respect, we distinguish between single agent deviations and group deviations. We first introduce the concepts for the hedonic game framework and translate them to the group activity selection problems afterwards. At the end of this section, we consider the stable invitation problem.

Starting with single agents' deviations, a very basic stability concept is Nash stability. In the hedonic game framework, a partition is Nash stable, if no agent benefits by moving from her coalition to another coalition. A partition is individually stable, if there is no agent such that the agent benefits by moving from her coalition to another coalition and all members of the new coalition agree with the agent joining. Finally, a partition is contractually individually stable, if it is not the case that a deviating agent is better off and both the old and new coalition agrees with the deviation of the agent.

In the group deviations case, the most famous solution concept is the core. Here, we distinguish between strong and weak group deviations. A *strong group deviation* is beneficial for each member of the deviating group; in a *weak group deviation* at least one member of the group is better off while the deviation does not harm any of the group members. These notions lead to the concepts of the core and the strict core, respectively.

Formally, in a *hedonic game*, a partition π is said to be

- *Nash stable*, if there is no $i \in N$ such that for some $S \in \pi \cup \{\emptyset\}$ we have $S \cup \{i\} \succ_i \pi(i)$.
- *individually stable*, if there is no $i \in N$ such that for some $S \in \pi \cup \{\emptyset\}$ we have $S \cup \{i\} \succ_i \pi(i)$ and $S \cup \{i\} \succsim_j S$ for each $j \in S$.
- *contractually individually stable*, if there is no $i \in N$ such that (i) for some $S \in \pi \cup \{\emptyset\}$ we have $S \cup \{i\} \succ_i \pi(i)$ and $S \cup \{i\} \succsim_j S$ for each $j \in S$ and (ii) $\pi(i) \setminus \{i\} \succsim_{j'} \pi(i)$ holds for each $j' \in \pi(i)$.
- *core stable*, if there is no subset $S \subseteq N$ such that for each $i \in S$ we have $S \succ_i \pi(i)$.
- *strictly core stable*, if there is no subset $S \subseteq N$ such that for each $i \in S$ we have $S \succsim_i \pi(i)$ and for at least one $j \in S$ we have $S \succ_j \pi(j)$.

For the setting of GASP these concepts translate as follows. An individually rational assignment π is called

- *Nash stable*, if

  (1) for every agent $i \in N$ with $\pi(i) \neq a_\emptyset$ and every $a_j \in A^* \setminus \{\pi(i)\}$ it holds that $(\pi(i), |\pi(i)|) \succsim_i (a_j, |\pi^j| + 1)$, and

  (2) for every agent $i \in N$ with $\pi(i) = a_\emptyset$ and every $a_j \in A^*$ it holds that $a_\emptyset \succsim (a_j, |\pi^j| + 1)$.

  For the approval-setting of a-GASP, note that an agent only has an incentive to deviate if she is currently assigned to $a_\emptyset$. In that setting the definition of Nash stability thus simplifies as follows: An individually rational assignment π is Nash stable, if for each $i \in N$ with $\pi(i) = a_\emptyset$ and each $a_j \in A^*$ it holds that $(a_j, |\pi^j| + 1) \notin S_i$.
  In the strict preference setting of o-GASP, the weak preferences in the above definition are replaced by strict preferences. With respect to Example 5.1 λ is not Nash stable, since, for instance, agent 2 wants to deviate from λ in order to join the single agent assigned to a (i.e., agent 2 prefers $(a, 2)$ to $(b, 4)$). In contrast, assignment π is Nash stable.

- *individually stable*, if
  (1) for every agent $i \in N$ with $\pi(i) \neq a_\emptyset$ and every $a_j \in A^* \setminus \{\pi(i)\}$ such that $(a_j, |\pi^j| + 1) \succ_i (\pi(i), |\pi(i)|)$ there exists an agent $i' \in \pi^j$ with $(a_j, |\pi^j|) \succ_{i'} (a_j, |\pi^j| + 1)$, and
  (2) for every agent $i \in N$ with $\pi(i) = a_\emptyset$ and every $a_j \in A^*$ such that $(a_j, |\pi^j| + 1) \succ_i a_\emptyset$ there exists an agent $i' \in \pi^j$ with $(a_j, |\pi^j|) \succ_{i'} (a_j, |\pi^j| + 1)$.

  For a-GASP, this boils down to the following definition: π is individually stable if for each agent $i \in N$ with $\pi(i) = a_\emptyset$ and each $a_j \in A^*$ it holds that $(a_j, |\pi^j| + 1) \notin S_{i'}$ for some $i' \in \pi^j \cup \{i\}$.

  As a particular example for the setting o-GASP, in Example 5.1 both λ and π are individually stable. For instance, considering assignment λ it is sufficient to verify that agent 1 has neither an incentive to leave activity a nor does she want other agents to join a.

- *contractually individually stable*, if
  (1) for every agent $i \in N$ with $\pi(i) = a_\ell$ for some $a_\ell \in A^*$ and every $a_j \in A^* \setminus \{\pi(i)\}$ such that $(a_j, |\pi^j| + 1) \succ_i (\pi(i), |\pi(i)|)$ there exists an agent $i' \in \pi^j$ with $(a_j, |\pi^j|) \succ_{i'} (a_j, |\pi^j| + 1)$ or an agent $i'' \in \pi^\ell \setminus \{i\}$ with $(a_\ell, |\pi(i)|) \succ_{i''} (a_\ell, |\pi(i)| - 1)$, and
  (2) for every agent $i \in N$ with $\pi(i) = a_\emptyset$ and every $a_j \in A^*$ such that $(a_j, |\pi^j| + 1) \succ_i a_\emptyset$ there exists an agent $i' \in \pi^j$ with $(a_j, |\pi^j|) \succ_{i'} (a_j, |\pi^j| + 1)$.

  Note that in a-GASP for individually rational assignments the concepts of contractual individual stability and individual stability coincide, because no agent assigned to a non-void activity has an incentive to deviate.

  As a particular example for the setting o-GASP, in Example 5.1 consider assignment μ with $\mu(1, 4) = a$, $\mu(2, 3) = b$ and $\mu(5) = a_\emptyset$. μ is not individually stable, because agent 4 would like to join b and agents $2, 3$ are better off with agent 4 joining since they prefer $(b, 3)$ over $(b, 2)$. However, μ is contractually individually stable, since agent 1 objects to agent 4 leaving a.

- *core stable* (or in the core), if there is no $E \subseteq N$, $E \neq \emptyset$, such that for some $a_j \in A^*$ with $\pi^j \subseteq E$ it holds that $(a_j, |E|) \succ_i (\pi(i), |\pi(i)|)$ for all $i \in E$.

  In the setting of a-GASP, given that the initial assignment is individually rational, only agents assigned to $a_\emptyset$ can benefit from forming a coalition (in order to deviate to a non-void activity). Hence, the above definition simplifies as follows: an individually rational assignment π is core stable if there is no $E \subseteq \pi^0$ such that for some $a_j \in A^*$ with $\pi^j = \emptyset$ it holds that $(a_j, |E|) \in S_i$ for all $i \in E$.

- *strictly core stable* (or in the strict core), if there is no $E \subseteq N$ such that for some $a_j \in A^*$ with $\pi^j \subseteq E$ it holds that $(a_j, |E|) \succsim_i (\pi(i), |\pi(i)|)$ for all $i \in E$ and $(a_j, |E|) \succ_i (\pi(i), |\pi(i)|)$ for some $i \in E$.

  For a-GASP, note that now also agents assigned to a non-void activity may be part of the deviating group E of agents as long as E contains at least one agent assigned to $a_\emptyset$ (which thus benefits from the deviation). In that setting, the definition can hence be simplified as follows: an individually rational assignment π is strictly core stable if there is no $E \subseteq N$ with $E \cap \pi^0 \neq \emptyset$ such that for some $a_j \in A^*$ with $\pi^j \subseteq E$ it holds that $(a_j, |E|) \in S_i$ for all $i \in E$. The condition $E \cap \pi^0 \neq \emptyset$ makes sure that at least one member of the deviating group E strictly prefers the new outcome.

We point out that in the setting of o-GASP the concepts of the core and the strict core coincide (Darmann, 2015). Consider again Example 5.1. Assignments π and λ are both core stable. In contrast, assignment η with $\eta(1) = a$, $\eta(3) = \eta(4) = b$ and $\eta(2) = \eta(5) = a_\emptyset$ is not core stable, since each member of the group $E = \{2, 3, 4, 5\} \supset \{3, 4\}$ is better off with $(b, |E|) = (b, 4)$.

Finally, in the stable invitation problem the concept of stability considered is the one of Nash stability. Formally, we have the following:

- In ASIP, an invitation S (i.e., a subset of agents) is *stable*, if it is individually rational and for each $i \in N \setminus S$ we have $|S| + 1 \prec_i 0$.
- In GSIP, an invitation S is *stable*, if it is individually rational and for each $i \in N \setminus S$ at least one of the following holds: $F_i \not\subseteq (S \cup \{i\})$, $R_i \cap (S \cup \{i\}) \neq \emptyset$, or $|S| + 1 \prec_i 0$.

Consider the instance of ASIP presented in Example 5.3. The invitation $\{1, 2, 3, 5\}$ is not stable, since agent 6 prefers a group size of 5 over 0. Similarly, neither is the invitation $\{2, 3, 4\}$ stable because agent 1 would like to join the group. However, it is not difficult to verify that the invitation $\{1, 2, 5, 6\}$ is stable.

5.4.3 Social Choice Based Concepts

In addition to these game-theory based stability concepts also concepts from social choice theory have been applied to GASP. These concepts are positional scores (in particular approval and Borda scores) on the one hand, and the Condorcet criterion on the other.

Given an instance of GASP, a scoring function f maps an assignment to a non-negative real number by means of $f(\pi) := \sum_{i \in N} f_i(\pi(i), |\pi_i|)$ with $f_i : X_i \to \mathbb{R}_0^+$. The value $f(\pi)$ is called *score of* π. The goal now would be to find an individually rational assignment of maximum total score.

In *Approval scores*, for $i \in N$ let $f_i(x) = 1$ if $x \in S_i$ and $f_i(x) = 0$ if $x \notin S_i$. Approval scores in the case $|S_i| = k$ for each $i \in N$ are called *k-approval scores* ($k \in \mathbb{N}$). In an instance of o-GASP, *Borda scores* are given by $f_i(x) = |\{x' \in X_i : x \succ_i x'\}|$ for $i \in N$.

Note that approval scores take back the setting to the one of a-GASP; in particular, an individually rational assignment of maximum approval score corresponds to a maximum individually rational assignment.

An alternative approach is to adapt the Condorcet criterion to GASP. This leads to the following two solution concepts: (i) a Condorcet-winner among the individually rational assignments, and (ii) a Condorcet-winner among the maximum individually rational assignments.

Comparing two individually rational assignments $\pi, \bar{\pi}$ in an instance (N, A, P) of GASP, we say that agent i *prefers* π over $\bar{\pi}$ (denoted by $\pi \rhd_i \bar{\pi}$), if $\pi(i) = a_j$ for some $a_j \in A^*$ and either

(1) $\bar{\pi}(i) = a_\emptyset$ and $(a_j, |\pi^j|) \succ_i a_\emptyset$ or

(2) $\bar{\pi}(i) = a_\ell$ for some $a_\ell \in A^*$ and $(a_j, |\pi^j|) \succ_i (a_\ell, |\bar{\pi}^\ell|)$ holds.

Then, an assignment π is called

- *IR-Condorcet*, if π is individually rational and for all individually rational assignments $\pi' \neq \pi$ we have $|\{i \in N : \pi \rhd_i \pi'\}| > |\{i \in N : \pi' \rhd_i \pi\}|$.
- *MIR-Condorcet*, if π is maximum individually rational and for all maximum individually rational assignments $\pi' \neq \pi$ we have $|\{i \in N : \pi \rhd_i \pi'\}| > |\{i \in N : \pi' \rhd_i \pi\}|$.

5.5 Computational Issues

In this section we provide some computational complexity results for GASP, and in particular a-GASP and o-GASP; we also refer to the stable invitation problem where appropriate. We begin with the concept of maximum individual rationality.

Maximum Individually Rational Assignments

Clearly, an individually rational assignment always exists; e.g., the assignment which assigns each agent to the void activity $a_\emptyset$ is individually rational. One natural goal, especially in the setting of a-GASP, is to assign the maximum number of agents to non-void activities. However, this task of finding a maximum individually rational assignment turns out to be computationally hard. In particular, even for restricted instances of a-GASP it is hard to decide whether a *perfect assignment* exists, i.e., an assignment that assigns each agent to a non-void activity.

Theorem 5.1 (Darmann et al., 2012). *It is* NP*-complete to decide whether* a-GASP *admits a perfect assignment, even when all activities in A^* are simple and all agents have increasing preferences.*

Theorem 5.2 (Darmann et al., 2012). *It is* NP*-complete to decide whether* a-GASP *admits a perfect assignment, even when all activities in A^* are simple and all agents have decreasing preferences.*

The latter theorem also holds if restricted to instances in which either (i) each agent $i \in N$, in any of her approved alternatives, accepts a group size of at most 2, or (ii) each agent i's approval set is "made up" of at most 3 different activities, i.e., $|\{a \mid (a, k) \in S_i \text{ for some } k \in \mathbb{N}\}| \leqslant 3$ holds for each $i \in N$.
On the positive side, it can be shown that a polynomial time algorithm to find a maximum individually rational assignment exists, if the number of agents or the number of activities are bounded by a constant ((Darmann et al., 2012); note that this also implies that in the anonymous stable invitation setting, a maximum

individually rational invitation can be determined efficiently). Alternatively, if the number of approved alternatives is bounded by a constant (i.e., in the case of k-approval scores), then for the case of all agents having increasing preferences a maximum individually rational assignment can be found efficiently; for the case of all agents having decreasing preferences this holds if and only if the number of approved alternatives is at most three for each agent (see Darmann, 2015).

Considering copyable activities, the decision problem whether a perfect assignment exists turns out to be NP-complete even when all activities in A^* are equivalent, i.e., A^* consists of a single infinitely copyable activity a only. However, on the positive side, there is a $\mathcal{O}(\sqrt{n})$ approximation algorithm in this case. For details and further results we refer the reader to Darmann et al. (2012).

Maximum Score Assignments and Condorcet Assignments

As mentioned in Section 5.4, a maximum individually rational assignment corresponds to one that maximizes total approval score among the individually rational assignments. In the setting of o-GASP it might seem plausible to use other solution concepts to compare different outcomes. For instance, other types of scores such as Borda scores could be applied. However, finding an individually rational assignment maximizing Borda score is NP-hard, both for the special cases of increasing and decreasing preferences (Darmann, 2016b).

Instead of using scores, an alternative solution concept from social choice theory would be the one of a Condorcet winner. This approach again leads to negative complexity results for restricted instances of o-GASP already: It turns out to be coNP-hard to decide whether an IR-Condorcet assignment or MIR-Condorcet assignment exists even in the case of increasing preferences. In contrast, in the case of decreasing preferences, an IR-Condorcet assignment is guaranteed to exist and can be determined efficiently. If a similar result holds also for MIR-Condorcet assignments is an interesting open question. For details we refer to Darmann (2016b).

Stable Assignments

For both hedonic and non-hedonic games, Ballester (2004) shows that it is NP-complete to decide whether a partition exists that is Nash stable, (contractually) individually stable, or core stable. We will not discuss the results for hedonic games in detail here, for an overview we refer to Aziz and Savani (2016).

In a-GASP, a Nash stable assignment does not always exist, as the following example (taken from Darmann et al. (2012)) shows.

Example 5.4. *Let $\mathcal{I} = (N, A, P)$ be an instance of* a-GASP *with $N = \{1, 2\}$, $A^* = \{a\}$, and induced approval votes $S_1 = \{(a, 1)\}$ and $S_2 = \{(a, 2)\}$. There are two individually rational assignments: π with $\pi(1) = a$ and $\pi(2) = a_\emptyset$ and λ with $\lambda(1) = \lambda(2) = a_\emptyset$. Neither of these assignments is Nash stable, since in π agent 2 would like to join a, and in λ agent 1 wants to engage in a.*

In particular, it turns out that the related decision problem whether a-GASP admits a Nash stable assignment is NP-complete (Darmann et al. (2012)). On the positive side, in a mixed increasing-decreasing instance of a-GASP a Nash stable assignment is guaranteed to exist; this case even allows for an efficient computation of such an assignment.

Theorem 5.3 (Darmann et al., 2012). *Given a mixed increasing-decreasing instance (N, A, P) of* a-GASP*, we can find a Nash stable assignment in polynomial time.*

However, in o-GASP an analogous result does not hold. In particular, even if all agents have increasing preferences a Nash stable assignment does not always exist as shown by the following example (taken from Darmann (2015)).

Example 5.5. *The following instance with 6 agents $N = \{1, 2, 3, 4, 5, 6\}$ and 3 activities $A^* = \{a, b, c\}$ is given by:*

1	2	3	4	5	6
$(b,6)$	$(a,6)$	$(c,6)$	$(b,6)$	$(a,6)$	$(c,6)$
$(b,5)$	$(a,5)$	$(c,5)$	$(b,5)$	$(a,5)$	$(c,5)$
$(b,4)$	$(a,4)$	$(c,4)$	$(b,4)$	$(a,4)$	$(c,4)$
$(b,3)$	$(a,3)$	$(c,3)$	$(b,3)$	$(a,3)$	$(c,3)$
$(a,6)$	$(a,2)$	$(b,6)$	$(b,2)$	$(c,6)$	$(c,2)$
$(a,5)$	$a_\emptyset$	$(b,5)$	$a_\emptyset$	$(c,5)$	$a_\emptyset$
$(a,4)$		$(b,4)$		$(c,4)$	
$(a,3)$		$(b,3)$		$(c,3)$	
$(a,2)$		$(b,2)$		$(c,2)$	
$(a,1)$		$(b,1)$		$(c,1)$	
$a_\emptyset$		$a_\emptyset$		$a_\emptyset$	

Consider an assignment π. Assume π is Nash stable. Then, π must assign each of the agents $1, 3, 5$ to a non-void activity, since otherwise the agents would like to join a, b and c respectively.

Assume agent 1 is assigned to b. *This implies that also agents $3, 4$ must be assigned to b. Agent 5, who has to be assigned to a non-void activity, cannot be assigned to a since this would imply that also agent 1 is assigned to a. Thus, agent 5 must be assigned to c. Hence, agent 6 must also be assigned to c due to the Nash stability of π. This, however, implies that agent 3 would like to deviate from π in order to join c because agent 3 prefers $(c, 3)$ to $(a, 3)$, and π cannot be Nash stable.*

Assume agent 1 is assigned to a. *By Nash stability, this implies that agent 2 has to be assigned to a, and in turn, also agent 5 needs to be assigned to a. Agent 3, who needs to be assigned to a non-void activity, must hence be assigned to c since agent 1 is already assigned to a (and not to b). This, however, implies that exactly 3 agents are assigned to c which is not possible since agent 5 is already assigned to a.*

In o-GASP, it turns out that even for increasing instances the decision problem whether a Nash stable assignment exists is NP-complete (Darmann, 2015). For the general setting of GASP, this decision problem is even W[1]-hard with respect to the number of activities (Igarashi et al., 2017).

Turning to the variant of GASP with only one activity, i.e., the stable invitation problem, we get different complexity results for the two versions ASIP and GSIP.

For the anonymous variant ASIP, it can be decided in polynomial time whether a (Nash) stable invitation exists; if it does, then a stable invitation maximizing the number of invitees can also be determined efficiently. In GSIP, however, in general the corresponding decision problem is computationally hard, even if the size of all rejection sets and acceptance sets is at most one. For details and further complexity results with respect to the size of these sets we refer the reader to Lee and Shoham (2015).

Considering further stability concepts, we point out that in a-GASP an individually stable assignment (and thus a contractually assignment) always exists and can be determined efficiently. For o-GASP, even in the restricted case of increasing preferences this does not hold, since in that special case individual stability coincides with Nash stability. In addition, recall that in o-GASP the concepts of core and strict core are equivalent; the core might be empty and the decision problem whether a core stable assignment exists is computationally hard even for increasing preferences. In contrast, for decreasing preferences, a stable assignment can be determined efficiently for each of these stability concepts. Also, we point out that in a-GASP both the core and the strict core are always non-empty, and an assignment in the core can be determined efficiently. For details we refer to Darmann et al. (2012) and Darmann (2015).

Finally, Igarashi et al. (2017) show that in GASP deciding whether a core stable assignment exists is NP-complete even for instances with only 4 activities. Further, they provide a number of complexity and fixed parameter tractability results for gGASP, for different types of underlying networks.

5.6 Strategic Issues

From a strategic viewpoint the question arises whether there is a deterministic mechanism (that for each instance of GASP outputs an assignment of agents to activities) which is robust against strategic manipulation. Darmann (2016a) shows that even in the single activity case a strategy-proof mechanism that outputs a maximum individually rational assignment in general does not exist. While it is not hard to see that restricting the single activity case to increasing preferences rules out strategic manipulation, also for increasing preferences there does not exist a strategy-proof mechanism that finds a maximum individually rational assignment if there are two or more activities. For further results and an analysis of strategic manipulability of the aggregation correspondence which outputs, for an instance of GASP, all maximum individually rational assignments, we refer to Darmann (2016a).

In a similar spirit, Lee and Shoham (2015) show that for ASIP there is no strategy-proof mechanism that always finds a stable assignment (if it exists); for the increasing preferences case, however, there is such a mechanism that even finds a stable invitation of maximum size in polynomial time.

5.7 Conclusion

We have provided an overview over variants of group activity selection problems, with the focus on the setting of GASP. We have discussed its relation to the literature and, in particular, hedonic games, and have shown how the stability concepts in hedonic games translate to GASP and its two variants a-GASP and o-GASP. Since it can also be understood as a kind of voting problem, we have adapted solution concepts from voting theory as well. For these concepts, both negative and positive computational complexity results have been provided; in addition, we have briefly pointed towards strategic issues.

We end by mentioning three areas that are related to group activity selection:

- *Congestion games* (Rosenthal, 1973): agents choose routes (which sort of play the same role as activities) and have decreasing preferences on the number of users who take the same route, which may conflict with their intrinsic preferences over routes (for instance, most agents prefer shorter routes, which then tend to be congested).

- *Committee elections* in the style of Monroe (1995): agents express preferences on single candidates, and a committee of k representatives has to be selected. The Monroe rule assigns every agent to one of the member of the committee, who is supposed to represent her; this is similar to group activity selection, with candidates playing the role of activities, and where agents have preferences that depend only on the activity they are assigned to, and not the number of participants.

- *Course assignment* (Gale and Shapley, 1962): Courses play a similar role as activities: each course has a capacity, we can only open a limited number of courses, and agents are assigned either to one course (in which case we have an instance of group activity selection with preferences over activities only) or to multiple courses, which can be seen as a generalization of group activity selection where the agent-activity assignment is many-to-many, and where agents have preferences over sets of activities.

Bibliography

H. Aziz and R. Savani. Hedonic games. In F. Brandt, V. Conitzer, U. Endriss, J. Lang, and A. D. Procaccia, editors, *Handbook of Computational Social Choice*, chapter 15. Cambridge University Press, 2016.

C. Ballester. NP-completeness in hedonic games. *Games and Economic Behavior*, 1(49):1–30, 2004.

S. Banerjee, H. Konishi, and T. Sönmez. Core in a simple coalition formation game. *Social Choice and Welfare*, 18:135–153, 2001.

A. Bogomolnaia and M. O. Jackson. The stability of hedonic coalition structures. *Games and Economic Behavior*, 38:201–230, 2002.

A. Darmann. Group activity selection from ordinal preferences. In *Proceedings of the 4th International Conference on Algorithmic Decision Theory (ADT)*, pages 35–51, 2015.

A. Darmann. Manipulation in group activity selection. *Available at SSRN: https://ssrn.com/abstract=2757100*, 2016a.

A. Darmann. Borda, Condorcet, and Pareto optimality in ordinal group activity selection. In *Proceedings of the 6th International Workshop on Computational Social Choice (COMSOC)*, 2016b.

A. Darmann, E. Elkind, S. Kurz, J. Lang, J. Schauer, and G. Woeginger. Group activity selection problem. In *Proceedings of the 8th Workshop on Internet and Network Economics (WINE)*, pages 156–169, 2012.

J. H. Drèze and J. Greenberg. Hedonic coalitions: Optimality and stability. *Econometrica*, 48(4):987–1003, 1980.

D. Gale and L. Shapley. College admissions and the stability of marriage. *The American Mathematical Monthly*, 69(1):9–15, 1962.

A. Igarashi, R. Bredereck, and E. Elkind. On parameterized complexity of group activity selection problems on social networks. In *Proceedings of the 16th International Conference on Autonomous Agents and Multiagent Systems (AAMAS)*, pages 1575–1577, 2017.

A. Igarashi, D. Peters, and E. Elkind. Group activity selection on social networks. In *Proceedings of the 31st AAAI Conference on Artificial Intelligence (AAAI)*, pages 565–571, 2017.

H. Lee and Y. Shoham. Stable group scheduling. In *Proceedings of the 13th International Conference on Autonomous Agents and Multiagent Systems (AAMAS)*, pages 1347–1348, 2014.

H. Lee and Y. Shoham. Stable invitations. In *Proceedings of the 29th AAAI Conference on Artificial Intelligence (AAAI)*, pages 965–971, 2015.

T. Lu and C. Boutilier. Matching models for preference-sensitive group purchasing. In *Proceedings of the 13th ACM Conference on Electronic Commerce (ACM EC)*, pages 723–740, 2012.

B. Monroe. Fully proportional representation. *The American Political Science Review*, 89(4):925–940, 1995.

R. Rosenthal. A class of games possessing pure-strategy Nash equilibria. *International Journal of Game Theory*, 2(1):65–67, 1973.

M. Spradling, J. Goldsmith, X. Liu, C. Dadi, and Z. Li. Roles and teams hedonic game. In *Proceedings of the 3rd International Conference on Algorithmic Decision Theory (ADT)*, pages 351–362, 2013.