Properties of Local Search PAV

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Abstract

Proportional Approval Voting (PAV) is a well-established voting rule for multi-winner approval voting with strong fairness guarantees: it is one of very few natural rules to satisfy the Extended Justified Representation axiom [19], and has optimal proportionality degree [20]. These guarantees extend to the bounded local search version of this rule, known as ε-ls-PAV, as long as $\varepsilon \leq \frac{n}{k^2}$ (where n is the number of voters and k is the target committee size). In this work, we provide a detailed study of the family of rules ε-ls-PAV. We show that for a suitable choice of ε (which still guarantees polynomial running time), these rules exhibit many desirable properties, such as 2-approximation to FJR and core, robustness to small changes, and linear-time verifiability. On the negative side, we show that if ε can be arbitrarily small, the running time of ε-ls-PAV may be super-polynomial, by proving a lower bound of $\Omega(k^{\log k})$ for non-adversarial better response; this resolves a question left open by [4]. To complement our lower bound, we provide an empirical comparison of two variants of local search—better-response and best-response—on several real-life data sets. Our experiments indicate that, in practice, better response exhibits faster running time than best response.

1 Introduction

A student organization with 200 members has funding for three annual magazine subscriptions, to be selected from a long list of equally-priced magazines. For simplicity, it has been decided to elicit preferences by means of approval ballots; however, the aggregation rule is yet to be determined. There are 102 students who are interested in sports, and all of them approve each of the three sports magazines on the list (and nothing else). However, there is also a group of 80 students who would all be interested in a subscription to The New Yorker, and it so happens that none of them are interested in any of the sports magazines.

If the students are to aggregate their preferences by means of counting the number of approvals for each magazine, the three sports magazines will get 102 votes each, while every other magazine will get fewer than 100 votes. Hence, the organization will end up subscribing to the three sports magazines, which seems unfair to the readers of The New Yorker, who constitute more than a third of the members of the organization. Hence, more sophisticated ways of aggregating approval ballots would be needed. In particular, the students may want to use a voting rule with the property that all sufficiently large groups of voters with cohesive preferences (such as, in our example, the readers of The New Yorker) obtain at least some representation.

There are many voting rules that use approval ballots, and a rich literature on group fairness in approval-based multiwinner voting [14]. In particular, the idea that large cohesive groups of voters should be represented in the winning set of alternatives (usually called a *committee*) is captured by the family of *justified representation* axioms, which includes (from least demanding to most demanding) justified representation (JR), proportional justified representation (PJR), extended justified representation (EJR), full justified representation (FJR), and the core [3, 19, 17].

One particularly attractive and well-established rule in this context is Proportional Approval Voting (PAV). This is an easy-to-explain voting rule that satisfies the fairly restrictive EJR axiom; it is also optimal with respect to another measure of group fairness, namely, *proportionality degree*

[20]. This rule proceeds by having voters assign scores to committees and selecting committees with the highest sum of scores; the score that a voter i assigns to a committee W is 0 if i does not approve any candidates in W and $1 + \frac{1}{2} + \cdots + \frac{1}{j}$ if i approves exactly j candidates in W. In our example, PAV would favor a committee that contains two sports magazines and The New Yorker over a committee that contains three sports magazines: the former would score $102 + 51 + 80$ points, while the latter would score $102 + 51 + 34$ points.

Unfortunately, while evaluating the PAV score of a given committee is easy, finding a committee with the maximum score is an NP-hard search problem [1]. To circumvent this issue, Aziz et al. [4] proposed a bounded local search version of this rule, which starts at an arbitrary committee and performs a sequence of local improvement steps. More specifically, at each step their rule checks whether the PAV score of the current committee can be improved by at least $\frac{n}{k^2}$ (where n is the number of voters and k is the target committee size) by exchanging a candidate currently in the committee with one who is not. It performs some such swap if it exists (and moves on to the next step), and it terminates when no $\frac{n}{k^2}$-improving swaps are available. Aziz et al. [4] show that the final committee is guaranteed to provide EJR; on the other hand, as the maximum PAV score of a size-k committee cannot exceed $O(nk \log k)$, the number of steps till convergence can be bounded by a polynomial in n and k. Thus, the resulting rule, which we will refer to as $\frac{n}{k^2}$-ls-PAV, is polynomial-time computable and provides EJR; indeed, historically, this is the first voting rule to be shown to have both of these properties.

Soon thereafter, Peters and Skowron [15] proposed another rule that is both polynomial-time computable and satisfies EJR, namely, the Method of Equal Shares (MES). This rule is intuitively appealing and preserves its attractive axiomatic properties when extended to the more general *participatory budgeting* setting [16]. Hence, $\frac{n}{k^2}$-ls-PAV has not received much attention in the computational social choice literature (see, however, the recent work of Halpern et al. [13], which considers this rule in an incomplete information scenario). The goal of our work is to argue that $\frac{n}{k^2}$-ls-PAV—and, more broadly, the rules that are based on the same principle but may use an arbitrary threshold ε in place of $\frac{n}{k^2}$—form an appealing family of multiwinner voting rule, which have desirable properties (some of which are not shared by MES), and therefore deserve further attention. To make our case, we explore the axiomatic and computational properties of the rules in this family for various values of ε.

Our Contribution We start by showing that $\frac{n}{k^2}$-ls-PAV provides a 2-approximation of the stronger FJR property (and this bound is tight). Similarly, just like the PAV itself, $\frac{n}{k^2}$-ls-PAV approximates the even more demanding *core stability* by a factor of 2 (again, this bound is tight). We also show that using $\varepsilon < \frac{n}{k^2}$ can be desirable: a smaller threshold makes ls-PAV more robust, in the sense that, even upon deleting or adding roughly $(1 - \varepsilon)\frac{n}{k^2}$ voters, we retain proportionality guarantees for the original instance. We also observe that $\frac{n}{k^2}$-ls-PAV has an attractive verifiability property, as formulated by Cevallos and Stewart [8]: given an output of this rule, we can quickly verify that it provides EJR and has optimal proportionality degree (even though it is NP-hard to check whether an arbitrary committee provides EJR [3]). This property is particularly desirable in the context of blockchain applications. Moreover, unlike the phragmms rule proposed by Cevallos and Stewart [8], the verification procedure for $\frac{n}{k^2}$-ls-PAV does not require any auxiliary information.

Now, while setting $\varepsilon = \frac{n}{k^2}$ guarantees both EJR and polynomial-time convergence, another natural choice would be to set ε to a very small positive value (which we will denote by 0^+), i.e., to perform a swap as long as it increases the PAV score. Clearly, all outputs of 0^+-ls-PAV provide EJR; moreover, as each swap increases the PAV score, the algorithm is guaranteed to terminate. However, the argument showing polynomial-time convergence no longer applies. Indeed, Aziz et al. [4] left it as an open problem whether 0^+-ls-PAV converges after polynomially many steps. We resolve this open problem in the negative. Specifically, for each $k \geq 0$ we construct a multiwinner election with target committee size k, in which the number of voters is polynomial in k, that has the property that 0^+-ls-PAV may require $\Omega(k^{\log k})$ steps until convergence. Our argument relies on an elementary, yet complicated combinatorial construction. As a warmup, we also show that $\frac{n}{k^2}$-ls-PAV has run-time $\Theta(k^2)$ in the worst case; the proof is simpler, but illustrates the main ideas behind the more complex construction.

Importantly, our lower bound only applies to the 'better response' variant of local search, in which the algorithm performs a swap as soon as it finds a pair of candidates such that swapping them results in the desired improvement. A natural alternative is a 'best response' version of this procedure, where, instead of performing the first suitable swap, the algorithm considers all pairs of candidates and performs a swap that offers the maximum improvement in the PAV score. While we conjecture that this variant of 0^+-ls-PAV may still require a super-polynomial number of steps to converge, we were unable to extend our proof to this case.

Note, however, that switching from better response to best response comes at a cost: when looking for a better response, we may be lucky to find an improving swap after checking just a few candidate pairs, whereas to find the best response, we need to consider all $k(m-k)$ possible swaps (where m is the total number of candidates). Thus, while we expect best response to converge after a smaller number of iterations, each individual iteration is more costly.

We conclude the paper by exploring this trade-off empirically, using several real-life datasets. We measure the performance of each algorithm on a given instance as the number of candidate swaps it needs to consider before termination (this is a useful proxy for running time as long as we do not have access to parallel processing hardware). Interestingly, on the datasets we investigate, better response considers fewer swaps than best response. Hence, while our theoretical worst-case results seem to suggest that best response may be preferable to better response, the empirical results paint the opposite picture.

Related Work There is, by now, a very substantial literature on approval-based multiwinner voting rules, their axiomatic properties, and algorithmic complexity [14]. In particular, while the basic JR axiom is easy to satisfy, there are only a few polynomial-time computable rules that satisfy the stronger axioms. Indeed, we are not aware of any polynomial-time computable voting rules that satisfy FJR, and it is a well-known open problem whether each multiwinner approval election has an outcome in the core. There are, however, a few voting rules with an efficient winner determination algorithms that satisfy EJR or its slightly less demanding cousin PJR, such as sequential Phragmén's rule [5] and Maximin support method [9] (which satisfy PJR), or Method of Equal Shares [15] (which satisfies EJR). Among these rules, MES is perhaps the most attractive: it is polynomial-time computable, is relatively easy to explain to voters, and satisfies EJR. Importantly, however, all these voting rules are based on entirely different principles than PAV: they are all formulated in terms of voters sharing the 'load' caused by the candidates in the committee. Hence, having a better understanding of PAV and its local search variants is important for building our intuition about group fairness in approval voting.

2 Preliminaries

An *approval election* is a 4-tuple $E = (N, C, (A_i)_{i \in N}, k)$, where $N = [n]$ is a set of *voters*, C is a set of *candidates*, $|C| = m$, $A_i \subseteq C$ is the *ballot* of voter $i \in N$, and k is a positive integer that satisfies $k \leq m$; we will refer to k as the *target committee size*. We associate the *size* of E with the total size of the voters' ballots: we write $|E| = \sum_{i \in N} |A_i|$. Subsets of C (not necessarily of size k) are called *committees*. Let $\mathcal{W}$ denote the set of all subsets of C, and let $\mathcal{W}(k)$ denote the set of all size-k subsets of C. If $c \in A_i$, we say that voter i *approves* candidate c, or *c receives an approval* from i. An *approval-based multiwinner voting rule* is a mapping that, given an approval election $(N, C, (A_i)_{i \in N}, k)$, outputs a non-empty subset of $\mathcal{W}(k)$; the elements of this subset are referred to as *winning committees.*

Proportionality Axioms We will now formulate several axioms that aim to capture the concept of proportionality in approval-based multiwinner voting.

Given an approval election $(N, C, (A_i)_{i \in N}, k)$ and positive integers $r \le \ell \le k$, we say that a group of voters $V \subset N$ is *ℓ-large* if $|V| \ge \ell \cdot \frac{n}{k}$; we say that V is *(ℓ, r)-cohesive* if there exists a set of candidates S with $|S| \le \ell$ such that $|A_i \cap S| \ge r$ for each $i \in V$.

Consider an election $(N, C, (A_i)_{i \in N}, k)$. We say that a committee $W \in \mathcal{W}$ provides

- *proportional justified representation (PJR)* if for each positive integer $\ell \le k$ and every ℓ-large (ℓ, ℓ)-cohesive group of voters V it holds that $|(\cup_{i \in V} A_i) \cap W| \ge \ell$.
- *extended justified representation (EJR)* if for each positive integer $\ell \le k$ and every ℓ-large (ℓ, ℓ)-cohesive group of voters V it holds that $|A_i \cap W| \ge \ell$ for some $i \in V$.
- *full justified representation (FJR)* if for each pair of positive integers (ℓ, r) with $r \le \ell \le k$ and every ℓ-large (ℓ, r)-cohesive group of voters V it holds that $|A_i \cap W| \ge r$ for some $i \in V$.

Furthermore, we say that W is in the *core* if there does not exist a group of voters $V \subseteq N$ and a set of candidates S such that $|S| \le \frac{|V|}{n} \cdot k$ and for each voter $i \in V$ we have $|A_i \cap S| > |A_i \cap W|$.

A voting rule is said to satisfy EJR/FJR/core stability if all committees in the output of this rule provide the respective property.

Given an election $(N, C, (A_i)_{i \in N}, k)$, let $\mathcal{W}_{\text{core}}$ be the set of all size-k committees that are in the core; similarly, let $\mathcal{W}_{\text{FJR}}$, $\mathcal{W}_{\text{EJR}}$, and $\mathcal{W}_{\text{PJR}}$ be the sets of all size-k committees that provide FJR, EJR, or PJR, respectively. It can be shown that

$$\mathcal{W}_{\text{core}} \subseteq \mathcal{W}_{\text{FJR}} \subseteq \mathcal{W}_{\text{EJR}} \subseteq \mathcal{W}_{\text{PJR}}.$$

Moreover, it is known that for every election the sets $\mathcal{W}_{\text{FJR}}$, $\mathcal{W}_{\text{EJR}}$, and $\mathcal{W}_{\text{PJR}}$ are non-empty.

Given an approval election $(N, C, (A_i)_{i \in N}, k)$ and a committee W, the *satisfaction* of voter i from W is measured as $|A_i \cap W|$, and the *average satisfaction* of a set of voters $V \subset N$ is measured as $avs_V(W) = \frac{1}{|V|} \sum_{i \in V} |A_i \cap W|$.

Proportional Approval Voting and its Local Search Variant

PAV Given an approval election $(N, C, (A_i)_{i \in N}, k)$ and a committee $W \subseteq C$, we define the *PAV-satisfaction* of voter i from W as $sat_i(W) = \sum_{j=1}^{|A_i \cap W|} \frac{1}{j}$. We extend this notation to sets of voters by setting $sat_V(W) = \sum_{i \in V} sat_i(W)$. The *PAV score* of $W \subseteq C$ is then defined as $sat(W) = sat_N(W)$. The PAV rule outputs all committees in $\mathcal{W}(k)$ that maximize the PAV score.

Given a committee W, a pair of candidates $c \notin W$, $c' \in W$, and a set of voters V, we denote by $\Delta_V(W, c, c')$ the change in the PAV-satisfaction of V that is caused by swapping c with c'; we write $\Delta^*_V(W)$ to denote the maximum increase in the PAV-satisfaction of V that can be accomplished by such a swap.

$$\Delta_V(W, c, c') = sat_V(W \cup \{c\} \setminus \{c'\}) - sat_V(W).$$

Overloading notation, we write $\Delta_V(W, c)$ to denote the increase in the PAV-satisfaction of V that results from adding c to W:

$$\Delta_V(W, c) = sat_V(W \cup \{c\}) - sat_V(W); \qquad \Delta^*_V(W) = \max_{c \in W, c' \notin W} \Delta_V(W, c, c').$$

For readability, we omit V from the notation when $V = N$, i.e., we write $\Delta(W, c, c') := \Delta_N(W, c, c')$, $\Delta^*(W) := \Delta^*_N(W)$, $\Delta(W, c) := \Delta_N(W, c)$.

Local Search PAV Given an approval election $(N, C, (A_i)_{i\in N}, k)$ and a starting committee $W \in \mathcal{W}(k)$, the bounded local search version of PAV (defined by Aziz et al. [4]) proceeds in rounds. In each round, it checks if there is a pair of candidates $c \notin W$, $c' \in W$ such that $\Delta(W, c, c') \geq \frac{n}{k^2}$; if yes, it updates the committee as $W := W \cup \{c\} \setminus \{c'\}$. If no such swap exists, it terminates and outputs W.

Building on this idea, we define a family of local search algorithms parameterized by a non-negative real value $\varepsilon \geq 0$ as follows.

ALGORITHM 1: ε-ls-PAV

Input: W: arbitrary initial committee
Output: W: final committee
while $\exists c \notin W, c' \in W$ *such that* $\Delta(W, c, c') \geq \varepsilon$ **do**
    $W \leftarrow (W \cup \{c\}) \setminus \{c'\}$
end

In this language, the algorithm of Aziz et al. [4] is $\frac{n}{k^2}$-ls-PAV.

The 'vanilla' local search algorithm, which performs a swap as long as $\Delta(W, c, c') > 0$, can be described as ε-ls-PAV for $\varepsilon \leq \min\{\Delta(W, c, c') : W \in \mathcal{W}(k), c \notin W, c' \in W, \Delta(W, c, c') > 0\}$. It can be shown that this condition can be satisfied by setting $\varepsilon = \text{lcm}([k])$, where for each $S \subset \mathbb{N}$ we denote by $\text{lcm}(S)$ the least common multiplier of the integers in S. In what follows, we denote this value of ε by 0^+. In the next section, we will see that values of ε other than $\frac{n}{k^2}$ and 0^+ may be of interest, too.

Better Response and Best Response Algorithm 1 does not explicitly prescribe which *pivoting rule* to use: if there are multiple pairs $c \notin W$, $c' \in W$ with $\Delta(W, c, c') \geq \varepsilon$, the algorithm may perform any of these swaps. We consider two natural pivoting rules: *better response* processes the pairs in $(C \setminus W) \times W$ lexicographically with respect to fixed order and performs the first suitable swap, while *best response* goes over all pairs in $(C \setminus W) \times W$ to identify a swap (c, c') with $\Delta(W, c, c') = \Delta^*(W)$, and performs some such swap (as long as it satisfies $\Delta(W, c, c') \geq \varepsilon$).

2.1 Verifiable and Robust Proportionality

Provided $\varepsilon \leq \frac{n}{k^2}$, we know that ε-ls-PAV outputs a committee that guarantuees every (ℓ, ℓ)-cohesive group average approval utility of more than $\ell - 1$, which is tight and in particular implies that it satisfies EJR. In Appendix A.1, we show that ε-ls-PAV approximates FJR and the core by a factor of 2, and this is tight in both cases. In this section, we highlight two desirable properties of multi-winner voting rules that ls-PAV satisfies effortlessly: The promise of proportionality, be it in the form of average satisfaction of more than $\ell - 1$ for (ℓ, ℓ)-cohesive groups or the 2-core property, is linear-time verifiable. On top of that, these guarantees offer some level of robustness in the presence of noisy elections.

2.1.1 Linear-time Verifiability

Some blockchain networks use multiwinner voting to appoint validators [7, 6, 11]. Validators are special roles that nodes can take on: they have to validate transactions and receive a monetary reward for doing so (or get punished for adversarial behavior). It is desirable to select validators (from the set of candidate nodes) in a proportional manner, both to increase voting nodes' satisfaction [12], and to avoid the centralization of power [10]. Unfortunately, many proportional multiwinner rules are computationally hard [2] or else have prohibitively slow polynomial running time and so nodes in the network with very limited computational resources such as an off-the-shelf computer may not be able to simulate the outcome. The innovation of decentralized networks such as blockchain networks is to eliminate the need for a central authority and so we cannot simply resort to a trusted party that computes an outcome and *promises* that it is proportional by some agreed-upon criterion. Instead, recent work by [8] proposes efficient verifiability as a solution. The observation is that the (expensive) computational task of choosing the validators can be performed by a non-trusted party ("off-chain") as long as the proportionality of the proposed solution can be efficiently checked by any node. Unfortunately, checking whether an arbitrary outcome satisfies the EJR axiom (or even the weaker PJR axiom) is NP-hard even in the setting of multiwinner voting [3, 4]. Cevallos and Stewart [8] propose an alternative approach. They argue that it suffices to have an algorithm whose output can be easily verified: given a winning committee (and possibly some auxiliary information), one should be able to quickly check that this committee satisfies selected proportionality axioms, such as PJR or EJR . Ideally, the verification algorithm should run in time linear in the input size (i.e., $O(|E|)$) and be parallelizable. Cervallos and Stewart put forward a new multiwinner voting rule, which they call phragmms. This rule provides PJR, and its output is linear-time verifiable with respect to PJR. In Appendix A.2 we argue that $\frac{n}{k^2}$-ls-PAV is linear-time verifiable with respect to stronger proportionality guarantees. Moreover, unlike phragmms, our verification algorithm does not use any auxiliary information.

2.1.2 Robustness

Having established that, for $\varepsilon \leq \frac{n}{k^2}$, ε-ls-PAV satisfies EJR and approximates the core and FJR by a factor of 2, we ask how robust these properties are to perturbations of the instance. E.g., suppose that the input to the algorithm is not the true election: perhaps some votes went missing, or were mistakenly added, or else contain errors. What can we say about our properties with respect to the original unperturbed instance? It turns out that, by executing ε-ls-PAV with $\varepsilon = \lambda\frac{n}{k^2}$, where $\lambda < 1$, we can allow some such errors without losing proportionality guarantees.

Let W be a committee output by $\frac{n}{sk^2}$-ls-PAV for some $s > 1$ on a set of voters V_1, $|V_1| = n$. Then W satisfies $\Delta^*_{V_1}(W) < \frac{n}{sk^2}$. Suppose a new batch of voters V_2 arrives. We now give conditions on the size of V_2 that imply that W still satisfies EJR and the 2-core property for $V_1 \cup V_2$ or $V_1 \setminus V_2$.

Theorem 1. *Consider two sets of voters V_1, V_2 and a committee W such that $\Delta^*_{V_1}(W) < \frac{|V_1|}{sk^2}$ for some $s > 1$. If $|V_2| \leq \frac{s-1}{s(k^2+1)} \cdot |V_1|$ then $\Delta^*_{V_1 \cup V_2}(W) \leq \frac{|V_1 \cup V_2|}{k^2}$ and $\Delta^*_{V_1 \setminus V_2}(W) \leq \frac{|V_1 \setminus V_2|}{k^2}$.*

Theorem 1 implies that any committee W that is in the output of $\frac{n}{sk^2}$-ls-PAV on V_1 is also in the output of $\frac{n}{k^2}$-ls-PAV on elections with sets of voters $V_1 \cup V_2$ and $V_1 \setminus V_2$, and hence provides EJR as well as other desirable properties for these elections.

3 Lower Bounds

Our main result in this section is Theorem 3, which shows that 0^+-ls-PAV with better response may make a super-polynomial number of swaps before it terminates. This formally confirms the suspicion voiced in Aziz et al. [4]. The lower bound construction is somewhat complicated;

To showcase the ideas behind our main lower bound in Theorem 3, we initially focus on the adversarial setting. We study lower bounds on the maximum length of a path in the directed graph whose vertices are committees and a directed edge (W, W') exists whenever W' can be obtained from W via a swap and W' has PAV score at least $\frac{n}{k^2}$ higher than W. We call this adversarial better response because these are the swaps that an agent that points out improvements of the existing state but acts adversarially might choose to show us. In this setting we prove a simpler result for $\frac{n}{k^2}$-ls-PAV. Specifically, we next establish that the upper bound on the number of iterations of this algorithm by [4] is tight up to a $\log k$ factor, by exhibiting an instance on which $\frac{n}{k^2}$-ls-PAV may make $\Omega(k^2)$ improvements before it reaches an equilibrium (recall that $O(k^2 \log k)$ iterations are sufficient for convergence).

3.1 Warm-up: Lower Bound for $\frac{n}{k^2}$-ls-PAV

Notation for the proof: Given a committee W, we say that a sequence of swaps $\mathbf{X} = (a_1, b_2), (a_2, b_2), \dots, (a_s, b_s)$ is *valid* if for each $i \in [s]$ the committee $W_i = W \cup \{b_1, \dots b_{i-1}\} \setminus \{a_1, \dots a_{i-1}\}$ satisfies $a_i \in W_i$, $b_i \notin W_i$. The length of sequence $\mathbf{X}$ is $|\mathbf{X}| = s$. We define the *inverse* (sequence) of $\mathbf{X}$ as $\mathbf{X}^{-1} = (b_s, a_s), (b_{s-1}, a_{s-1}), \dots, (b_1, a_1)$. Given two finite sequences of swaps $\mathbf{X}$ and $\mathbf{Y}$, we define their *concatenation* $\mathbf{X} \oplus \mathbf{Y}$ as the the sequence with prefix $\mathbf{X}$ followed by suffix $\mathbf{Y}$. For the proofs in this section it will be useful to have an arbitrarily large pool of 'dummy' candidates. We therefore define $D_k = \{d_1, \dots, d_k\}$ so that $D_{k+1} = D_k \cup \{d_{k+1}\}$ and $D_0 = \varnothing$.

Theorem 2. *$\frac{n}{k^2}$-ls-PAV with adversarial better response requires $\Omega(k^2)$ iterations in the worst case.*

Proof. Let $t = \lfloor \frac{k}{4} \rfloor$. We define the election $E = (N, C, (A_i)_{i \in N}, k)$ as follows.

$$C = C_1 \cup C_2 \cup D_{k-2}, C_1 = \{c_1^1, \dots, c_{t+1}^1\}, C_2 = \{c_1^2, \dots, c_k^2\};$$
$$N = V_1 \cup V_2 \cup_{i=1}^{k} S_i \cup U \text{ where } V_i = \{v_1^i, \dots, v_t^i\}, i \in [2], |U| = \lfloor \frac{k^2}{4} - \frac{k}{2} \rfloor, |S_i| = t, i \in [k].$$

The approval sets of voters v_i^1 and v_i^2 are given by

$$A_{v_i^1} = \{c_{i+1}^1, \dots, c_{t+1}^1\} \cup \{c_j^2 \in C_2 \mid j \text{ is even}\} \text{ and } A_{v_i^2} = \{c_1^1, \dots, c_i^1,\} \cup \{c_j^2 \in C_2 \mid j \text{ is odd}\},$$

where $1 \leq i \leq t$. Each of the t voters $s_j \in S_i$ approves candidates $A_{s_j} = \{c_i^2, \dots, c_k^2\}$. Each $u \in U$ has approval set $A_u = D_{k-2}$. Intuitively, voters in U are dummy voters and candidates D_{k-2} are dummy candidates. The sequence of swaps we will exhibit only affects voters in $N \setminus U$ and candidates in $C_1 \cup C_2$ and no other voters or candidates.

Set-up Consider $\frac{n}{k^2}$-ls-PAV on the above instance with an initial committee

$$W_0 = D_{k-2} \cup \{c_1^1, c_1^2\}.$$

Since each swap must increase the PAV-score by at least $\frac{n}{k^2}$, we bound its value:

$$n = |V_1| + |V_2| + \sum_{i=1}^{k} |S_i| + |U| = \left\lfloor \frac{k}{4} \right\rfloor + \left\lfloor \frac{k}{4} \right\rfloor + k \times \left\lfloor \frac{k}{4} \right\rfloor + \left\lfloor \frac{k^2}{4} - \frac{k}{2} \right\rfloor \leq \frac{k^2}{2} \implies \frac{n}{k^2} \leq \frac{1}{2}.$$

For the purpose of this proof, we will call a swap (a, b) a *good swap for W* if $\Delta(W, a, b) \geq \frac{n}{k^2}$, or simply a *good swap* if W is clear from context. So every valid swap that increases the PAV score by at least $\frac{1}{2}$ (recall that $\Delta(W, a, b) \geq \frac{1}{2}$) is a good swap.

Sequence of Swaps Define $\mathbf{Y} = \oplus_{i=1}^{t} (c_i^1, c_{i+1}^1)$. We aim to show that

$$\mathbf{X} = \oplus_{i=1}^{k-1} \mathbf{Y}^{(-1)^{i-1}} \oplus (c_i^2, c_{i+1}^2)$$

is a sequence of $\Omega(k^2)$ good swaps, each swap increasing the PAV-score by at least $\frac{1}{2}$. That $|\mathbf{X}| = \Omega(k^2)$ follows since $|\mathbf{Y}| = |\mathbf{Y}^{-1}| = t$ and each of the $k-1$ such $\mathbf{Y}$'s and $\mathbf{Y}^{-1}$'s is followed by an additional single swap, and so $|\mathbf{X}| = (k-1)\cdot(t+1) = \Omega(k^2)$.

To show that $\mathbf{X}$ is a sequence of good swaps for initial committee W_0, we split up the analysis into the following four cases.

1. For committee $W = D_{k-2} \cup \{c_i^1\} \cup \{c_j^2\}$, if j is odd and $i \leq t$, (c_i^1, c_{i+1}^1) is a good swap
2. For committee $W = D_{k-2} \cup \{c_i^1\} \cup \{c_j^2\}$, if j is even and $i > 1$, (c_i^1, c_{i-1}^1) is a good swap.
3. For committee $W = D_{k-2} \cup \{c_{t+1}^1\} \cup \{c_j^2\}$, if $j < k$ is odd, (c_j^2, c_{j+1}^2) is a good swap.
4. For committee $W = D_{k-2} \cup \{c_1^1\} \cup \{c_j^2\}$, if $j < k$ is even, (c_j^2, c_{j+1}^2) is a good swap.

Once we have proven the above four items, it will follow that $\mathbf{X}$ is a sequence of good swaps: From item 1, if $W = D_{k-2} \cup \{c_1^1\} \cup \{c_j^2\}$ and j is odd, then $\mathbf{Y}$ is a sequence of good swaps. Furthermore, the sequence of swaps $\mathbf{Y}$ results in a committee $W = D_{k-2} \cup \{c_{t+1}^1\} \cup \{c_j^2\}$ satisfying the condition in item 3, so if $j < k$, then (c_j^2, c_{j+1}^2) is a good swap. This results in a committee $W = D_{k-2} \cup \{c_{t+1}^1\} \cup \{c_{j+1}^2\}$ satisfying the condition in item 2. This implies that $\mathbf{Y}^{-1}$ is a sequence of good swaps, resulting in committee $W = D_{k-2} \cup \{c_1^1\} \cup \{c_{j+1}^2\}$. This committee in turn is as described in item 3, so if $j+1 < k$, then (c_{j+1}^2, c_{j+2}^2) a good swap. This results in $W = D_{k-2} \cup \{c_1^1\} \cup \{c_{j+2}^2\}$, which again satisfies the condition in item 1. Thus, since $W_0 = D_{k-2} \cup \{c_1^1, c_1^2\}$ is as in item 1, using the previous chain of arguments we conclude that the sequence $\mathbf{X}$ is a sequence of good swaps.

Consider the first case, i.e., the committee $W = D_{k-2} \cup \{c_i^1\} \cup \{c_j^2\}$ where j is odd and $1 \leq i \leq t$. Note that v_i^2 approves c_i^1 and not c_{i+1}^1, and conversely for v_i^1, while every other voter either approves both or neither of c_i^1 and c_{i+1}^1. Neither v_i^1 nor v_i^2 approve any candidates in D_{k-2}. Further, the remaining committee members are $W \setminus D_{k-2} = \{c_i^1, c_j^2\}$ where j is odd, so by construction both of these are approved by v_i^2 and none by v_i^1, implying that $|A_{v_i^2} \cap W| = 2$ and $|A_{v_i^1} \cap W| = 0$. We conclude that (c_i^1, c_{i+1}^1) is a good swap because $\Delta(W, c_i^1, c_{i+1}^1) = +1 - \frac{1}{2} = \frac{1}{2}$.

For the second case, consider $W = D_{k-2} \cup \{c_i^1\} \cup \{c_j^2\}$, where $1 < i \leq t+1$ and j, $1 \leq j \leq k$, is even. Since $|A_{v_{i-1}^1} \cap W| = 2$, while $|A_{v_{i-1}^2} \cap W| = 0$ and every other voter approves either both of c_{i-1}^1 and c_i^1 or neither of them, by the same argument as above it holds that $\Delta(W, c_i^1, c_{i-1}^1) = +1 - \frac{1}{2} = \frac{1}{2}$.

For the third case, consider $W = D_{k-2} \cup \{c_{t+1}^1\} \cup \{c_i^2\}$ where $i < k$ is odd. Each voter in S_{i+1} approves c_{i+1}^2 and not c_i^2. Every voter v_j^2, $1 \leq j \leq t$, approves c_i^2, but not c_{i+1}^2, while every voter v_j^1, $1, \leq j \leq t$, approves c_{i+1}^2 and not c_i^2. By construction, the remaining voters (i.e. the voter in S_j, $j \neq i+1$, and the voters in U) approve either both of c_i^2 and c_{i+1}^2 or neither. For $s \in S_{i+1}$, their satisfaction is $|A_s \cap W| = 0$, while for every $1 \leq j \leq t$, $|A_{v_j^1} \cap W| = 1$ as v_j^1 for every $1 \leq j \leq t$ approves only c_{t+1}^1 in W, and $|A_{v_j^2} \cap W| = 1$, because v_j^2 approves only c_i^2 as i is odd. So

$$\Delta(W, c_i^2, c_{i+1}^2) = +|S_{i+1}| + \frac{1}{2} \cdot |V_1| - |V_2| = t + \frac{t}{2} - t = \frac{t}{2} \geq \frac{1}{2},$$

provided k and hence t is large enough. This shows that (c_i^2, c_{i+1}^2) is a good swap. Again by an analogous argument we can show that if $i < k$ is even and $W = D_{k-2} \cup \{c_1^1\} \cup \{c_i^2\}$ then $\Delta(W, c_i^2, c_{i+1}^2) \geq \frac{1}{2}$ since the instance is symmetric. This concludes the proof. □

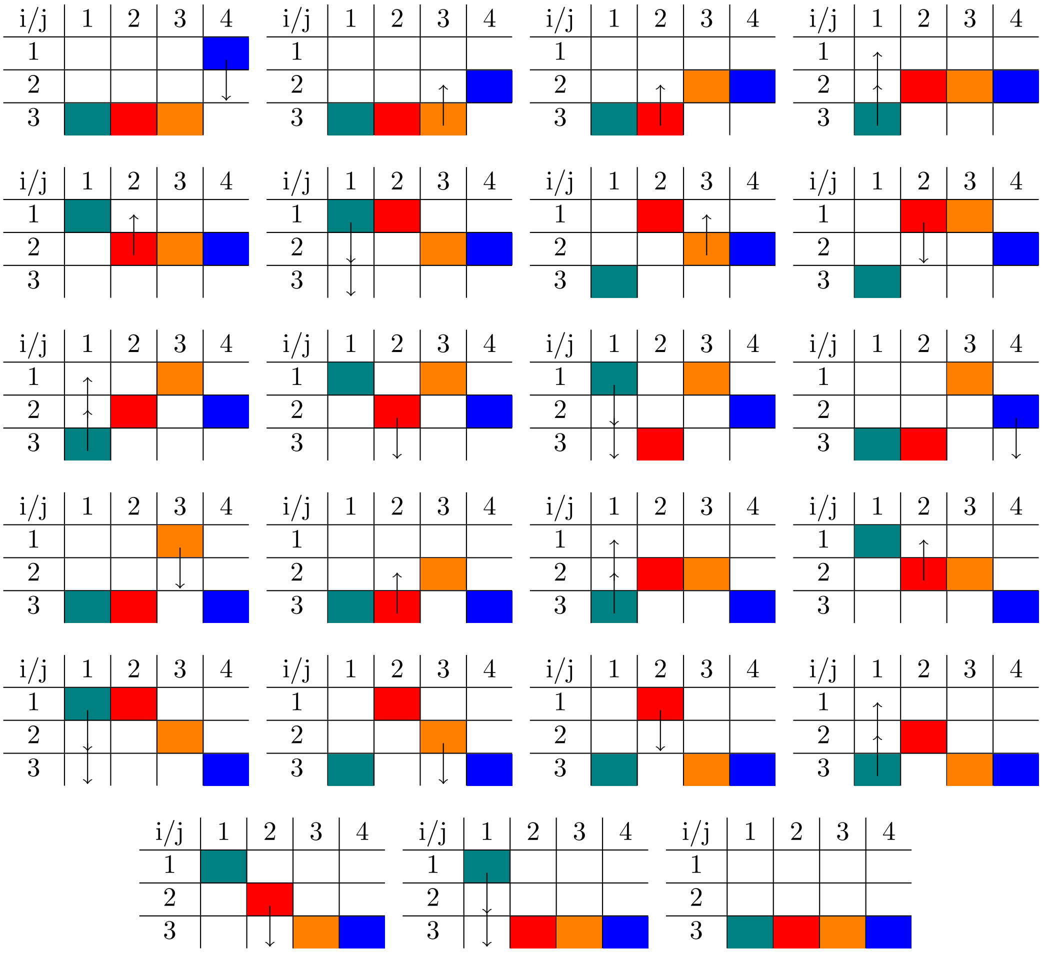


Figure 1: This example illustrates the sequence of swaps in the proof of Theorem **??** via a small example. Each one of the 4×5 grids represent committees: Each of the 12 cells represents a candidate and the 4 coloured cells represent candidates that are in the committee. The candidates in column i are $c_{i,1}, c_{i,2}, c_{i,3}$, ordered from top to bottom. We omit the dummy candidates in this depiction and let $k' = 4$ and $t + 1 = 3$, as larger t is necessary for the sequence length in the proof only. An arrow indicates the swap that will take place from the current committee to the next committee. The top left initial committee is $\{c_{1,3}, c_{2,3}, c_{3,3}, c_{4,1}\}$ and the bottom right final committee is $\{c_{1,3}, c_{2,3}, c_{3,3}, c_{4,3}\}$.

3.2 Main result

We now state our main result.

Theorem 3. *0^+-ls-PAV with better response needs $\Omega(k^{\log k})$ iterations in the worst case.*

The proof this lower bound is in many ways similar to the proof of Theorem 2. The full proof of Theorem 3 can be found in Appendix B, but we give an overview along with intuition for the proof in this section.

We will construct election E along with a committee W of size k along with a sequence of good swaps of super-polynomial length $\Omega(k^{\log k})$. As we will show, 0^+-ls-PAV executes this sequence when initialized on W. Just as in the proof of Theorem 2, this sequence of swaps will leave most members of the initial committee untouched. Most of the *action* will take place in a small minority of committee positions, namely in the first $k_1 = O(\log k)$ committee spots, while the candidates in the remaining committee spots stay in place throughout the whole sequence of swaps. Each of the first k_1 committee spots, i.e. committee spot i, $1 \leq i \leq k_1$ is assigned its own set of of candidates C^i. Intuitively, a candidate in C^i (inhabiting position i) will only ever be replaced by another candidate from C^i in our constructed sequence.

Furthermore, each of these committee positions has their corresponding voters and between different committee positions, the voters are initially very unequally satisfied; Voters in the first one are most happy and voters corresponding to later committee positions are increasingly unhappy. The sequence of swaps will again start off by making the most happy voters happier and then move on to less happy voters, making them better off. *Thereby it will undo all the work it has done so far and will have to repeat it.*
Consider Figure 1. Each individual board in Figure 1 represents a committee. The Figure should be read left to right and top to bottom, where the next board/ committee results from the previous one if certain swaps are made. More precisely, within a board each square corresponds to a candidate, and the candidates in the column numbered 1 are the candidates C_1, the candidates in column 2 are the candidates C_2 and so on. The colored squares mark the candidates that are in the committee and arrows between squares indicate swaps between candidates. The result of the swap(s) can be seen in the following board.
The reader may not yet observe any pattern in Figure 1 and should revisit it after having read the proof of Theorem 18. However, for now we use Figure 1 to illustrate how we build up our instance in several steps, creating increasingly larger building blocks. Observe that swaps only occur along a column (corresponding to candidates C_i for some i) in Figure 1, consistent with the previously mentioned property of our construction that swaps can only replace candidates in C_i by other candidates in C_i.

1. Zooming in on a single swap, the voters responsible for this swap form the atomic building block of our construction. This building block, Election $E(j,k)$ is given in Section B.1. In Lemma 14 we show that the corresponding swap increases the PAV-score by exactly

$$\delta(j,k) = \frac{j!}{\prod_{i=0}^{j}(k-i)}.$$

2. Zooming out to just the ith column, the Election responsible for the dynamics along the ith column $E^t(j,k)$ (here $t+1$ is the number of candidates in C_i). We discuss how to construct it from $E(j,k)$ in Section B.2.

3. Finally, the entire board (roughly and for a small example) corresponds to the entire election E, constructed in Section B.3, out of the building blocks $E^t(j,k)$. However j is carefully picked to depend on i where i is the corresponding column or spot in the committee.

With the constructed election E in hand, in Section B.4 we exhibit an initial committee and a sequence of positive swaps of length superpolynomial in k, so that 0^+-ls-PAV under adversarial better response executes said sequence of swaps. Finally in Section B.5 we show how to modify the instance to show that even with a fixed pivoting rule ls-PAV may make super-polynomially many swaps, by showing that a large subset of the swap sequence from Theorem 18 preserves the pivoting rule.

4 Better Response vs Best Response: Empirical Analysis

We have seen that, in the worst case, 0^+-ls-PAV with better response may require a number of iterations that is super-polynomial in the committee size. Intuitively, best response short-cuts the search, and may therefore require fewer iterations. On the other hand, best response necessarily considers $k(m-k)$ swaps in each iteration, while better response may be able to identify a sufficiently good swap after considering just a few candidate pairs. Thus, it is not clear which of these pivoting rules should be prefered in practice. In this section, we take an experimental approach and compare better and best response on various data sets available from PrefLib[1]. As a proxy for running time, we consider the number of evaluations of the quantity $\Delta(W, c, c')$ during the execution of each algorithm. For simplicity, in all our experiments, we choose a committee with the maximum number of approvals (i.e., $W \in \arg\max \sum_{i \in N} |A_i \cap W|$) as our starting point.

AAMAS 2015 / 2016 Bidding Data This pair of datasets contains the bids of reviewers over papers from the 2015 and 2016 editions of Autonomous Agents and Multi-agent Systems Conference. Inclusion in these datasets was explicitly opt-in. The 2015 dataset contains 9,817 bids of 201 reviewers on 613 papers; this represents about 40% of the actual $22,360$ bids of 281 reviewers over 670 papers. The 2016 dataset contains 161 out of 393 reviewers with bids over 442 out of 550 papers. The bidding language for these conferences consisted of four options for each paper: 'yes', 'maybe', 'no', and 'conflict'. We merge answers categories 'yes' and 'maybe' to get an approval vote, so a voter approves a paper if and only if she selected 'yes' or 'maybe'.

AI Conference Bidding Data We also consider three datasets that contain the bidding data from other computer science conferences. They contain the bids of all reviewers (apart from a small number of opt-outs) over a subset of papers at the conference. We will refer to the three individual datasets as 'CS bidding data 1','CS bidding data 2' and 'CS bidding data 3'. 'CS bidding data 1' contains 31 voters and 54 candidates. 'CS bidding data 2' contains 24 voters and 52 candidates. 'CS bidding data 3' contains 146 voters and 176 candidates. The committee size is not part of the input, so we vary it from 1 to 10 or 30, depending on the instance size.

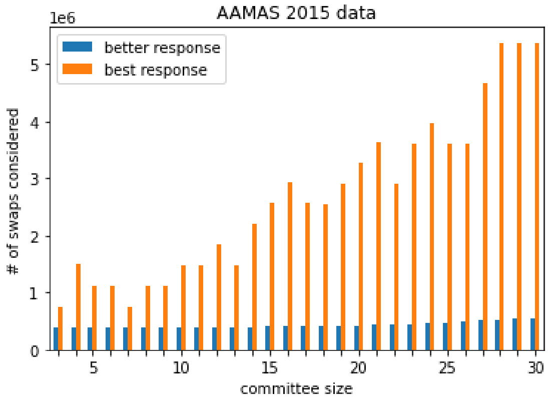
Across our experiments we observe that for larger elections, better response outperforms best response. As we see in Figures 3a, on the AAMAS 2015 data set, the number of swaps considered by the best response is up to 6 times larger than the number of swaps considered by the better response. Further, as we increase the committee size from 3 up to 30, for the better response the increase (if any) in the number of swaps considered is very slow. In contrast, for the best response the number of swaps considered increases over 5-fold in both AAMAS datasets. For the medium-size CS conferences, this effect is not as strong: though better response is consistently faster, increasing very slowly with k, for larger committee sizes there appears to be a decreasing trend for best response. However, in the range that we consider, best response is still slower.

5 Conclusion

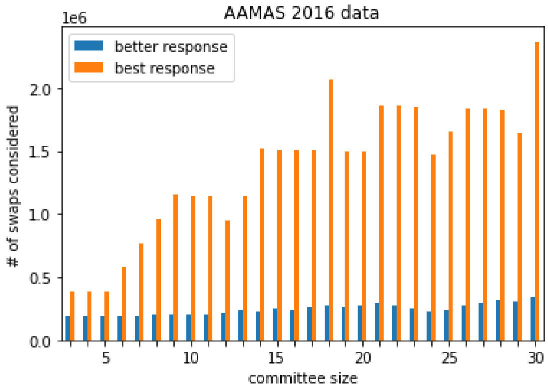
We have shown that ε-ls-PAV rules have many desirable proportionality properties, even if ε is large enough to ensure polynomial runtime. On the other hand, while smaller ε provides additional robustness, decreasing it all the way to 0^+ comes at a cost in terms of the worst-case running time.

More broadly, we believe that ε-ls-PAV is an important element of our toolbox for designing multiwinner voting rules, as it provides flexibility not shared by iterative rules. Indeed, it can adapt quickly when the input election is modified: even if the changes are too large to be captured

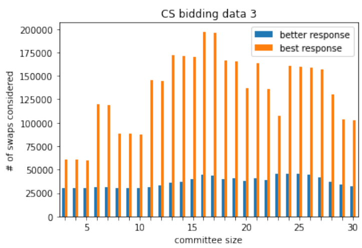
[1] https://www.preflib.org/



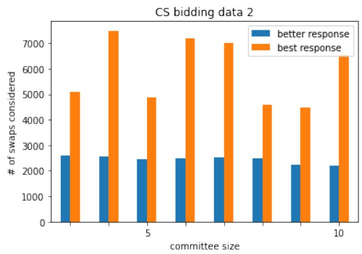
(a) Number of swaps ls-PAV considers on AAMAS 2015 bidding data



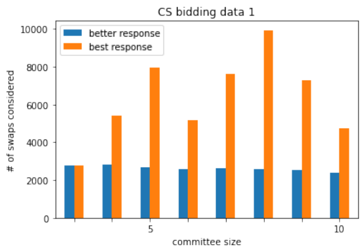
(b) Number of swaps ls-PAV considers on AAMAS 2016 bidding data



(c) Number of swaps ls-PAV considers on CS bidding data 3



(d) Number of swaps ls-PAV considers on CS bidding data 2



(e) Number of swaps ls-PAV considers on CS bidding data 1

Figure 2: 0^+-ls-PAV with best response vs. 0^+-ls-PAV with better response. On the x-axis, we vary the committee size, and on the y-axis we display the number of swaps considered.

by our robustness result, we can still re-start the local search and expect it to converge in a few iterations, whereas MES would have to be re-started from scratch. Furthermore, for large enough ε, we can expect that the local search will converge quickly in most cases, and hence the final committee will be similar to the initial committee. Hence, if we choose the initial committee using application-specific criteria, and then run ε-ls-PAV to ensure proportionality, we can hope to keep some or the original desiderata. Proving formal guarantuees of this nature is an interesting future direction. We leave open whether the run-time of 0^+-ls-PAV with best response has polynomial run-time or not.

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A Omitted proofs from Section A.1

A.1 Proportionality Guarantees

Recall that FJR is a stronger property than EJR: it guarantees representation to cohesive groups under a much weaker notion of cohesiveness. It is currently not known whether any polynomial-time computable multiwinner voting rule satisfies FJR. We will now argue that ε-ls-PAV with $\varepsilon \leq \frac{n}{k^2}$ satisfies FJR in an approximate sense.

Throughout this section, let W be a committee in the output of ε-ls-PAV for $\varepsilon \leq \frac{n}{k^2}$. We will use the following lemma by Halpern et al. [13].

Lemma 4 ([13], Lemma 3.6)**.** *For $c \notin W$, it holds that $\max_{c' \in W} \frac{\Delta(W,c,c')}{n} \geq \frac{1}{k}\left((k+1)\frac{\Delta(W,c)}{n} - 1\right)$.*

As W is an output of ε-ls-PAV with $\varepsilon \leq \frac{n}{k^2}$, for each $c \notin W$ we have $\max_{c' \in W} \Delta(W,c,c') < \frac{n}{k^2}$. Hence, Lemma 4 implies that $\frac{1}{k} > (k+1)\frac{\Delta(W,c)}{n} - 1$, or, simplifying, $\Delta(W,c) < \frac{n}{k}$.

Our next lemma has a useful interpretation. If a candidate c is popular, but is not in W, then we can 'justify' this with the fact that the supporters of c are, on average, sufficiently satisfied. This lemma (the proof, as well as all other omitted proofs, can be found in the appendix) generalizes and simplifies Theorem 1 in the work of Aziz et al. [4].

Lemma 5. *If there exists a set of voters V such that $(\cap_{i \in V} A_i) \setminus W \neq \varnothing$ and $|V| \geq \frac{sn}{k}$ for some $s > 0$ then $avs_V(W) > s - 1$.*

Proof of Lemma 5. Consider a set of voters V satisfying the statement of the lemma for some $s > 0$, and let c be some candidate in $(\cap_{i \in V} A_i) \setminus W$. We have the following sequence of (in)equalities, where the first transition follows from Lemma 4 and the third transition follows from the inequality between the arithmetic mean and the harmonic mean:

$$\frac{n}{k} > \Delta(W,c) \geq \sum_{i \in V} \frac{1}{w_i + 1} \geq \frac{|V|^2}{\sum_{i \in V}(w_i + 1)} = \frac{|V|}{avs_V(W) + 1} \geq \frac{sn}{k} \cdot \frac{1}{avs_V(W) + 1}.$$

Rearranging the terms, we obtain $avs_V(W) > s - 1$. □

Suppose there exists a set of candidates S, $|S| \leq \ell$, and an ℓ-large group of voters V such that members of V approve at least r members from S on average

Proposition 6. *Consider an ℓ-large set of voters V and a set of candidates $S \subset C$ with $|S| \leq \ell$ for some $\ell > 0$. Suppose that S and V satisfy $r + 1 > avs_V(S) \geq r$ for some integer $r \leq \ell$. Let $S' = \{c \in S : |\{i \in V : c \in A_i\}| \geq \frac{rn}{k}\}$. Then $S' \neq \varnothing$, and either $S' \subseteq W$ or there exists a voter $i \in V$ such that $|A_i \cap W| \geq r$.*

Proof. Suppose that $S' = \varnothing$, i.e., $|\{i \in V : c \in A_i\}| < \frac{rn}{k}$ for each $c \in S$. Then

$$avs_V(S) = \frac{1}{|V|}\sum_{i \in V}\sum_{c \in S} \mathbb{1}_{c \in A_i} = \sum_{c \in S}\frac{1}{|V|}\sum_{i \in V} \mathbb{1}_{c \in A_i} < \frac{|S|}{|V|} \cdot \frac{rn}{k} \leq r,$$

a contradiction (here, $\mathbb{1}_X$ takes value 1 if condition X is true and 0 otherwise).

Suppose $S' \not\subseteq W$. Consider a candidate $c \in S' \setminus W$. This candidate is approved by at least $\frac{rn}{k}$ voters in V. Applying Lemma 5 to V, we conclude $avs_V(W) > r - 1$ and hence by the pigeonhole principle $|A_i \cap W| > r - 1$ for some $i \in V$. As $|A_i \cap W|$ is an integer, we have $|A_i \cap W| \geq r$. □

Lemma 5 is also used in the proof of the following result, which shows that we can always guarantee more than $\frac{r}{2} - 1$ representatives for every (ℓ, r)-cohesive group. We can interpret this result as saying that W provides a nearly 2-approximation to FJR. Our result extends to a slightly more general definition of a cohesive group: Rather than requiring that every voter approves at least r candidates from S, all we need is that S receives at least $r \cdot |V|$ approvals from V in total.

Theorem 7. *Consider an ℓ-large set of voters V and a set of candidates $S \subset C$ with $|S| \leq \ell$ for some $\ell > 0$. Suppose that $\sum_{i \in V} |A_i \cap S| \geq r \cdot |V|$. Then $|A_i \cap W| > \frac{r}{2} - 1$ for some $i \in V$.*

Proof. Suppose $|A_i \cap W| \leq \frac{r}{2} - 1$ for all $i \in V$. In total, voters in V allocate at least $|V|r$ approvals to candidates in S. If at most $|V|(\frac{r}{2} - 1)$ of these approvals go to candidates in W, the number of approvals allocated by V to candidates in $S \setminus W$ is at least

$$|V|r - |V|\left(\frac{r}{2} - 1\right) = |V|\left(\frac{r}{2} + 1\right).$$

In particular, there is a candidate c in $S \setminus W$ that has at least

$$\frac{|V|(\frac{r}{2}+1)}{|S \setminus W|} \geq \frac{|V|\frac{r}{2}}{|S|} \geq \frac{\ell n}{k\ell} \cdot \frac{r}{2} = \frac{r}{2} \cdot \frac{n}{k}$$

supporters in V. Denote this set of voters by $V' \subset V$. By Lemma 5 it follows that $avs_{V'}(W) > \frac{r}{2} - 1$, and so by the pigeonhole principle we have $|A_i \cap W| > \frac{r}{2} - 1$ for some $i \in V$, a contradiction. ☐

We show that this bound is tight, by exhibiting an instance with a group of voters V, $|V| \geq \frac{sn}{k}$, and a set of candidates S of size $s < k$, such that every voter in V approves at least r members of S, but only $\frac{r}{2}$ (where r is even) members of the PAV-committee.

Theorem 8. *PAV (and therefore ε-ls-PAV) may output a committee in which every member of an (ℓ, r)-cohesive group of voters has at most $\frac{r}{2}$ representatives.*

Proof of Theorem 8. Let r be an even integer that we will lower bound later. Let $\ell = r^2$ and $k = \ell^2$ and in particular $k > \ell > r$. The set of voters will have size $n = 2k(k - \frac{r}{2})(\ell - \frac{r}{2})(r + 2)$. We construct an election with a set of voters $N = N_1 \cup N_2$, where $|N_1| = \frac{\ell n}{k}$ and $|N_2| = \frac{(k-\ell)n}{k}$, and a set of candidates $C = C_1 \cup C_2 \cup C_3$, where $|C_1| = \frac{r}{2}$, $|C_2| = \ell - \frac{r}{2}$, $|C_3| = k - \frac{r}{2}$.

All voters in N_1 approve of all candidates in C_1. In addition, each voter in N_1 approves $\frac{r}{2}$ candidates from C_2 do that votes are spread as evenly as possible, i.e. each candidate in C_2 receives at most $\lceil \frac{|N_1| \cdot \frac{r}{2}}{|C_2|} \rceil = \lceil \frac{s \cdot \frac{r}{2}}{\ell - \frac{r}{2}} \cdot \frac{n}{k} \rceil$ approvals. Observe in particular that N_1 is an (ℓ, r)-cohesive group since $|N_1| = \frac{\ell n}{k}$, every voter approves in N_1 and at least r voters in $C_1 \cup C_2$ and $|C_1 \cup C_2| = \ell$.

The voters in N_2 are split into $|C_3| = k - \frac{r}{2}$ disjoint groups that differ in size by at most 1, with each group approving a distinct candidate in C_3.

We claim that every committee output by PAV contains all candidates in C_1. Indeed, the marginal contribution of a candidate in C_1 to any committee not containing her is at least $|N_1| \cdot \frac{2}{r}$, while each candidate in C_3 receives at most $\lceil \frac{|N_2|}{k - \frac{r}{2}} \rceil \leq \frac{|N_2|}{k - \frac{r}{2}} + 1$ approvals and

$$|N_1| \cdot \frac{2}{r} > \frac{|N_2|}{k - \frac{r}{2}} + 1 \iff \frac{2 \cdot \ell}{r} > \frac{(k - \ell)}{k - \frac{r}{2}} + \frac{k}{n} \tag{1}$$

$$\iff 2\ell k - r\ell > rk - r\ell + \frac{r(k - \frac{r}{2})k}{n} \iff 2\ell k > rk + \frac{r(k - \frac{r}{2})k}{n} \tag{2}$$

which clearly holds since $\ell > r$ and $n = 2k(k - \frac{r}{2})(\ell - \frac{r}{2})(r + 2) > r(k - \frac{r}{2})k$.
However, we show next that PAV would choose every candidate in C_3 over any candidate in C_2

with our choice of $\ell = r^2$ and $k = r^4$. First every candidate in C_2 is approved by at most $\frac{|N_1|\cdot\frac{r}{2}}{\ell-\frac{r}{2}}+1$ voters in N_1 and each of these adds no more than $\frac{1}{\frac{r}{2}+1}$ to the PAV score. Each candidate in C_3 receives at least $\frac{|N_2|}{k-\frac{r}{2}} - 1$ approvals

$$
\begin{aligned}
&(\frac{|N_1|}{\ell-\frac{r}{2}} \cdot \frac{r}{2} + 1) \cdot \frac{1}{\frac{r}{2}+1} < \frac{|N_2|}{k-\frac{r}{2}} - 1 \\
\iff &\frac{|N_1|}{\ell-\frac{r}{2}} \cdot \frac{\frac{r}{2}}{\frac{r}{2}+1} + \frac{1}{\frac{r}{2}+1} < \frac{|N_2|}{k-\frac{r}{2}} - 1 \\
\iff &\frac{|N_1|}{\ell-\frac{r}{2}} \cdot \frac{\frac{r}{2}}{\frac{r}{2}+1} + \frac{\frac{r}{2}+2}{\frac{r}{2}+1} < \frac{|N_2|}{k-\frac{r}{2}} \\
\iff &\frac{\ell}{\ell-\frac{r}{2}} \cdot \frac{1}{\frac{2}{r}+1} + \frac{\frac{r}{2}+2}{\frac{r}{2}+1} \cdot \frac{k}{n} < \frac{k-\ell}{k-\frac{r}{2}} \\
\iff &\frac{\ell}{\ell-\frac{r}{2}} \cdot \frac{r}{r+2} + \frac{\frac{r}{2}+2}{\frac{r}{2}+1} \cdot \frac{k}{n} < \frac{k-\ell}{k-\frac{r}{2}} \\
\iff &\frac{\ell r}{\ell-\frac{r}{2}} + \frac{\frac{r}{2}+2}{\frac{r}{2}+1} \cdot \frac{k}{n} \cdot (r+2) < (r+2)\frac{k-\ell}{k-\frac{r}{2}} \\
\iff &\ell r(k-\frac{r}{2})\ell + \frac{\frac{r}{2}+2}{\frac{r}{2}+1} \cdot \frac{k(k-\frac{r}{2})(\ell-\frac{r}{2})}{n} \cdot (r+2) < (r+2)(k-\ell)(\ell-\frac{r}{2})
\end{aligned}
$$

and since $\frac{\frac{r}{2}+2}{\frac{r}{2}+1} < 2$ we have $\frac{\frac{r}{2}+2}{\frac{r}{2}+1} \cdot \frac{k(k-\frac{r}{2})(\ell-\frac{r}{2})(r+2)}{n} < 2 \cdot \frac{k(k-\frac{r}{2})(\ell-\frac{r}{2})(r+2)}{n} = 1$, it suffices to show that

$$
\begin{aligned}
&\ell r(k-\frac{r}{2})\ell + 1 < (r+2)(k-\ell)(\ell-\frac{r}{2}) \\
\iff &\ell r(k-\frac{r}{2})\ell + 1 < (rk+2k-r\ell-2\ell)(\ell-\frac{r}{2}) \\
\iff &r\ell k - \frac{\ell r^2}{2} + 1 < r\ell k + 2k\ell - \ell^2 r - 2\ell^2 - k\frac{r^2}{2} - kr + \frac{\ell r^2}{2} + \ell r \\
\iff &k\frac{r^2}{2} + kr + \ell^2 r + 2\ell^2 + 1 < 2k\ell + \ell r^2 + \ell r \\
\iff &\frac{r^6}{2} + 2r^5 + 2r^4 + 1 < 2r^6 + r^4 + r^3,
\end{aligned}
$$

which clearly holds for sufficiently large r.
This examples shows that the committee output by PAV is $W = C_1 \cup C_3$. The voters N_1 are (ℓ, r)-cohesive, but each voter in N_1 approves exactly $\frac{r}{2}$ committee members.

□

Recall the notion of the core, as defined in Section 2. Its approximate version is defined as follows (see, e.g., [15]). For $\alpha \geq 1$, we say that a committee W is in the *α-core* if there does not exist a coalition of voters $V \subset N$ and a set of candidates S such that $\frac{|S|}{k} \leq \frac{|V|}{n}$ and for each voter $i \in V$ it holds that $|A_i \cap S| > \max\{\alpha \cdot |A_i \cap W|, 1\}$. A voting rule has the *α-core property* if it always outputs a committee in the α-core. Peters and Skowron [15] show that PAV has the 2-core property. We will now show that this result extends to ε-ls-PAV with $\varepsilon \leq \frac{n}{k^2}$; our proof is more succinct than the one in the original paper.

Theorem 9. *For every $\varepsilon \leq \frac{n}{k^2}$ the voting rule ε-ls-PAV has the 2-core property.*

Proof. Let W be a committee in the output of ε-ls-PAV with $\varepsilon \leq \frac{n}{k^2}$. Suppose there exists a set of voters V and a set of candidates S with $|V| \geq |S| \cdot \frac{n}{k}$ such that for every $i \in V$ it holds that $|A_i \cap S| > 2|A_i \cap W|$, or, equivalently, $|A_i \cap S| \geq 2|A_i \cap W| + 1$. Then

$$\sum_{c \in S \setminus W} \Delta(W, c) \geq \sum_{i \in V} \frac{|A_i \cap S| - |A_i \cap W|}{|A_i \cap W| + 1} \geq \sum_{i \in V} \frac{2|A_i \cap W| + 1 + |A_i \cap W|)}{|A_i \cap W| + 1} = |V|.$$

Hence, by the pigeonhole principle, there is a candidate $c \in S \setminus W$ with $\Delta(W, c) \geq \frac{|V|}{|S \setminus W|} \geq \frac{|V|}{|S|} \geq \frac{n}{k}$. By Lemma 4, it follows that W cannot be an outcome of ε-ls-PAV with $\varepsilon \leq \frac{n}{k^2}$, a contradiction. □

A.2 Verifiability

We first note that $O(k|E|)$-time verifiability with respect to EJR (and any other property implied by the condition of Lemma 10 in this section) follows from the definition of this rule and the fact that every output of $\frac{n}{k^2}$-ls-PAV provides EJR. To get $O(|E|)$ verifiability, consider the following verification algorithm, which takes as input a committee W in the output of $\frac{n}{k^2}$-ls-PAV. In time $O(|E|)$, we compute $|A_i \cap W|$ for all $i \in V$ in one sweep: Iterating over the at most $|E|$ approval pairs (i, c) where $i \in V$ and $c \in A_i \cap W$, we add 1 to the current satisfaction score of i. Next, iterating over approval pairs (i, c') where $i \in V$ and $c' \in A_i \setminus W$, we add $\frac{1}{|A_i \cap W| + 1}$ to the current (lower bound) estimate for $\Delta(W, c')$ (where $c' \notin W$). If this estimate exceeds $\frac{n}{k}$ for any c', we reject, else accept.

By Lemma 4, for an output of $\frac{n}{k^2}$-ls-PAV, the verification algorithm never rejects. Now, suppose the verification algorithm accepts. Then W provides EJR, as implied by the following lemma of Halpern et al. [13].

Lemma 10 ([13], Lemma 3.5)**.** *If a committee W satisfies $\max_{c \in C} \Delta(W, c) < \frac{n}{k}$ then it provides EJR.*

Similarly, our verification algorithm can be used to check that a committee guarantuees (ℓ, ℓ)-cohesive group average satisfaction at least $\ell - 1$ [4]or is in the 2-core. The former case follows since it is included in the complete statement of Lemma 3.5 in [13]. The latter follows by the proof of Theorem 9.

A.3 Robustness

Proof of Theorem 1. By assumption, we have

$$|V_2| \leq \frac{s-1}{s} \frac{|V_1|}{k^2 + 1} \leq \frac{s-1}{s} \frac{|V_1|}{k^2 - 1}.$$

We can rewrite this as

$$\frac{s-1}{s} \cdot |V_1| \geq (k^2 - 1) \cdot |V_2|,$$

or, equivalently, as

$$\frac{1-s}{sk^2} \cdot |V_1| \leq \left(\frac{1}{k^2} - 1\right) \cdot |V_2|.$$

This can be further rewritten as

$$\frac{|V_1|}{sk^2} + |V_2| \leq \frac{|V_1|}{k^2} + \frac{|V_2|}{k^2}.$$

As V_1 satisfies $\Delta^*_{V_1}(W) < \frac{|V_1|}{sk^2}$, using the easy fact that $\Delta^*_V(W) \leq |V|$, we obtain

$$\Delta^*_{V_1 \sqcup V_2}(W) \leq \Delta^*_{V_1}(W) + \Delta^*_{V_2}(W) < \frac{|V_1|}{sk^2} + |V_2| \leq \frac{|V_1|}{k^2} + \frac{|V_2|}{k^2} = \frac{|V_1 \cup V_2|}{k^2}.$$

To bound $\Delta^*_{V_1 \setminus V_2}(W)$, we define $\overline{\Delta}_V(W) = \min_{c \in W, c' \notin W} sat_V(W \cup \{c'\} \setminus \{c\}) - sat_V(W)$. Using the easy fact that $-\overline{\Delta}_V(W) \leq |V|$, we obtain

$$\begin{aligned} \Delta^*_{V_1 \setminus V_2}(W) &\leq \Delta^*_{V_1}(W) - \overline{\Delta}_{V_2}(W) < \frac{|V_1|}{sk^2} + |V_2| \leq \frac{|V_1|}{sk^2} + \frac{s-1}{s}\frac{|V_1|}{k^2+1} \\ &= \frac{k^2|V_1| + |V_1| + sk^2|V_1| - k^2|V_1|}{sk^2(k^2+1)} = \frac{sk^2|V_1| + s|V_1|}{sk^2(k^2+1)} - \frac{s|V_1| - |V_1|}{sk^2(k^2+1)} \leq \frac{|V_1| - |V_2|}{k^2}. \end{aligned}$$

To complete the proof, it remains to observe that $|V_1 \setminus V_2| = |V_1| - |V_2|$. □

A.4 Candidate Monotonicity

Given a (committee) election $E = (N, C, (A_i)_{i \in N}, k)$, we write E_{i+X} to denote the election E where voter i additionally approves the candidates from X.

Definition 11 (Support monotonicity, [18])**.** *A committee election rule $\mathcal{R}$ satisfies support monotonicity if for every election instance E, $i \in N$, and candidate set $X \subset C$ it holds that*

1. *if $X \subset W$ for all $W \in \mathcal{R}(E)$ then $X \subset W'$ for all $W' \in \mathcal{R}(E_{i+X})$ and*
2. *if $X \subset W$ for some $W \in \mathcal{R}(E)$ then $X \subset W'$ for some $W' \in \mathcal{R}(E_{i+X})$.*

Rules that satisfy this axiom only for singleton sets X are said to satisfy *candidate monotonicity.* We show that if arbitrary initialization of the committee is taken, then local search PAV does not satisfy candidate monotonicity. To put it simply, a committee W may be a local maximum with respect to its PAV-score, however if an existing voter additionally approves $a \in W$, it is possible that local search PAV outputs a committee W' that does not contain a. Arguably, that means the committee containing a was bad in the first place. We note that our example holds even if best response is used. It is an interesting future direction to check if candidate monotonicity for additional voters holds.

Proposition 12. *0^+-ls-PAV does not satisfy candidate monotonicity with arbitrary initialization with best response.*

Proof. Consider an election with $k = 5$ and $C = \{a, b, c, d, e, f, g\}$. We define $S(c)$ to be the set of voters ("supporters") who approve $c \in C$ as the candidate point of view will be more helpful for the purpose of this proof. Then the approvals are defined as follows:

$$\begin{aligned} S(a) &= \{a_1, \ldots, a_4\}, i \geq 2 \\ S(f) &= \{f_1, f_2, a_1, b_1, x_1, x_2\} \\ S(z) &= \{x_1, \ldots, x_4, z_1, z_2, z_3\}, z \in \{b, c, d, e\} \\ S(g) &= \{g_1, \ldots g_4\} \end{aligned}$$

We claim that the committee $W_1 = \{a, b, c, d, e\}$ is a local optimum. We first show that no swap $(z, g), z \in W$ is good. Note that

$$sat(W_1) - sat(W_1 \setminus \{z\}) = 4 \cdot \frac{1}{4} + 3 \cdot 1 = 4$$

for $z \in \{b,c,d,e\} \subset W_1$. Similarly, $sat(W_1) - sat(W_1 \setminus \{a\}) = 4 \cdot 1 = 4$. So replacing one of these with g whose set of supporters has size 4 and is disjoint from $\cup_{c \in W} S(c)$, does not change the PAV score.
Case: Swap (b,f).
Observe that $S(b) \cap S(f) = \{x_1, x_2, b_1\}$, so subtracting these, the loss from removing b is 2.5 from x_3, x_4, b_2, b_3 in the case of b), the marginal gain from adding f is 2.5, 2 from f_1,f_2, 0.5 from a_1. So $\Delta(W_1, b, f) = 0$ and (b,f) does not increase the PAV score.
Case: Swap (z,f), where $z \in \{c,d,e\}$
Note that $S(z) \cap S(f)$, so subtracting these, the loss from removing x would be 3.5 from x_3, x_4, z_1, z_2, z_3, the gain from adding f is 3, 2 from f_1,f_2, 0.5 from a_1 and 0.5 from b_1. So $\Delta(W_1, x, f) = -0.5$ and hence (z,f) is not a good swap.
Case swap (a,f):
Since $S(a) \cap S(f) = \{a_1\}$, the loss from removing a is 3, and the gain from adding f is 3, 2 from f_1, f_2 and 0.5 from x_1, x_2 and 0.5 from b_1. So $\Delta(W_1, z, f) = 0$ and also (z,f) is not a good swap. We conclude that W_1 is a local maximum.

Now suppose a is additionally supported by x_3 so $S(a) = \{a_1, \dots, a_4, x_3\}$. Again consider the initial committee W_1 as above. We claim that $(b,f),(a,g)$ is a sequence of good best response swaps. Note that neither f nor g is approved by x_3, their contribution to each swap score calculated above does not change. The marginal loss suffered from removing $z \in \{b,c,d,e\}$ changes by $-\frac{1}{5} + \frac{1}{4}$ as $sat_{x_3}(W_1) = 5$ under the modified instance. Hence the previously calculated scores all increase by $\frac{1}{20}$ and in particular (b,f) is a (non-unique) best-response swap.

The new committee is $W_2 = \{a,f,c,d,e\}$. Due to the presence of f, the loss from removing a decreases since f is also supported by a_1: In particular $\Delta(W_1, a, g) = 0 + 1 - 0.5 - \frac{1}{4} = \frac{1}{4}$ since the contribution of a_1 changes from, -1 to -0.5 while x_3 adds $-\frac{1}{4}$, so overall a loss of $\frac{1}{4}$. Mean while $g's$ marginal gain remains 4, making (a,g) a good swap. To argue that it is a best response, not that the removal of b has increased the contribution of c,d,e to the score of the committee W_2 compared to W, so since the supports are disjoint $\Delta(z,g) < 0$ for $z \in \{c,d,e\}$. Similarly $\Delta(b,z) < 0$ since b and z are clones apart from the fact that b_1 also approves $f \in W_2$. Finally, $\Delta(W_2, a, b) = -3 - \frac{1}{2} + \frac{1}{4} + 2 \cdot \frac{1}{5} + 2 + \frac{1}{2} = -1 + \frac{13}{20} < 0$. So now (a,g) is the unique best response to W_2.
We claim that the committee $W_3 = \{c,d,e,f,g\}$ is a local maximum. Clearly (g,a) is not a good swap. Since $\Delta(W_1, z, g) \leq 0$, also $\Delta(W_2, z, a) \leq 0$, $z \in W_1 \setminus \{a\}$, as otherwise $(a,g)(z,a)$ is a sequence of good swaps making (z,g) a good swap. Since $\Delta(W_1, a, b) < 0$ we can conclude that $\Delta(W_2, g, b) < 0$, as otherwise $(a,g)(g,b)$ was a good sequence of swaps for W_1 and therefore (a,b) was a good swap. Furthermore, $\Delta(W_1, z, b) = \Delta(W_2, z, b) \leq 0$ for $z \in \{c,d,e\}$ since the addition of g does not affect this score due to disjoint supports, while the removal of a does not either as $S(a) \cap S(z) = S(a) \cap S(b) = \{x_3\}$. Finally, $\Delta(f,b) = -3(f_1, f_2, a_1) + 2(b_2, b_3) + 2 \cdot \frac{1}{4}(x_2, x_4) = -0.5 < 0$. So we conclude that W_3 is indeed a local maximum for the modified instance and it does not contain a.

□

B Proof of Theorem 3

Having given an informal overview of the proof of Theorem 3 in Section 3, we now move on to its formal justification. We will build up our instance in several steps, creating increasingly larger building blocks. We now introduce a family of elections which forms the smallest building block of our instance.

B.1 Election $E(j,k)$

We now introduce a family of elections which forms the smallest building block of our instance. For committee size $k \in \mathbb{N}$, we will inductively (on j) construct an election $F(j,k) = (N, C, (A_i)_{i\in N}, k)$ with $|N| = 2^j, j < k$ and $C = D_{k-1} \cup \{a, b\}$. We will frequently say an election E has structure X if it is isomorphic to (Election) X. We first describe how to construct $F(j,k)$ iteratively.

Construction 1 [Election $F(j,k)$] Let $k \in \mathbb{N}$ satisfying $k \geq j+1$. So in particular, since $j \geq 1$, also $k \geq 2$. For $j = 1$, let $F(j,k) = ([2], C, (A_i)_{i\in N}, k)$ where $C = D_{k-1} \cup \{a, b\}$, $A_1 = D_{k-1} \cup \{a\}$ and $A_2 = D_{k-2} \cup \{b\}$.
For $j > 1$ (and $k \geq j+1$), $F(j,k)$ is constructed as follows. Consider Elections $F(j-1,k) = (N_1, C_1, (A_i)_{i\in N_1}, k)$ and $F(j-1,k-1) = (N_2, C_2, (A_i)_{i\in N_2}, k-1)$, where $N_1 \cap N_2 = \emptyset$ (so we relabel the voters to be distinct) and where $C_1 = D_{k-1} \cup \{a_1, b_1\}$ and $C_2 = D_{k-2} \cup \{a_2, b_2\}$. Furthermore, we identify $a_1 = b_2$ and $b_1 = a_2$, which means that $C_1 = C_2 \cup \{d_{k-1}\}$. Setting $a := a_2$ and $b := b_2$, we obtain $F(j,k) = (N, C, (A_i)_{i\in N}, k)$, where $N = N_1 \cup N_2$ and $C = C_1$.

Observe that the number of voters in $F(j,k)$ (for any $k \geq 1$) is exactly 2^j; this follows easily from induction since the set of voters in election $F(1,k)$ has size 2, and for $j > 1$, elections $F(j-1,k)$ and $F(j-1,k-1)$ have disjoint sets of voters each of size 2^{j-1}. Furthermore, observe that $F(j-1,k)$ can be obtained from $F(j-1,k-1)$ by an approval from voter v to candidate d_l if previously $l-1$ was the maximum such that v approved $d_{\ell-1}$. This gives a natural bijection between voters in $F(j-1,k-1)$ and voters in $F(j-1,k)$.

Construction 2 [Election $E(j,k)$] Election $E(j,k)$ is of the same spirit as $F(j,k)$; We introduce two new candidates x and y, and add approvals from $v \in N_1$ to x and $v \in N_2$ to y, and further we delete an approval from each voter in $v \in N_1$ to the dummy candidate with the largest index in A_v.
Formally, consider $F(j-1,k-1) = (N_1, C_1, (A_i)_{i\in N_1}, k)$ (this is in contrast to $F(j-1,k)$ in the previous construction) and $F(j-1,k-1) = (N_2, C_2, (A_i)_{i\in N_2}, k)$ and define $(N, C, (A_i')_{i\in N_1\cup N_2}, k)$ where $N = N_1 \cup N_2$ and $C = D_{k-2} \cup \{x, y\} \cup \{a, b\}$, again identifying $a_1 = b_2$ and $b_1 = a_2$ ($a_1, b_2 \in C_1$ and $a_2, b_2 \in C_2$). Now $A_v' = A_v \cup \{x\}$ if $v \in N_1$ and $A_v' = A_v \cup \{y\}$ if $v \in N_2$. Let $s : N_1 \mapsto N_2$ be the natural bijection (between the two subelections $E(j-1,k-1)$ and $E(j-1,k)$ that $E(j,k)$ consists of). We make the following two simple observations.

Proposition 13. *Let $k \geq j+2 \geq 3$. In $E(j,k)$, the following hold*

1. $A_v \cap D = A_{s(v)} \cap D = D_j$, $1 < j \leq k-1$.
2. *v approves $a \in C$ ($b \in C$ resp.) if and only if $s(v)$ approves $b \in C$ ($a \in C$ resp.).*

Consider the committee $W = D_{k-2} \cup \{x\} \cup \{a\}$. With respect to committee W, the election $E(j,k)$ satisfies two important properties that we will need in the proof of Theorem **??** and we state them in the following lemma. To make the lemma easier to use, we define $\delta : \mathbb{N} \times \mathbb{N} \mapsto \mathbb{R}$ by

$$\delta(j,k) = \frac{j!}{\prod_{i=0}^{j}(k-i)} = \sum_{i=0}^{j} \frac{(-1)^{i+1}}{k-i}\binom{j}{i}$$

Lemma 14. *Let $j < k$. Election $E(j,k)$ satisfies that*

1. $\Delta(W, a, b) = \delta(j, k)$*, and*

2. for every voter in $v \in N$ *has* $sat_v(W) \geq k - (j+1)$*.*

Proof. To prove the claim for $E(j, k)$, it suffices to prove the claim for $F(j, k)$ with $W = D_{k-1} \cup \{a\}$ as d_{k-1} is a clone (approved by the same voters) of x which it replaces while y does not affect $\Delta(W, a, b)$ since $y \notin W$. We prove both claims by induction on j.

Base case: For $j = 1$, $sat_1(W) = k$ and $sat_2(W) = k - 2$, since voter 1 approves all members of W, while voter 2 approves everyone but d_{k-1} and a. So (2) holds. For (1) observe that

$$\Delta(W, a, b) = +\frac{1}{k-1} - \frac{1}{k} = \frac{1}{k(k-1)} = \delta(1, k).$$

Inductive Step: Let $j \geq 1$ and suppose the statements of the Lemma holds for j and arbitrary k. Consider elections $F(j, k)$ and $F(j, k-1)$ with $F(j, k) = (N_1, C_1, (A_i)_{i \in N_1}, k)$ and $F(j, k-1) = (N_2, C_2, (A_i)_{i \in N_2}, k-1)$. By construction, $N_1 \cap N_2 = \emptyset$. Let $W_1 = D_{k-1} \cup \{a_1\}$ and $W_2 = D_{k-2} \cup \{a_2\}$. Applying the inductive hypothesis to $F(j, k)$ with committee W_1 and to $F(j, k-1)$ with committee W_2, we see that

$$\Delta_{N_1}(W_1, a_1, b_1) = \delta(j, k) = \frac{j!}{\prod_{i=0}^{j}(k-i)} \text{ and}$$
$$\Delta_{N_2}(W_2, a_2, b_2) = \delta(j, k-1) = \frac{j!}{\prod_{i=0}^{j}(k-1-i)}.$$

As in the construction of $F(j+1, k)$ in Section B.1 we can obtain election $F(j+1, k) = (N, C, (A_i)_{i \in N_1 \cup N_2}, k)$ with $= N_1 \cup N_2$ and $a = a_2 = b_1$ and $b = b_2 = a_1$ from elections $F(j, k)$ and $F(j, k-1)$. We observe that for $W = D \cup \{a\}$.

$$\Delta_{N_1}(W, a, b) = \Delta_{N_1}(W, b_1, a_1) = -\Delta_{N_1}(W_1, a_1, b_1) = -\frac{j!}{\prod_{i=0}^{j}(k-i)} \text{ and} \quad (3)$$

$$\Delta_{N_2}(W, a, b) = \Delta_{N_2}(W_2, a_2, b_2) = \frac{j!}{\prod_{i=0}^{j}(k-1-i)} \quad (4)$$

We conclude that in election $F(j+1, k)$ for committee W,

$$\begin{aligned} \Delta_N(W, a, b) &= \Delta_{N_1}(W, a, b) + \Delta_{N_2}(W, a, b) = \\ &- \frac{j!}{\prod_{i=0}^{j}(k-i)} + \frac{j!}{\prod_{i=0}^{j}(k-1-i)} = \frac{(k-(k-1-j))j!}{\prod_{i=0}^{j+1}(k-i)} \\ &= \frac{(j+1)!}{\prod_{i=0}^{j+1}(k-i)} = \delta(j+1, k), \end{aligned}$$

using Equalities 3 and 4. This concludes the proof of (1). To see (2), note that by the inductive hypothesis in election $F(j, k)$, voters $v \in N_1$ satisfy $sat_v(W_1) \geq k - (j+1)$. To obtain W_1 from W, we replace a with b and so we can conclude that in the worst case $v \in N_1$ approves one less member of W than of W_1 and so $sat_v(W) \geq k - 1 - (j+1) = k - (j+2)$. Voters $v \in N_2$ in election $F(j, k-1)$ satisfy $sat_v(W_2) \geq k - 1 - (j+1)$ by the inductive hypothesis and since $W_2 \subset W$, $sat_v(W) \geq k - 1 - (j+1) = k - (j+2)$. We conclude that therefore in election $F(j+1, k)$, every $v \in N$ has satisfaction $sat_v(W) \geq k - (j+2)$, as desired.

□

Observe that by construction of the election $E(j,k)$ exactly half of the voters approve a (and not b) and half of the voters approve b and not a.

Corollary 15. *Consider $W = D_{k-2} \cup \{x, a\}$. Then $\Delta(W, a, b) = \delta(j,k)$ and (a,b) is a good swap. If $W = D_{k-1} \cup \{y, b\}$, then $\Delta(W, b, a) = \delta(j,k)$ and (b,a) is a good swap.*

B.2 Level up: Election $E^t(j,k)$

Construction 3 [Election $E^t(j,k)$]. Let $t \in \mathbb{N}$. Consider for each i, $1 \leq i \leq t$, $E_i = (N_i, C_i, (A_l)_{l \in N_i}, k)$ where E_i has structure $E(j,k)$ with $C_i = D_{k-2} \cup \{a_i, b_i\} \cup \{x_i, y_i\}$. For each i, $1 \leq i < t$, we identify $b_i = a_{i+1}$, so that $\cup_{i=1}^t C_i = \{a_1, \dots, a_t, b_t\}$. Furthermore, we identify x_i and y_{i+1} as well as y_i and x_{i+1}, where $i < t$ and we write $x = x_1 = y_2 = \dots = y_{t-1} = x_t$ and $y = y_1 = x_2 = \dots = x_{t-1} = y_t$. We relabel $a_1, \dots, a_t, b_t$ as $c_1, c_1 \dots c_{t+1}$ and write $C = \{c_1, c_2 \dots, c_{t+1}, x, y\}$. For each $v \in N_i$ we define

$$A'_v = A_v \cup \{c_{i+1}, \dots, c_{t+1} \mid c_l \in A_v, l > i\} \cup \{c_1, \dots, c_i \mid c_l \in A(v), l \leq i\}.$$

This means that from the point of view of a voter $v \in N_i$, the candidates $c_1, \dots c_i$ are *clones*: she either approves all or none of them, and similarly $c_{i+1}, \dots, c_{t+1}$ are clones for v. Finally, define $E^t(j,k) = (N, D_{k-2} \cup C, (A'_l)_{l \in N}, k)$ where $N = \cup_{i=1}^t N_i$. Since each $E(j,k)$ is an election with 2^j voters we observe the following.

Proposition 16. *The number of voters in $E^t(j,k)$ is $t2^j$.*

Consider an election E with structure $E^t(j,k)$ and voters $\cup_{i=1}^t N_i$ as in the above construction. Since in election E_i with structure $E(j,k)$, each voter v in N_i approves exactly one out of a_i and b_i, in $E^t(j,k)$, v also approves exactly one of c_i and c_{i+1}. So by the above construction in $F^t(j,k)$, voter v either approves exactly $c_1, \dots, c_i$ out of C or she approves exactly $c_{i+1}, \dots, c_{t+1}$ in C. In particular, in election $E^t(j,k)$ a swap (c_i, c_{i+1}) or (c_{i+1}, c_i) only changes the satisfaction of voters in $N_i \subset N$.
We can therefore make some useful observations on good swaps with respect to the committee $W = \{d_1, \dots, d_{k-2}, z, c_i\}$ where $z \in \{x, y\}$ in election $E^t(j,k)$.

Proposition 17.
1. *If $W_1 = D_{k-2} \cup \{x, c_1\}$, then $(c_1, c_2), (c_2, c_3), \dots, (c_t, c_{t+1})$ is a sequence of good swaps, increasing the PAV-score by $\sum_{i=1}^t \Delta(W_1, c_i, c_{i+1}) = t\delta(j,k)$.*
2. *If $W_2 = D_{k-2} \cup \{y, c_{t+1}\}$, then $(c_{t+1}, c_t), (c_t, c_{t-1}), \dots, (c_2, c_1)$ is a sequence of good swap, increasing the PAV-score by $\sum_{i=1}^t \Delta(W_2, c_i, c_{i+1}) = t\delta(j,k)$.*
3. *If $W_3 = D_{k-2} \cup \{x, c_{t+1}\}$ then $\Delta(W_3, x, y) = -t\delta(j,k)$.*
4. *If $W_4 = D_{k-2} \cup \{y, c_1\}$, then $\Delta(W_4, y, x) = -t\delta(j,k)$.*

Proof. Item 1 and 2 follow from by applying Corollary 15 for each $i \in [t]$: We previously argued that the swaps (c_i, c_{i+1}) or (c_{i+1}, c_i) only change the satisfaction of voters $N_i \subset N$. Furthermore $c_j, j \neq i, i+1$ are neither in the committee at any point of the sequence, so we may ignore them: But restricting $E^t(j,k)$ to N_i and candidates $\{d_1, \dots, d_{k-1}, x, y, c_{i+1}, x_{i+1}\}$ recovers an election isomorphic to $E(j,k)$ so that we can apply Corollary 15 t times to see that $\sum_{i=1}^t \Delta(W_1, c_i, c_{i+1}) = t\delta(j,k)$ and $\sum_{i=1}^t \Delta(W_2, c_i, c_{i+1}) = t\delta(j,k)$. For items 3 and 4, note that by symmetry $\Delta(W_3, x, y) = \Delta(W_4, y, x)$ and furthermore

$$\sum_{i=1}^t \Delta(W_1, c_i, c_{i+1}) + \Delta(W_3, x, y) + \sum_{i=1}^t \Delta(W_2, c_i, c_{i+1}) + \Delta(W_4, y, x) = 0,$$

(as by executing this sequence of swaps we end up where we started, namely with committee W_1, implying that $\Delta(W_4, y, x) = \Delta(W_3, x, y) = -t\delta(j,k)$ ☐

B.3 Final Election Instance E

Let k be the desired committee size. Let $k_1 = \lfloor \log k \rfloor$ and $k_2 = k - k_1$. We will construct an election instance $E = (N, C, (A_i)_{i \in N}, k)$ together with an initial committee W_0 such that if we run 0-ls-PAV on E with initial committee W_0, it makes a super-polynomial in k number of swaps.

Constructing the Instance/Election We will now show how to construct the election E and committee W_0. Informally, Election E will be the result of combining k_1 elections of the form $E^t(j, k_2)$ with some modification of the approvals. More formally, let $t \geq k$ be even and as small as possible and for each i, $1 \leq i \leq k_1$ consider

$$E_i = (N_i, C_i \cup D_{k_2}, (A_i)_{l \in N_i}, k_2 + 2) \text{ with structure } E^t(2k_1 - 2(i-1), k_2 + 2) \quad (5)$$

$$\text{where } N_i = N_{i,1} \sqcup N_{i,2} \text{ (partition into sibling sets)} \quad (6)$$

$$\text{and } C_i = \{c_{i,1}, \ldots, c_{i,t+1}, x_i, y_i\}. \quad (7)$$

Note that the committee size above is $k_2 + 2$ so that the corresponding committee as discussed in the previous section can contain D_{k_2} as well as exactly one of x_i, y_i and exactly one of $c_{i,1}, \ldots, c_{i,t+1}$.

We define E by merging these elections in the natural way, with the caveat that we additionally modify some approvals: Furthermore we remove all candidates x_i, y_i and, intuitively, instead let the candidates in $C_{i+1} \setminus \{x_{i+1}, y_{i+1}\}$ take on the roles of x_i and y_i for voters N_i. Let $i < k_1$. In election E voters in N_i who approve x_i in E_i instead approve of all $c_{i+1,j}$ where j is odd. Similarly, in election E voters in N_2 approve of $c_{i+1,j}$ whenever j is even if they approve y_i in E_i.

Formally, let

$$E = (N, C, (A'_i)_{i \in N}, k_1 + k_2) \text{ where } N = \cup_{i=1}^{k_1} N_i,\ C = D_{k_2} \cup \cup_{i=1}^{k_1} C_i \text{ and for } v \in N_i:$$
$$A'_v = A_v \cup \{c_{i+1,j} \mid j \text{ is odd }, x_i \in A_v\} \cup \{c_{i+1,j} \mid j \text{ is even}, y_i \in A_v\} \setminus \{y_i, x_i\}$$

We justify why the election E has size polynomial in k. From Proposition 16, it follows that $|N_i| \leq t 2^{\lfloor \log k \rfloor} \leq k(k+1)$ and so $|N| = \sum_{i=1}^{k_1} |N_i| = O(poly(k))$. Furthermore, the number of alternatives is $|C| = k - 1 + k_1 \cdot t \leq 2k \log k$.

B.4 Adversarial Better Response

We prove a lower bound of $\Omega(k^{\log(k)})$ on the maximum length of a path in the directed graph whose vertices are committees and a directed edge (W, W') exists whenever W' can be obtained from W via a swap and W' has higher PAV score than W. We call it adversarial because these are the swaps that an agent that points out improvements of the existing state but acts adversarially might choose to show us.

Notation for intermediate Committees In the following we consider committees as ordered k-tuples, as it will be useful to number the committee positions. We will only consider committees of the form

$$(a_1, a_2, \ldots, a_{k_1}, d_1, d_2, \ldots, d_{k_2})$$

where $a_i \in C_i \setminus D_{k_2}$, $1 \leq i \leq k_1$, so in particular every committee in the following will be of this form. We will refer to a_i as the C_i-candidate of the committee, i.e. the candidate in committee position i, and say that the C_i-candidate has odd (resp. even) index if i is odd (resp. even). Our super-polynomial swap sequence will exclusively contain swaps of the form (a, b) where

$a, b \in C_i \setminus D_{k_2}$ for some $1 \leq i \leq k_1$. Note the committee that results from this swap sequence may not be a valid stable point of 0^+-ls-PAV, but then we may simply append any sequence of good swaps that does result in an output of 0^+-ls-PAV.

Given a committee W of the form above, we will say that voters N_i, where $i < k_1$, are *stable* if

1. W contains $c_{i,1}$ and $c_{i+1,j}$ where j is even or
2. W contains $c_{i,t+1}$ and $c_{i+1,j}$ and j is odd.

The justification for this definition is that as a consequence of Proposition 17 in this case we have that no swap (a, b) with $a, b \in C_i$ is a good swap. For N_{k_1} we will say it is stable only if candidate $c_{k_1,t}$ is on the committee. We no define the initial committee for out swap sequence as

$$W_0 = (c_{1,t+1}, c_{2,t+1}, \ldots, c_{k_1-1,t+1}, c_{k_1,1}, d_1, \ldots d_{k_2})$$

so $|W_0| = k_1 + k_2$. Note that in W_0, all of the voter groups N_i, $i < k_1$ are stable and N_{k_1} is unstable. We will exhibit a sequence of good swaps which results in the final committee

$$(c_{1,t+1}, c_{2,t+1}, \ldots, c_{k_1,t+1}, d_1, \ldots, d_{k_2})$$

where all voters groups N_i are stable since $t + 1$ is odd. We will call N_{i-1} the *predecessor* group of N_i. We will say a swap sequence destabilises N_i if prior to the execution it was stable and upon the execution N_i was no longer stable.

Sequence of super-polynomial length We define the sequence of swaps

$$\mathbf{Y}_i = \oplus_{j=1}^{t}(c_{i,j}, c_{i,j+1}) \textit{ for } 1 \leq i \leq k_1.$$

Let $\mathbf{X}_1^1 = \mathbf{Y}_1$ and $\mathbf{X}_1^0 = \mathbf{Y}_1^{-1}$

$$\mathbf{X}_i^0 = \oplus_{j=1}^{t}(c_{i,t-j+2}, c_{i,t-j+1})\mathbf{X}_{i-1}^{j-1 \mod 2} \text{ and }.$$
$$\mathbf{X}_i^1 = \oplus_{j=1}^{t}(c_{i,j}, c_{i,j+1})\mathbf{X}_{i-1}^{j-1 \mod 2} \text{ for } i > 1$$

Our goal in this proof is to show that 0-ls-PAV will perform the sequence of swaps $\mathbf{X}_{k_1}$ when run on election E with initial committee W_0. That is, $\mathbf{X}_{k_1}$ is a sequence of good swaps. Once we have shown that, since the length of $|\mathbf{X}_i^l|$, $l \in \{0, 1\}$ is $t(|\mathbf{X}_{i-1}^l| + 1)$ and $|\mathbf{X}_1^l| = t \geq k$ this implies that $\mathbf{X}_{k_1}^1$ has length $\Omega(t^{k_1}) = \Omega(k^{\log k})$, giving us the desired sequence of good swaps with super-polynomial length in k.

Theorem 18. *0^+-ls-PAV with adversarial better response needs $\Omega(k^{\log k})$ iterations in the worst case.*

Proof. To show that the sequence $\mathbf{X}_{k_1}^1$ is a sequence of good swaps, we use induction $i < k$ to show the following. Suppose the groups of voters $N_1, \ldots N_{i-1}$ are stable and N_i is the group of voters with smallest index that is not stable (so if $i = 1$, simply N_1 is not stable). We want to prove inductively, that in this case one of the swap sequences $\mathbf{X}_i^1$ and $\mathbf{X}_i^0$ is good.

To prove the claim for i we split into two cases: Consider committee W of the form

$$W = (c_{1,t+1}, \ldots c_{i-1,t+1}, c_{i,j}, c_{i+1,j'}, \ldots a_{k_1-1}, a_{k_1}, d_1, \ldots, d_{k_2})$$

where $j = 1$ and j' is odd or $j = t + 1$ and j' is even. As before $a_l \in C_l \setminus D_{k_2}$, $k_1 \geq l > i + 1$. We show that in the former case, i.e. if the C_{i+1}-candidate in W has odd index, then $\mathbf{X}_i^1$ is a sequence of good swaps and in the latter case, i.e. if the C_{i+1}-candidate in W has even index, then $\mathbf{X}_i^0$ is a sequence of good swaps.

We prove the claim inductively on i. Suppose N_1 is not stable. If $c_{1,1} \in W$, then since only voters in C_1 approve candidates in C_1, applying Proposition 17 the swap sequence

$$\mathbf{X}_1^1 = (c_{1,1}, c_{1,2}), \ldots, (c_{1,t-1}, c_{1,t+1})$$

is a sequence of good swaps, increasing the PAV score by $t\delta(\lceil \log(k) \rceil, k)$. If $c_{1,t} \in W$, then by stability also $c_{2,j} \in W$ where j is even. In this case, also by Proposition 17,

$$\mathbf{X}_1^0 = (c_{1,t+1}, c_{1,t-1}), \ldots, (c_{1,2}, c_{1,1})$$

is a sequence of good swaps.

Now suppose $i > 1$ and N_i is unstable and the voters $N_1, \ldots N_{i-1}$ are stable. We will consider two cases, and in both we show that there is a good swap involving the C_i candidate that destabilises the group N_{i-1}, allowing us to apply the inductive hypothesis.
Consider first the case where

$$W = (c_{1,t+1}, \ldots c_{i-1,t+1}, c_{i,j}, c_{i+1,j'}, \ldots, , a_{k_1}, d_1, \ldots, d_{k_2})$$

where $j \leq t$ and both j and j' are odd. We prove that $c_{i,j}, c_{i,j+1}$ is a good swap, i.e. $\Delta(W, c_{i,j}, c_{i,j+1}) > 0$. From Lemma 14, it follows that $\Delta_{N_i}(W, c_{i,j}, c_{i,j+1}) = \delta(\lfloor \log(k) \rfloor - 2(i-1), k)$. Furthermore, by construction of E, none of the groups N_j, $j \notin \{i, i-1\}$ approve either of $c_{i,j}, c_{i,j+1}$ so that $\Delta_{N_j}(W, c_{i,j}, c_{i,j+1}) = 0$. So showing that $(c_{i,j}, c_{i,j+1})$ is a good swap therefore boils down to showing that

$$\begin{aligned} &\Delta_N(W, c_{i,j}, c_{i,j+1}) = \Delta_{N_{i-1} \sqcup N_i}(W, c_{i,j}, c_{i,j+1}) \\ &= \delta(2\lfloor \log(k) \rfloor - 2(i-1)), k_2 + 2) - t\delta(2\lfloor \log(k) \rfloor - 2(i-2)), k_2 + 2) > 0. \end{aligned}$$

We do this in the following lemma.

Lemma 19. $\delta(\lfloor \log(k) \rfloor - 2(i-1)), k_2 + 2) > t\delta(\lfloor \log(k) \rfloor - 2(i-2)), k_2 + 2)$

Proof. By construction $2(i-1) \leq 2\lceil \log(k) \rceil$. Also, $t \leq k+1$ and for large enough k, we see that

$$t \frac{(2\lfloor \log k \rfloor - 2(i-2))!}{\prod_{i=0}^{2\lfloor \log k \rfloor - 2(i-2)} k_2 + 2 - i} < \frac{(2\lfloor \log k \rfloor - 2(i-1))!}{\prod_{i=0}^{2\lfloor \log k \rfloor - 2(i-1)} k_2 + 2 - i} \tag{8}$$

$$\iff t(2\lfloor \log k \rfloor - 2(i-2))(2\lfloor \log k \rfloor - 2i + 1) < (k_2 + 2 - 2\lfloor \log k \rfloor + 2(i-2))(k_2 + 2 - 2\lfloor \log k \rfloor + 2i - 3), \tag{9}$$

so since the leading term on the LHS is $O(k \log^2 k)$ as $t \leq k+1$ and the leading term on the RHS is $\Omega(k^2)$, the inequality holds for sufficiently large k. □

So $(c_{i,j}, c_{i,j+1})$ is a good swap and results in the committee

$$W = (c_{1,t}, \ldots c_{i-1,t}, c_{i,j+1}, c_{i+1,j'}, \ldots a_{k_1-1}, a_{k_1}, d_1, \ldots, d_{k_2})$$

which is not stable for N_{i-1} as the C_i candidate has even index $j+1$. By the inductive hypothesis $\mathbf{X}_{i-1}^0$ is a sequence of good swaps and the results in the committee

$$(c_{1,t}, \ldots, c_{i-2,t}, c_{i-1,1}, c_{i,j+1}, c_{i+1,j'}, \ldots a_{k_1-1}, a_{k_1}, d_1, \ldots, d_{k_2}).$$

Now suppose the current committee is

$$W = (c_{1,t}, \dots, c_{i-2,t}, c_{i-1,1}, c_{i,j}, c_{i+1,j'}, \dots a_{k_1-1}, a_{k_1}, d_1, \dots, d_{k_2})$$

where $j \leq t$ is even and j' is odd. The argument that $\Delta_N(W, c_{i,j}, c_{i,j+1}) > 0$ is the same as before; $\Delta_N(W, c_{i,j}, c_{i,j+1}) = \Delta_{N_{i-1} \cup N_i}(W, c_{i,j}, c_{i,j+1})$
$= \delta(2\lfloor \log(k) \rfloor - 2(i-1)), k_2 + 2) - t\delta(\log(k) - 2(i-2)), k) > 0$ by Lemma 19. So we move to the committee

$$W = (c_{1,t}, \dots, c_{i-2,t} c_{i-1,1}, c_{i,j+1}, c_{i+1,j'}, \dots a_{k_1-1}, a_{k_1}, d_1, \dots, d_{k_2}).$$

However in this case $j+1$ is odd and so by the inductive hypothesis $\mathbf{X}^1_{i-1}$ is a sequence of good swaps and resulting in the committee

$$(c_{1,t}, \dots c_{i-1,t}, c_{i,j+1}, c_{i+1,j'}, \dots a_{k_1-1}, a_{k_1}, d_1, \dots, d_{k_2}).$$

Note that in the resulting committees N_i is unstable, and we can keep repeating our argument until the C_i candidate has index t. This proves that if $j = 1$ (and j' is odd), then the sequence of swaps

$$\mathbf{X}^1_i = \oplus_{j=1}^{t-1} (c_{i,j}, c_{i,j+1}) \mathbf{X}_{i-1}^{j-1 \mod 2}$$

is a sequence of good swaps resulting in the committee

$$W = (c_{1,t}, c_{2,t}, \dots, c_{i,t}, c_{i+1,j'}, \dots, d_1, \dots, d_{k-1}).$$

The case that j' is even can be shown similarly. In particular, consider the committee

$$W = (c_{1,t}, \dots c_{i-1,t}, c_{i,j}, c_{i+1,j'}, \dots, , a_{k_1}, d_1, \dots, d_{k_2})$$

where $j > 0$ is odd and j' is even. Again, the swap $(c_{i,j}, c_{i,j-1})$ is a good swap and destabilises the voters N_{i-1}: Since by construction only voters in N_i and N_{i-1} can distinguish $c_{i,j}$ and $c_{i,j-1}$

$$\Delta(W, c_{i,j}, c_{i,j-1}) = \Delta_{N_{i-1} \sqcup N_i}(W, c_{i,j}, c_{i,j-1})$$

and so we can apply Proposition 17 and Corollary 15 to the induced instances together with Lemma 19 to conclude that $\Delta(W, c_{i,j}, c_{i,j-1}) = \delta(\lfloor \log(k) \rfloor - 2(i-1)), k_2 + 2) - t\delta(\lfloor \log(k) \rfloor - 2(i-2)), k_2 + 2) > 0$. Then, by the inductive hypothesis the sequence of swaps $\mathbf{X}^1_{i-1}$ is a good sequence of swaps.
If instead, the committee initially is

$$W = (c_{1,t}, \dots c_{i-1,1}, c_{i,j}, c_{i+1,j'}, \dots, , a_{k_1}, d_1, \dots, d_{k_2})$$

where $j > 0$ is even and j' are even, then by an analogous argument the swap $(c_{i,j}, c_{i,j-1})$ is good and destabilises N_{i-1}, so by the inductive hypothesis $\mathbf{X}^0_{i-1}$ is a sequence of good swaps.

Finally, note that Lemma 19 also applies to $i = k_1$, meaning that if we have a committee W with $c_{k_1,i} \in W, i \leq t+1$ which is stable for $N_1, \dots, N_{k_1-1}$ the swap $(c_{k_1,j}, c_{k_1,j+1})$ is good. If $i+1$ is even, we just proved that the sequence of swaps $\mathbf{X}^0_{k_1-1}$ is a sequence of good swap and if $i+1$ is odd, then by the inductive hypothesis $\mathbf{X}^0_{k_1-1}$ is a sequence of good swaps. This proves that indeed $\mathbf{X}^1_{k_1}$ is a sequence of good swaps for initial committee W_0 and we are done. □

B.5 Beyond adversarial better response: Extension to a fixed pivoting rule

We adapt our proof of Theorem 18 to a natural non-adversarial setting: An intuitive method to select swaps is consider some fixed ordering on the candidates $C = \{c_1, \dots, c_m\}$, for example

$c_1 < \ldots < c_m$. Then to select a good swap (c', c) we search over the candidates in $C \setminus W$ to find a candidate c to add to the committee. For each such c we search over candidates in W to find c' so that (c', c) is a good swap. In each search over a subset of C we respect the order $<$. In other words we consider a lexicographic ordering on pairs (c, c') (where c is to be added and c' is to be removed from the committee) induced by the order $<$.

Theorem 3. *0^+-ls-PAV with better response needs $\Omega(k^{\log k})$ iterations in the worst case.*

Proof. Consider election E the proof of Theorem 18 and the linear order $<$ on the set of candidates that satisfies

$$c_{i,j} < c_{i,j+1}, c_{i,j} < d_1 < \ldots, d_{k_2} \text{ and } c_{i,j} < c_{i+1,j'}.$$

We will slightly modify E by adding $O(poly(k))$ voters. In particular, these voters cannot distinguish between the candidates within a set C^i, so that any swaps involving two candidates in C^i are unaffected by these new voters. Their purpose is to prevent swaps (a, b) where $a \in C^i \cup D_{k_2}, b \in C^j$, $j \neq i$ from being executed by ls-PAV with the lexicographic pivoting rule. We observe the trivial fact that $c \in W, c' \notin W \implies \Delta(W, c, c') \leq n = O(poly(k))$, where n is the number of voters in E. Assuming for now we have such a bound, we add voters $\cup_{i=1}^{k} V_i$ with $|V_i| = \lceil 2n \rceil$. All the voters in V_i approve of all candidates in C^i for $1 \leq i \leq k_1$, while voters in $V_i, i > k_1$ approve d_{i-k_1}. Let $c_i \in C^i, c_j \in C^j$ and $i \neq j$ and assume $c_i \in W$ where W is the result of the execution of some prefix of the swap sequence $\mathbf{X}_{k_1}^1$. This means that W contains exactly one candidate from C^i, namely c_i, and exactly one candidate from C^j but not c_j. We observe that

$$\Delta_{V_i \cup V_j}(W, c_i, c_j) = -\lceil 2n \rceil + \frac{1}{2}\lceil 2n \rceil = -\frac{\lceil 2n \rceil}{2} \leq -n,$$

so

$$\Delta_{N \cup_{l=1}^{k} V_l}(W, c_i, c_j) = \Delta_N(W, c_i, c_j) + \Delta_{V_i \cup V_j}(W, c_i, c_j) \leq n - n = 0,$$

implying that (as a result of the presence of the additional voters) the swap (c_i, c_j) is not a good swap. Similarly for $i > k_1$,

$$\Delta_{N \cup_{l=1}^{k} V_i}(W, d_i, c_j) = \Delta_N(W, d_i, c_j) + \Delta_{V_i \cup V_j}(W, d_i, c_j) \leq n - n = 0,$$

so (d_i, c_j) is not a good swap for W. Finally, note that $| \cup_{i=1}^{k} V_i| = k\lceil 2n \rceil$, so clearly the modified instance still has size $poly(k)$.

We will now argue that 0^+-ls-PAV will execute a swap sequence of superpolynomial length on this modified instance. We claim 0^+-ls-PAV will not execute the same swaps $\mathbf{X}_{k_1}^1$ but instead *shortcuts* this sequence of swaps. More formally, we say that a subsequence $(a_1, a_2)(a_2, a_3), \ldots, (a_{l-2}, a_{l-1}), (a_{l-1}, a_l)$ is shortcut, if in the new sequence it is replaced by the new subsequence (a_1, a_l). We write

$$(a_1, a_2)(a_2, a_3), \ldots, (a_{l-2}, a_{l-1}), (a_{l-1}, a_l) \xrightarrow{replace} (a_1, a_l).$$

We will argue that this type of short cutting does not happen too much, so that even with the better response pivoting rule the number of iterations remains superpolynomial. Moreover, the set of intermediate committees obtained by executing any prefix of the swap sequence are a subset of the set of intermediate committees obtained in $\mathbf{X}_{k_1}^1$.

To obtain the new sequence of swaps we shortcut subsequences in $\mathbf{X}_{k_1}^1$ as follows:

We iterate from $l = k_1$ to $l = 1$ and for a subsequence $\mathbf{X}_l^0$ of $\mathbf{X}_{k_1}^1$ update the sequence via

$$\mathbf{X}_l^0 \xrightarrow{replace} (c_{l,t+1}, c_{l,1})$$

Proceeding this way, once we replace a subsequence $\mathbf{R} = \mathbf{X}_i^0$ with the single swap $(c_{i,t+1}, c_{i,1})$, $\mathbf{R}$ is not the subsequence of sequence of the form $\mathbf{X}_l^0$, $l > i$, as by design of the procedure $\mathbf{X}_l^0$ was already replaced earlier in the process. We call the sequence obtained from $\mathbf{X}_i^1$ by replacing sequences in this manner $\mathbf{Z}_i^1$. Formally,

$$\mathbf{Z}_i^1 = \oplus_{j=1}^t (c_{i,j}, c_{i,j+1}) \mathbf{Z}_{i-1}^{j-1 \mod 2} \text{ for } i > 1$$

where $\mathbf{Z}_{i-1}^0 = (c_{i,t+1}, c_{i,1})$ and $\mathbf{Z}_1^1 = \mathbf{X}_1^1$. To conclude our argument, we need to show the following two items.

1. 0^+-ls-PAV executes $\mathbf{Z}_{k_1}^1$ on E when using better response with the fixed order $<$ and
2. $\mathbf{Z}_{k_1}^1$ has superpolynomial length.

For item 1, note first that all swaps in $\mathbf{Z}_{k_1}^1$ are of the form (a, b) with $a, b \in C_i$ for some $i \in [k_1]$. Indeed, we previously showed that for any committee W resulting from executing a prefix of $\mathbf{X}_{k_1}^1$ any swaps of the form

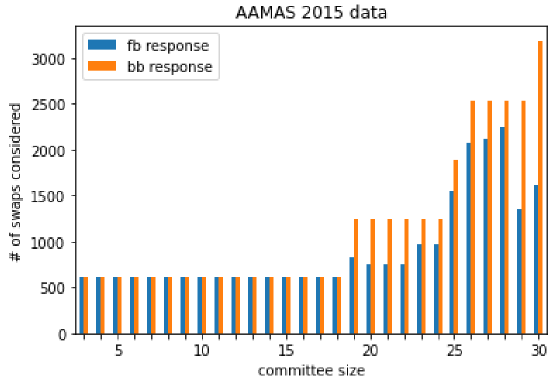
1. (c_i, c_j), $c_i \in C^i$ and $c_j \in C^j, i \neq j$
2. (d_i, c_j), $c_j \in C^j, i \neq j$,

can not increase the PAV score. Moreover, we also argued that any committee W that results from executing a prefix of $\mathbf{Z}_{k_1}^1$ is also a committee resulting from executing some prefix of the swap sequence $\mathbf{X}_{k_1}^1$, and hence the claim translates to the new swap sequence $\mathbf{Z}_{k_1}^1$. It remains to prove that for committee W resulting from the execution of some prefix of $\mathbf{Z}_{k_1}^1$, the following holds: If the next swap is $(a, b), a, b$ in C_i, then no other swap (c, d) that lexicographically preceeds (a, b) is a good swap. With this in mind, the important point to observe for any swap (a, b) in $\mathbf{X}_{k_1}^1$ where $a, b \in C_i$, then the committee W resulting from the prefix in $\mathbf{X}_{k_1}^1$ preceeding (a, b) is stable for any $N_j, j < i$. This stability implies that no swap $(c, d), c, d \in C_j$ is good. $\mathbf{Z}_{k_1}^1$ inherits this property since it inherits the intermediate committees from $\mathbf{X}_{k_1}^1$. Furthermore, by construction of $\mathbf{Z}_{k_1}^1$, (a, b) is either $(c_{i,j}, c_{i,j+1})$ for some j or $(c_{i,t+1}, c_{i,1})$. The latter is clearly lexicographically minimal. For the former observe that since $(c_{i,j}, c_{i,j+1})$ is good, this implies that voters N_i are unstable with i minimum, so either $i = k_1$ or else $c_{i+1,p} \in W$ where p is odd. So any swap $(c_{i,j}, c_{i,l})$, $l < j$, is bad. This concludes the proof of item 1.

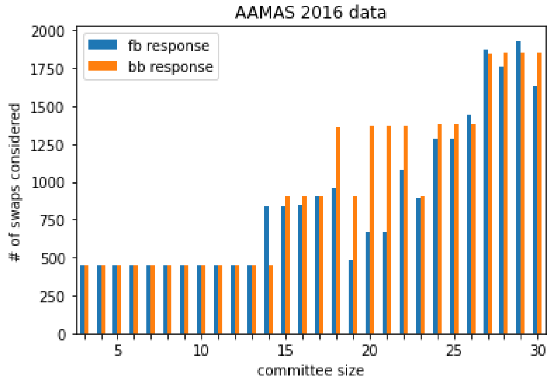
To prove item (2), we now show that $\mathbf{X}_{k_1}^1$ still has length $\Omega(k^{\lfloor \log(k) \rfloor - 1})$. Consider $\mathbf{X}_2^1$ (which is not the subsequence of any $\mathbf{X}_l^0$, $l > 2$). Under the lexicographic pivoting rule, the sequence $\mathbf{X}_1^0$ following a swap $(c_{2,j}, c_{2,j+1})$ where j is odd gets replaced with the single swap $(c_{1,t+1}, c_{1,1})$, but at least half the swaps remain intact. So $|\mathbf{Z}_2^1| \geq \frac{1}{2}|\mathbf{X}_2^1| = \frac{t^2}{2}$. Similarly, $|\mathbf{Z}_{i+1}^1| = |\oplus_{j=1}^t (c_{i,j}, c_{i,j+1}) \mathbf{Z}_i^{j-1 \mod 2}| \geq \frac{t}{2}|\mathbf{Z}_i^1|$ (remember that t is even) so that $|\mathbf{Z}_{k_1}^1| \geq \frac{t}{2}^{k_1 - 2}|\mathbf{Z}_2^1| \geq \frac{t}{2}^{k_1-2}\frac{t^2}{2} = \Omega(\frac{t}{2}^{k_1}) = \Omega(k^{k_1 - 1})$ since $t \geq k$. □

C Further Material for Section 4

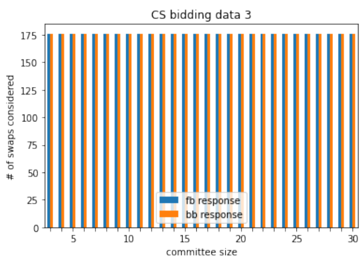
We can run the same experiments for $\frac{n}{k^2}$-ls-PAV. In the case of $\frac{n}{k^2}$-ls-PAV we can use the termination condition from Lemma 4, which can be evaluated in time $O(|E|)$. Halpern et al. [13] use



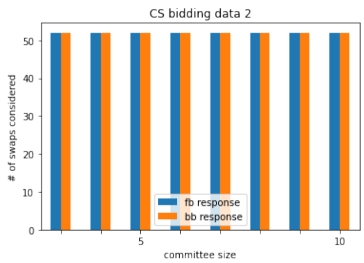
(a) Number of swaps $\frac{n}{k^2}$-ls-PAV considers on AAMAS 2015 bidding data



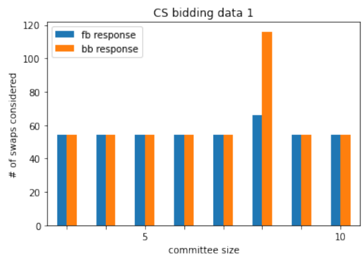
(b) Number of swaps $\frac{n}{k^2}$-ls-PAV considers on AAMAS 2016 bidding data



(c) Number of swaps $\frac{n}{k^2}$-ls-PAV considers on CS bidding data 3



(d) Number of swaps $\frac{n}{k^2}$-ls-PAV considers on CS bidding data 2



(e) Number of swaps $\frac{n}{k^2}$-ls-PAV considers on CS bidding data 1

Figure 3: $\frac{n}{k^2}$-ls-PAV with best response and $\frac{n}{k^2}$-ls-PAV with better response on various Computer Science conference bidding data. On the x-axis, we vary the committee size and on the y-axis we display the number of swaps considered by $\frac{n}{k^2}$-ls-PAV.

best response, i.e., they find the candidate that has the largest marginal contribution to the committee and then remove the candidate in the committee so as to maximize the PAV score resulting from this swap. However, one can also consider a better response version. Specifically, we define better response as first searching for the first candidate outside the committee that increases the PAV-score by at least $\frac{n}{k}$, then for the first candidate in the committee such that the overall swap increases the committee score by at least $\frac{n}{k^2}$. The experiments are analogous to those in Section 4 and are displayed in Figure 3, where best response is denoted as 'bb' and better response is denoted as 'fb'. We can see that also in this case better response tends to be faster under our choice of initialization, but the difference is much less pronounced than for 0^+-ls-PAV.