# Refined Characterizations of Approval-Based Committee Scoring Rules

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**Abstract**

In approval-based committee (ABC) elections, the goal is to elect a fixed size subset of the candidates, a so-called committee, based on the voters' approval ballots over the candidates. One of the most popular classes of ABC voting rules are ABC scoring rules, which have recently been characterized by Lackner and Skowron [23]. However, this characterization relies on a model where the output is a ranking of committees instead of a set of winning committees and no full characterization of ABC scoring rules exists in the latter standard setting. We address this issue by characterizing two important subclasses of ABC scoring rules in the standard ABC election model, thereby both extending the result of Lackner and Skowron [23] to the standard setting and refining it to subclasses. In more detail, we characterize *(i)* the prominent class of Thiele rules and *(ii)* a new class of ABC voting rules called ballot size weighted approval voting. Both of these results are driven by a consistency notion analogous to the one of Young [37]. Based on these theorems, we also derive characterizations of three well-known ABC voting rules, namely multi-winner approval voting, proportional approval voting, and satisfaction approval voting.

## 1 Introduction

An important problem for multi-agent systems is collective decision making: given the voters' preferences over a set of alternatives, a common decision has to be made. This problem has traditionally been studied by economists for settings where a single candidate is elected (see, e.g., [1]), but there is also a multitude of applications where a fixed number of the candidates needs to be elected. The archetypal example for this is the election of a city council, but there are also technical applications such as recommender systems [32, 20]. In social choice theory, this type of elections is typically called *approval-based committee (ABC) elections* and has recently attracted significant attention [e.g., 3, 17, 16, 25]. In more detail, the research on these elections focuses on *ABC voting rules*, which are functions that choose a set of winning committees (i.e., subsets of the candidates of a fixed size) based on the voters' approval ballots (i.e., the subsets of candidates that the voters find acceptable).

Maybe the most prominent class of ABC voting rules are ABC scoring rules. These rules rely on a scoring function $s$ to compute the winning committees and each voter assigns $s(x, y)$ points to a committee if she approves $x$ candidates in the committee and $y$ in total. An ABC scoring rule then chooses the committees with maximal total score. There are many well-known ABC scoring rules, e.g., multi-winner approval voting (`AV`), satisfaction approval voting (`SAV`), Chamberlin-Courant approval voting (`CCAV`), and proportional approval voting (`PAV`). While these rules have rather different behavior, they are all consistent: if some common committees are chosen for two disjoint elections, precisely these common committees are chosen in a joint election. Indeed, all ABC scoring rules satisfy this axiom and they can thus be seen as an equivalent to single-winner scoring rules, which have prominently been characterized by Young [37] based on an analogous consistency condition.

In a recent breakthrough result, Lackner and Skowron [23] have managed to formalize the relation between ABC scoring rules and single-winner scoring rules by characterizing ABC scoring rules with almost the same axioms as Young [37] uses for his characterization of single-winner scoring rules. In more detail, Lackner and Skowron [23] show that ABC scoring rules are the only ABC ranking rules that satisfy anonymity, neutrality, continuity, weak efficiency, and consistency. However, this result discusses ABC ranking rules, which return transitive rankings of committees, whereas the literature on ABC elections typically focuses on sets of committees as output. While Lackner and Skowron [24] also present a result for the latter setting, we believe the proof of this result to be incomplete.[1] Moreover, even when the proof would be correct, this result is not a full characterization of ABC scoring rules as it requires a technical axiom called 2-non-imposition. This axiom postulates that every pair of committees is the outcome for some profile and is, e.g., violated by `AV` and `SAV`. Hence, characterizations of important ABC scoring rules and, more generally, tools that allow us to easily infer such results are still missing for the standard ABC setting. Lackner and Skowron also acknowledge this shortcoming by writing that "a full characterization of ABC scoring rules within the class of ABC choice rules remains as important future work" [24, p. 16].

**Our contribution.** We address this problem by presenting full axiomatic characterizations of two subclasses of ABC scoring rules, namely Thiele rules and ballot size weighted approval voting (BSWAV) rules, in the standard ABC election setting. Hence, our results refine the result of Lackner and Skowron [23] and extend it to the standard ABC voting setting while avoiding technical auxiliary conditions. *Thiele rules* are ABC scoring rules that do not depend on the ballot size and have attracted significant attention [e.g., 3, 32, 10, 22]. On the other hand, *BSWAV rules* are a new generalization of multi-winner approval voting where the voters are weighted depending on the size of their ballots. For example, `PAV` and `CCAV` are Thiele rules, `SAV` is a BSWAV rule, and `AV` is in both classes. Moreover, every ABC scoring rule that has been studied in the literature is in one of our two classes.

For our characterization of Thiele rules, we rely on the axioms of Lackner and Skowron [23] and additionally require *independence of losers*. This axiom demands that a winning committee $W$ stays winning if some voters change their ballot by disapproving candidates outside of $W$ as, intuitively, the quality of $W$ should only depend on its members. Similar conditions are well-known in single-winner elections [e.g., 7] and this axiom has recently been adapted to ABC elections by Dong and Lederer [12]. Based on this axiom, we show that *an ABC voting rule is a Thiele rule if and only if it satisfies anonymity, neutrality, consistency, continuity, and independence of losers (Theorem 1).*

In order to characterize BSWAV rules, we introduce a new axiom called *choice set convexity*. This condition requires that if two committees are chosen, then all committees "in between" those committees are chosen, too: if $W$ and $W'$ are chosen, then all committees $W''$ with $W \cap W' \subseteq W'' \subseteq W \cup W'$ are also chosen. We believe that this axiom is reasonable for excellence-based elections (which only focus on the individual quality of the candidates) because a tie between committees indicates that they are equally good and the candidates in $W \setminus W'$ and $W' \setminus W$ are thus exchangeable. We then prove that *an ABC voting rule is a BSWAV rule if and only if it satisfies anonymity, neutrality, consistency, continuity, weak efficiency, and choice set convexity (Theorem 2).*

While our theorems are intuitively related to the results of Lackner and Skowron [23, 24], they are logically independent as *all* BSWAV rules (including `AV`) fail 2-non-imposition. In particular, Theorem 2 allows, in contrast to the result of Lackner and Skowron, to characterize `AV` and `SAV`. Moreover, it is easy to infer full characterizations of specific Thiele rules based on Theorem 1 as we can apply this result simply to known (partial) characterizations within the

[1] Roughly, the proof of the main result of Lackner and Skowron [24] works by constructing an ABC ranking rule $g$ based on an ABC voting rule $f$ that satisfies the given axioms. Then, Lackner and Skowron [24] show that the axioms of $f$ inherit to $g$, so $g$ must be an ABC scoring rule (in the ranking setting). This implies that $f$ is an ABC scoring rule (in the choice setting). However, the authors never show that the rankings returned by $g$ are transitive, which is required by definition of ABC ranking rules, and proving this seems surprisingly difficult.

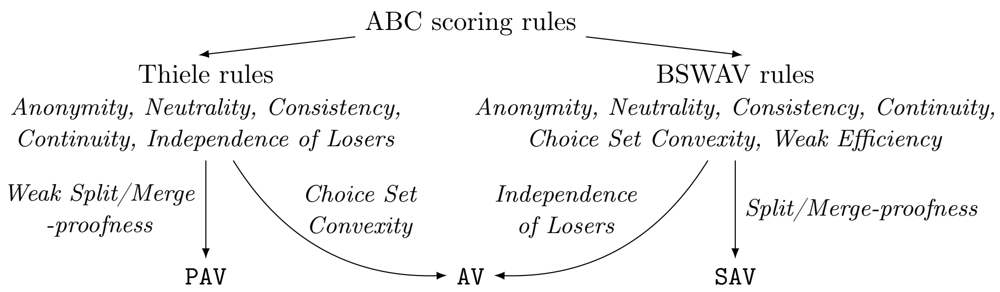


Figure 1: Overview of our results. An arrow from $X$ to $Y$ means that $Y$ is a subset or an element of $X$. The axioms below Thiele rules and BSWAV rules characterize these classes of ABC voting rules. The axioms on the arrows to AV, PAV, and SAV characterize these rules within the set of non-trivial Thiele rules and non-trivial BSWAV rules, respectively.

class of Thiele rules. We demonstrate these points also by characterizing three well-known ABC voting rules in Section 4. In more detail, we first obtain a characterization of AV as it is essentially the only rule that is both a BSWAV rule and a Thiele rule. Secondly, we also characterize PAV and SAV by considering party-list profiles (where candidates are partitioned into parties and each voter approves all candidates of a single party) and analyzing whether it is advantageous for candidates to compete by themselves or to form a party. An overview of our results is shown in Figure 1.

**Related work.** The lack of characterizations of ABC voting rules is one of the major open problems in the field [see, e.g., 25, Q1], and there are thus only few closely related papers. Maybe the most important one is due to Lackner and Skowron [23] who characterize ABC scoring rules in the context of ABC ranking rules; however, this result does not allow for characterizations of ABC voting rules in the standard setting. The (erroneous) follow-up paper by Lackner and Skowron [24] tries to fix this issue by characterizing ABC scoring rules in the standard setting, but it requires a technical auxiliary condition that rules out important rules such as AV and SAV. Finally, Dong and Lederer [12] characterize committee monotone ABC rules, which can be seen as greedy approximations of ABC scoring rules. It should also be mentioned that committee scoring rules have been analyzed for strict rankings as input, but these results are also restricted to characterizations in the context of committee ranking rules [33] or characterizations within the class of committee scoring rules [16, 18].

Furthermore, a large amount of papers studies axiomatic properties of ABC scoring rules [e.g., 21, 3, 31, 10, 22]. For instance, Aziz et al. [3] investigate Thiele rules with respect to how fairly they represent groups of voters with similar preferences, and Sánchez-Fernández and Fisteus [31] study various monotonicity conditions for a number of ABC voting rules, including Thiele rules. Another important aspect of these rules is their computational complexity. In particular, it is known that all Thiele rules but AV are NP-hard to compute on the full domain [2, 32]. There is thus significant work on how to compute these rules by, e.g., restricting the domain of preference profiles [14, 28, 35], considering approximation algorithms for Thiele rules [13, 5], or designing FPT algorithms [9]. For a more detailed overview on ABC scoring rules, we refer to the survey by Lackner and Skowron [25].

Finally, in the broader realm of social choice, there are many results that are conceptually similar to ours as they rely on consistency axioms: Young [37] has characterized scoring rules for single-winner elections based on this axiom (see also [34, 26, 30]), numerous characterizations of single-winner approval voting rely on consistency [19, 7], Young and Levenglick [38] have characterized Kemeny's rule with the help of this axiom, and Brandl et al. [8] characterize a randomized voting rule called maximal lotteries based on this condition.

## 2 Preliminaries

Let $\mathbb{N} = \{1, 2, \dots\}$ denote an infinite set of voters and let $\mathcal{C} = \{c_1, \dots, c_m\}$ denote a set of $m \geq 2$ candidates. Intuitively, we interpret $\mathbb{N}$ as the set of all possible voters, and a concrete electorate $N$ is a finite and non-empty subset of $\mathbb{N}$. To this end, we define $\mathcal{F}(\mathbb{N}) = \{N \subseteq \mathbb{N} \colon N \textit{ is non-empty and finite}\}$ as the set of all possible electorates. Given an electorate $N \in \mathcal{F}(\mathbb{N})$, we assume that each voter $i \in N$ reports her preferences over the candidates as *approval ballot* $A_i$, i.e., as a non-empty subset of $\mathcal{C}$. $\mathcal{A}$ is the set of all possible approval ballots. An *approval profile* $A$ is a mapping from $N$ to $\mathcal{A}$, i.e., it assigns an approval ballot to every voter in the given electorate. Moreover, we define $\mathcal{A}^* = \bigcup_{N \in \mathcal{F}(\mathbb{N})} \mathcal{A}^N$ as the set of all possible approval profiles. For every profile $A \in \mathcal{A}^*$, $N_A$ denotes the set of voters that submit a ballot in $A$. Furthermore, two approval profiles $A, A'$ are called *disjoint* if $N_A \cap N_{A'} = \emptyset$ and for disjoint profiles $A, A' \in \mathcal{A}^*$, we define the profile $A'' = A + A'$ by $N_{A''} = N_A \cup N_{A'}$, $A''_i = A_i$ for $i \in N_A$, and $A''_i = A'_i$ for $i \in N_{A'}$.

Given an approval profile, our aim is to elect a *committee*, i.e., a subset of the candidates of predefined size. We denote the target committee size by $k \in \{1, \dots, m-1\}$ and the set of all size $k$ committees by $\mathcal{W}_k = \{W \subseteq \mathcal{C} \colon |W| = k\}$. For determining the winning committees for a given preference profile, we use *approval-based committee (ABC) voting rules* which are mappings from $\mathcal{A}^*$ to $2^{\mathcal{W}_k} \setminus \{\emptyset\}$. Note that ABC voting rules are defined for a fixed committee size and may return multiple committees. The latter indicates that chosen committees are tied for the final victory in the election, which is necessary to satisfy basic fairness conditions; e.g., if all voters approve all candidates, all committees are equally acceptable and a fair voting rule cannot distinguish between them.

### 2.1 ABC Voting Rules

We focus in this paper on three classes of ABC voting rules: ABC scoring rules, Thiele rules, and BSWAV rules.

**ABC scoring rules.** ABC scoring rules rely on a scoring function according to which voters assign points to committees and choose the committees with maximal score. Formally, a *scoring function* $s(x, y)$ is a mapping from $\{0, \dots, k\} \times \{1, \dots, m\}$ to $\mathbb{R}$ such that $s(x, y) \geq s(x', y)$ for all $x, x' \in \{\max(0, k + y - m), \dots, \min(k, y)\}$ with $x \geq x'$. We define the score of a committee $W$ in a profile $A$ as $\hat{s}(A, W) = \sum_{i \in N_A} s(|A_i \cap W|, |A_i|)$. Then, an ABC voting rule $f$ is an *ABC scoring rule* if there is a scoring function $s$ such that $f(A) = \{W \in \mathcal{W}_k \colon \forall W' \in \mathcal{W}_k \colon \hat{s}(A, W) \geq \hat{s}(A, W')\}$. Note that the set $\{\max(0, k+y-m), \dots, \min(k, y)\}$ contains all "active" intersection sizes: a committee of size $k$ and a ballot of size $y$ intersect at least in $\max(0, k + y - m)$ candidates and at most in $\min(k, y)$ candidates.

**Thiele rules.** Arguably the most prominent subclass of ABC scoring rules are Thiele rules. Their namesake Thiele [36] proposed them with a simple argument: if the elected committee contains $x$ of the approved candidates of a voter, the voter should have some benefit $s(x)$ from the committee. Hence, Thiele rules are defined by a non-decreasing *Thiele scoring function* $s : \{0, \dots, k\} \to \mathbb{R}$ with $s(0) = 0$, and choose the committees that maximize the total score. Formally, an ABC voting rule $f$ is a *Thiele rule* if there is a Thiele scoring function $s$ such that $f(A) = \{W \in \mathcal{W}_k \colon \forall W' \in \mathcal{W}_k \colon \hat{s}(A, W) \geq \hat{s}(A, W')\}$, where $\hat{s}(A, W) = \sum_{i \in N_A} s(|A_i \cap W|)$. There are numerous important Thiele rules such as multi-winner approval voting (`AV`; defined by $s_{AV}(x) = x$), proportional approval voting (`PAV`; defined by $s_{\texttt{PAV}}(x) = \sum_{z=1}^{x} \frac{1}{z}$ for $x > 0$), and Chamberlin-Courant approval voting (`CCAV`; defined by $s_{CCAV}(x) = 1$ for $x > 0$).

**BSWAV rules.** Ballot size weighted approval voting rules form a new subclass of ABC scoring rules which generalize `AV` by weighting voters based on their ballot size. Formally, a *ballot size weighted approval voting (BSWAV) rule* $f$ is defined by a weight vector $\alpha \in \mathbb{R}^m_{\geq 0}$ and chooses for a profile $A$ the committees $W$ that maximize $\hat{s}(A,W) = \sum_{i\in N_A} \alpha_{|A_i|}|A_i \cap W|$. We note that the score of a committee $W$ for a BSWAV rule can be represented as the sum of the scores of individual candidates $c \in W$ since $\sum_{i\in N_A} \alpha_{|A_i|}|A_i \cap W| = \sum_{c\in W} \sum_{i\in N_A:\, c\in A_i} \alpha_{|A_i|}$. AV is clearly part of this class by setting $\alpha_x = 1$ for all $x \in \{1,\dots,m\}$. Another well-known BSWAV rule is satisfaction approval voting (SAV) defined by $\alpha_x = \frac{1}{x}$ for $x \in \{1,\dots,m\}$. SAV has been suggested by Brams and Kilgour [6] and can be motivated by the "one man, one vote" principle as every voter distributes a budget of 1 to her approved candidates.

We note that Thiele rules and BSWAV rules are diametrically opposing subclasses of ABC scoring rules: Thiele rules do not depend on the ballot size at all, whereas BSWAV rules only depend on this aspect. Consequently, if $k < m-1$, the sets of BSWAV rules and Thiele rules only intersect in AV and the trivial rule TRIV (which always chooses all size $k$ committees). So, AV is the only non-trivial ABC voting rule that is in both classes; *non-triviality* means here that there is a profile $A$ such that $f(A) \neq \texttt{TRIV}(A)$. Moreover, both classes are proper subsets of the set of ABC scoring rules if $1 < k < m-1$. In contrast, the set of BSWAV rules is equivalent to the set of ABC scoring rules if $k \in \{1, m-1\}$.

## 2.2 Basic Axioms

Next, we introduce the axioms used for our characterizations.

**Anonymity.** Anonymity is one of the most basic fairness properties and requires that all voters should be treated equally. Formally, we say an ABC voting rule $f$ is *anonymous* if $f(A) = f(\pi(A))$ for all profiles $A \in \mathcal{A}^*$ and permutations $\pi : \mathbb{N} \to \mathbb{N}$. Here, we denote by $A' = \pi(A)$ the profile with $N_{A'} = \{\pi(i)\colon i \in N_A\}$ and $A'_{\pi(i)} = A_i$ for all $i \in N_A$.

**Neutrality.** Similar to anonymity, *neutrality* is a fairness property for the candidates. This axiom requires of an ABC voting rule $f$ that $f(\tau(A)) = \{\tau(W)\colon W \in f(A)\}$ for all profiles $A \in \mathcal{A}^*$ and permutations $\tau : \mathcal{C} \to \mathcal{C}$. This time, $A' = \tau(A)$ denotes the profile with $N_{A'} = N_A$ and $A'_i = \tau(A_i)$ for all $i \in N_A$.

**Weak Efficiency.** Weak efficiency requires that unanimously unapproved candidates can never be "better" than approved ones. Formally, we say an ABC voting rule $f$ is *weakly efficient* if $W \in f(A)$ for a committee $W \in \mathcal{W}_k$ with $c \in W \setminus (\bigcup_{i\in N_A} A_i)$ implies that $(W \cup \{c'\}) \setminus \{c\} \in f(A)$ for all candidates $c' \in \mathcal{C} \setminus W$.

**Continuity.** The intuition behind continuity is that a large group of voters should be able to enforce that some of its desired outcomes are chosen. Hence, an ABC voting rule $f$ is *continuous* if for all profiles $A, A' \in \mathcal{A}^*$, there is $\lambda \in \mathbb{N}$ such that $f(\lambda A + A') \subseteq f(A)$. Here, $\lambda A$ denotes the profile consisting of $\lambda$ copies of $A$; the names of the voters in $N_{\lambda A}$ will not matter as we will focus on anonymous rules.

**Consistency.** The central axiom for our results is consistency. This condition states that if some committees are chosen for two disjoint profiles, then precisely those committees are chosen in the joint profile. Formally, an ABC voting rule $f$ is *consistent* if $f(A + A') = f(A) \cap f(A')$ for all disjoint profiles $A, A' \in \mathcal{A}^*$ with $f(A) \cap f(A') \neq \emptyset$. Consistency and the previous four axioms have been introduced by Lackner and Skowron [24] for ABC elections. Moreover, except consistency, all of these axioms are very mild and satisfied by almost all commonly considered ABC voting rules.

**Independence of Losers.** Independence of losers has been adapted to ABC elections by Dong and Lederer [12] and requires of an ABC voting rule $f$ that a winning committee $W$ should still be a winning committee if voters disapprove candidates outside of $W$. Formally, we say that $f$ is *independent of losers* if $W \in f(A)$ implies $W \in f(A')$ for all profiles $A, A' \in \mathcal{A}^*$ and committees $W \in \mathcal{W}_k$ such that $N_A = N_{A'}$ and $W \cap A_i = W \cap A'_i$ and

$A'_i \subseteq A_i$ for all $i \in N_A$. The motivation for this axiom is that the quality of $W$ should only depend on the candidates in $W$. So, if the voters disapprove candidates $x \notin W$, this does not affect the quality of $W$ and a chosen committee $W$ should stay chosen. All commonly studied ABC voting rules that are independent of the ballot size (e.g., Thiele rules, sequential Thiele rules, and Phragmen's rule) satisfy this axiom, whereas all BSWAV rules except `AV` fail it.

**Choice Set Convexity.** Finally, we introduce a new condition called choice set convexity: an ABC voting rule $f$ is *choice set convex* if $W, W' \in f(A)$ implies that $W'' \in f(A)$ for all committees $W, W', W'' \in \mathcal{W}_k$ and profiles $A \in \mathcal{A}^*$ such that $W \cap W' \subseteq W'' \subseteq W \cup W'$. More informally, this axiom states that if a rule chooses two committees $W$ and $W'$, then all committees "between" $W$ and $W'$ should also be chosen. We believe that choice set convexity is reasonable in elections in which only the individual quality of the elected candidates matters. For instance, if we want to hire 3 applicants for a job based on the interviewer's preferences, it seems unreasonable that the sets $\{c_1, c_2, c_3\}$ and $\{c_1, c_4, c_5\}$ are good enough to be hired but $\{c_1, c_2, c_4\}$ is not. More generally, we can interpret the membership of a candidate in a chosen committee as certificate for its quality and all candidates $c \in (W \setminus W') \cup (W' \setminus W)$ are then equally good. Many commonly considered voting rules fail this axiom, but it is always possible to compute the "convex hull" of a choice set.

# 3 Characterizations of Classes of ABC Voting Rules

We now turn to our characterizations of Thiele rules and BSWAV rules, which are discussed in Section 3.2 and Section 3.3, respectively. The proofs of these results are rather involved, so we defer them to the appendix. Since we nevertheless want to showcase our proof technique, we revisit the result of Lackner and Skowron [24] in Section 3.1 for $k \in \{1, m-1\}$ as this allows us to explain our ideas while avoiding challenging technical details.

## 3.1 ABC Scoring Rules

As mentioned in the introduction, Lackner and Skowron [24] partially characterize ABC scoring rules: they show that a 2-non-imposing ABC voting rule is an ABC scoring rule if and only if it satisfies anonymity, neutrality, consistency, continuity, and weak efficiency. We will here revisit this result for $k \in \{1, m-1\}$ to showcase our proof idea for deriving Theorems 1 and 2. Note that we omit 2-non-imposition as it is not necessary if $k \in \{1, m-1\}$.

Our techniques are inspired by those of Young [37] and Skowron et al. [33] as we will use the separating hyperplane theorem for convex sets to show our results. To further explain our approach, let $f$ denote an ABC voting rule that satisfies anonymity, neutrality, consistency, and non-imposition. The last condition means that for every committee $W \in \mathcal{W}_k$, there is a profile $A \in \mathcal{A}^*$ such that $f(A) = \{W\}$. This axiom is no restriction for our analysis as all non-trivial ABC voting rules that we consider are non-imposing. As first step, we will change the domain of $f$ from approval profiles to a numerical space. For this, we use that $f$ is anonymous and thus only depends on the number of voters who submit a specific ballot. Thus, let $B \colon \{1, \dots, |\mathcal{A}|\} \to \mathcal{A}$ denote an enumeration of all approval ballots and define $v(A)$ as the vector whose $\ell$-th entry counts how often the ballot $B(\ell)$ appears in the profile $A$. By anonymity, there is a function $g : \mathbb{N}^{|\mathcal{A}|} \to 2^{\mathcal{W}_k} \setminus \{\emptyset\}$ such that $f(A) = g(v(A))$ for all profiles $A \in \mathcal{A}^*$. Moreover, this function inherits neutrality ($g(\tau(v)) = \{\tau(W) \colon W \in g(v)\}$ for all vectors $v$ and permutations $\tau : \mathcal{C} \to \mathcal{C}$) and consistency ($g(v + v') = g(v) \cap g(v')$ for all vectors $v, v'$ with $g(v) \cap g(v') \neq \emptyset$) from $f$. Here, $\tau(v)$ denotes the vector such that $\tau(v)_i = v_j$ for all $i, j$ with $B(i) = \tau(B(j))$. Next, we extend the domain of $g$ from $\mathbb{N}^{|\mathcal{A}|}$ to $\mathbb{Q}^{|\mathcal{A}|}$ while preserving all desirable properties. The proof of this claim can be found in the appendix.

**Lemma 1.** *Let $f$ denote a non-imposing ABC voting rule that satisfies anonymity, neutrality, and consistency. There is a function $\hat{g} : \mathbb{Q}^{|\mathcal{A}|} \to 2^{\mathcal{W}_k} \setminus \{\emptyset\}$ that satisfies neutrality, consistency, and $\hat{g}(v(A)) = f(A)$ for all $A \in \mathcal{A}^*$.*

Since $\hat{g}$ fully describes $f$, we aim to represent $\hat{g}$ by a scoring function. For this, we define an arbitrary order $W^1, \dots, W^{|\mathcal{W}_k|}$ over the committees and let $R_i^f = \{v \in \mathbb{Q}^{|\mathcal{A}|} \colon W^i \in \hat{g}(v)\}$. Moreover, $\bar{R}_i^f$ is the closure of $R_i^f$ with respect to $\mathbb{R}^{|\mathcal{A}|}$. It is easy to see that the sets $\bar{R}_i^f$ are convex and their interiors are disjoint because of the properties of $\hat{g}$. We can thus apply the separating hyperplane theorem for convex set to derive a non-zero vector $u^{i,j}$ such that $u^{i,j}v \geq 0$ if $v \in \bar{R}_i^f$ and $u^{i,j}v \leq 0$ if $v \in \bar{R}_j^f$ for every pair of sets $\bar{R}_i^f, \bar{R}_j^f$ ($u^{i,j}v = vu^{i,j}$ denotes throughout the paper the standard scalar product). Our next lemma shows that there are symmetric non-zero vectors that fully describe the sets $\bar{R}_i^f$.

**Lemma 2.** *Let $f$ denote a non-imposing ABC voting rule that satisfies anonymity, neutrality, and consistency. There are non-zero vectors $\hat{u}^{i,j}$ that satisfy the following conditions for all $W^i, W^j \in \mathcal{W}_k$:*

1. *$\bar{R}_i^f = \{v \in \mathbb{R}^{|\mathcal{A}|} \colon \forall j' \in \{1, \dots, |\mathcal{W}_k|\} \setminus \{i\} \colon \hat{u}^{i,j'}v \geq 0\}$.*
2. *$\hat{u}^{i,j} = -\hat{u}^{j,i}$.*
3. *$\hat{u}^{i',j'} = \tau(\hat{u}^{i,j})$ if $\tau(W^i) = W^{i'}$ and $\tau(W^j) = W^{j'}$.*

Based on Lemma 2, we now show how to derive the score function of the considered rule.

**Proposition 1.** *Assume $k = 1$ or $k = m - 1$. An ABC voting rule is an ABC scoring rule if and only if it satisfies anonymity, neutrality, consistency, continuity, and weak efficiency.*

*Proof.* It is easy to check that ABC scoring rules satisfy all given axioms. So, we focus on the converse and let $f$ denote an ABC voting rule that satisfies all given axioms for $k = 1$; the case that $k = m - 1$ follows from similar arguments. First, if $f$ is trivial, it is the ABC scoring rule induced by the score function $s(x, y) = 0$. We hence suppose that $f$ is non-trivial. We will first show that $f$ is non-imposing. For this, we note that there is a ballot $A \in \mathcal{A}$ such that $f(A) \neq \mathcal{W}_k$ because of non-triviality and consistency. Let $c, d$ denote candidates such that $\{c\} \in f(A)$, $\{d\} \notin f(A)$ and consider a permutation $\tau : \mathcal{C} \to \mathcal{C}$ with $\tau(c) = c$. By neutrality, $\{c\} \in f(\tau(A))$, $\{\tau(d)\} \notin f(\tau(A))$. Next, consider the profile $A^*$ that consists of a ballot $\tau(A)$ for every permutation $\tau$ with $\tau(c) = c$. By consistency, we infer that $f(A^*) = \bigcap_{\tau:\mathcal{C}\to\mathcal{C} \colon \tau(c)=c} f(\tau(A)) = \{\{c\}\}$. Neutrality implies now that $f$ is non-imposing

Next, we use Lemma 1 to obtain the function $\hat{g} : \mathbb{Q}^{|\mathcal{A}|} \to 2^{\mathcal{W}_k} \setminus \{\emptyset\}$ and define the sets $R_i^f = \{v \in \mathbb{Q}^{|\mathcal{A}|} \colon W^i \in \hat{g}(v)\}$. In turn, Lemma 2 entails the existence of symmetric non-zero vectors $\hat{u}^{i,j}$ such that $\bar{R}_i^f = \{v \in \mathbb{R}^{|\mathcal{A}|} \colon \forall j \in \{1, \dots, |\mathcal{W}_k|\} \setminus \{i\} \colon \hat{u}^{i,j}v \geq 0\}$. Now, consider committees $W^i, W^j, W^{i'}, W^{j'} \in \mathcal{W}_k$ with $W^i \neq W^j$ and $W^{i'} \neq W^{j'}$. Since $k = 1$, this means that $|W^i \setminus W^j| = |W^{i'} \setminus W^{j'}| = 1$. Moreover, let $B(\ell)$, $B(\ell')$ denote two ballots such that $|B(\ell)| = |B(\ell')|$, $|B(\ell) \cap W^i| = |B(\ell') \cap W^{i'}|$, and $|B(\ell) \cap W^j| = |B(\ell') \cap W^{j'}|$. These assumptions imply that there is a permutation $\tau : \mathcal{C} \to \mathcal{C}$ such that $\tau(B(\ell)) = B(\ell')$, $\tau(W^i) = W^{i'}$, and $\tau(W^j) = W^{j'}$. Condition (3) of Lemma 2 then shows that $\hat{u}_{\ell'}^{i',j'} = \tau(\hat{u}^{i,j})_{\ell'} = \hat{u}_\ell^{i,j}$. This means that there are functions $s^1(x, y, z)$ such that $\hat{u}_\ell^{i,j} = s^1(|W^i \cap B(\ell)|, |W^j \cap B(\ell)|, |B(\ell)|)$ for all committees $W^i, W^j$ and ballots $B(\ell)$. Next, consider two committees $W^i$ and $W^j$ and a permutation $\tau$ such that $\tau(W^i) = W^j$, $\tau(W^j) = W^i$, and $\tau(x) = x$ for all $x \in \mathcal{C} \setminus (W^i \cup W^j)$. By Conditions (2) and (3) of Lemma 2, $-\hat{u}_{\ell'}^{i,j} = \hat{u}_{\ell'}^{j,i} = \tau(\hat{u}^{i,j})_{\ell'} = \hat{u}_\ell^{i,j}$ for all ballots $B(\ell)$ and $B(\ell') = \tau(B(\ell))$. Now, if $W^i \cup W^j \subseteq B(\ell)$ or $B(\ell) \cap (W^i \cup W^j) = \emptyset$, then $\tau(B(\ell)) = B(\ell)$ and this inequality simplifies to $-\hat{u}_\ell^{i,j} = \hat{u}_\ell^{i,j}$. This implies that $\hat{u}_\ell^{i,j} = 0$, so $s^1(x, x, z) = 0$ for all $x \in \{0, 1\}$, $z \in \{1, \dots, m\}$. On the other hand, if $|W^i \cap B(\ell)| = 1 > 0 = |W^j \cap B(\ell)|$, then $|W^i \cap B(\ell')| = 0 < 1 = |W^j \cap B(\ell')|$ and $s^1(1, 0, z) = -s^1(0, 1, z)$ for all $z \in \{1, \dots, m\}$.

We can now infer the score function $s(x,z)$ from $s^1(x,y,z)$: we define $s(0,z)=0$ and $s(1,z)=s^1(1,0,z)$ for all $z\in\{1,\dots,m\}$. It is easy to check that $\hat{u}^{i,j}v=\sum_{\ell\in\{1,\dots,|\mathcal{A}|\}} v_\ell s^1(|W^i\cap B(\ell)|,|W^j\cap B(\ell)|,|B(\ell)|)=\sum_{\ell\in\{1,\dots,|\mathcal{A}|\}} v_\ell(s(|W^i\cap B(\ell)|,|B(\ell)|)-s(|W^j\cap B(\ell)|,|B(\ell)|))$ for all $W^i$, $W^j\in\mathcal{W}_k$ and $v\in\mathbb{R}^{|\mathcal{A}|}$. Next, we define $\hat{s}(v,W)=\sum_{\ell\in\{1,\dots,|\mathcal{A}|\}} v_\ell(s(|W^i\cap B(\ell)|,|B(\ell)|)$ and infer that $\bar{R}_i^f=\{v\in\mathbb{R}^{|\mathcal{A}|}\colon \forall j\in\{1,\dots,|\mathcal{W}_k|\}\setminus\{i\}\colon \hat{u}^{i,j}\geq 0\}=\{v\in\mathbb{R}^{|\mathcal{A}|}\colon \forall j\in\{1,\dots,|\mathcal{W}_k|\}:\hat{s}(v,W^i)\geq\hat{s}(v,W^j)\}$. Hence, $f(A)=\hat{g}(v(A))\subseteq\{W\in\mathcal{W}_k\colon \forall W'\in\mathcal{W}_k:\hat{s}(A,W)\geq\hat{s}(A,W')\}:=f'(A)$ for all $A\in\mathcal{A}^*$.

Next, we will show that this subset relation is an equality. Suppose for this that there is a profile $A$ such that $f(A)\subsetneq f'(A)$ and let $\{d\}\in f'(A)\setminus f(A)$. We note that $f'$ is consistent and non-trivial, so an analogous argument as for $f$ shows that it is non-imposing. Thus, there is a profile $A'$ such that $f'(A')=\{\{d\}\}$. By the consistency of $f'$ and the above subset relation, we have that $f(\lambda A+A')=f'(\lambda A+A')=\{\{d\}\}$ for all $\lambda\in\mathbb{N}$. However, this contradicts the continuity of $f$, which requires that there is $\lambda\in\mathbb{N}$ such that $f(\lambda A+A')\subseteq f(A)$. So, $f$ is the ABC scoring rule induced by $s$. Finally, we show that $s$ is non-decreasing. Otherwise, there is a ballot size $y\in\{1,\dots,m-1\}$ such that $0=s(0,y)>s(1,y)$. Now, consider a single ballot $A$ of size $y$. By definition of $s$ and $f$, $f(A)=\{W\in\mathcal{W}_k\colon W\not\subseteq A\}$. However, this outcome violates weak efficiency, so $s$ needs to be non-decreasing in its first argument. □

## 3.2 Thiele Rules

We now turn to our first full characterization: Thiele rules are the only ABC voting rules that satisfy anonymity, neutrality, consistency, continuity, and independence of losers. We note here that, compared to the results of Lackner and Skowron [23], we only need to replace weak efficiency with independence of losers.

**Theorem 1.** *An ABC voting rule is a Thiele rule if and only if it satisfies anonymity, neutrality, consistency, continuity, and independence of losers.*

*Proof Sketch.* First, suppose that $f$ is a Thiele rule and let $s(x)$ denote its Thiele scoring function. Clearly, $f$ is anonymous, neutral, consistent, and continuous as all ABC scoring rules satisfy these axioms. So, we will only show that $f$ is independent of losers. For this, consider two profiles $A,A'\in\mathcal{A}^*$ and a committee $W\in f(A)$ such that $N_A=N_{A'}$ and $A'_i\subseteq A_i$ and $W\cap A'_i=W\cap A_i$ for all $i\in N_A$. It holds that $\hat{s}(A',W)=\hat{s}(A,W)$ since $W\cap A'_i=W\cap A_i$ for all $i\in N_A$. On the other hand, $\hat{s}(A,W')\geq\hat{s}(A',W')$ for all $W'\in\mathcal{W}_k$ as $s(x)$ is non-decreasing. Finally, since $W\in f(A)$, $\hat{s}(A,W)\geq\hat{s}(A,W')$ for all $W'\in\mathcal{W}_k$ and we conclude that $\hat{s}(A',W)=\hat{s}(A,W)\geq\hat{s}(A,W')\geq\hat{s}(A',W')$ for all committees $W'\in\mathcal{W}_k$. So, $W\in f(A')$ and $f$ satisfies independence of losers.

For the other direction, suppose that $f$ is an ABC voting rule that satisfies all axioms of the theorem. Since this direction is much more involved, we only give a rough proof sketch. Now, if $f$ is trivial, it is the Thiele rule defined by $s(x)=0$ for all $x$. Hence, suppose that $f$ is non-trivial. As the first step, we then show that $f$ is non-imposing, so we can use Lemmas 1 and 2 to derive that $f$ (resp. the function $\hat{g}$) can be described by non-zero vectors $\hat{u}^{i,j}$. Moreover, due to independence of losers, we get that $\hat{u}^{i,j}_\ell=\hat{u}^{i,j}_{\ell'}$ for all committees $W^i,W^j\in\mathcal{W}_k$ and ballots $B(\ell),B(\ell')\in\mathcal{A}$ with $|X\cap B(\ell)|=|X\cap B(\ell')|$ for $X\in\{W^i\cap W^j,W^i\setminus W^j,W^j\setminus W^i\}$, regardless of $|B(\ell)|$ and $|B(\ell')|$. Now, if $k=1$ or $k=m-1$, we can infer the claim with an analogous reasoning as in the proof of Proposition 1. In contrast, if $1<k<m-1$, we need to relate the vectors $\hat{u}^{i,j}$ and $\hat{u}^{i',j'}$ for committees $W^i,W^j,W^{i'},W^{j'}\in\mathcal{W}_k$ with $|W^i\setminus W^j|\neq|W^{i'}\setminus W^{j'}|$.

For doing so, consider two arbitrary committees $W^i$ and $W^j$ and suppose that $|W^i\setminus W^j|=t>1$. Next, we construct a sequence of committees $W^{j_0},\dots,W^{j_t}$ by replacing the candidates in $W^i\setminus W^j$ one after another with those in $W^j\setminus W^i$. Hence, $W^i=W^{j_0}$, $W^j=W^{j_t}$, and

$|W^{j_{x-1}} \setminus W^{j_x}| = 1$ for all $x \in \{1, \dots, t\}$. Our main goal is to show that $\hat{u}^{i,j} = \delta \sum_{x=1}^{t} \hat{u}^{j_{x-1},j_x}$ for some $\delta > 0$. For proving this, we investigate the linear independence of the vectors $\hat{u}^{i,j}$ and $\hat{u}^{j_{x-1},j_x}$ for $x \in \{1, \dots, t\}$, and prove that the set $\{\hat{u}^{j_0,j_1}, \dots, \hat{u}^{j_{t-1},j_t}\}$ is linearly independent but the set $\{\hat{u}^{j_0,j_1}, \dots, \hat{u}^{j_{t-1},j_t}, \hat{u}^{i,j}\}$ is not. So, $\hat{u}^{i,j}$ is a linear combination of the vectors $\hat{u}^{i_x,i_{x+1}}$ and we only need to derive the coefficients to show our claim.

Based on this insight, we now define our score function. To this end, we let $s^1(x, y) = \hat{u}_\ell^{i,j}$ for two arbitrary committees $W^i$, $W^j$ with $|W^i \setminus W^j| = 1$ and a ballot $B(\ell)$ such that $|B(\ell) \cap W^i| = x$ and $|B(\ell) \cap W^j| = y$. Then, we define the score function $s(x)$ by $s(0) = 0$ and $s(x) = s(x-1) + s^1(x, x-1)$ for $x \geq 1$. By the additivity of the vectors $\hat{u}^{i,j}$, it follows that $\hat{u}_\ell^{i,j} = \delta(s(|W^i \cap B(\ell)|) - s(|W^j \cap B(\ell)|))$, so $\bar{R}_i^f = \{v \in \mathbb{R}^{|\mathcal{A}|} : \forall W^j \in \mathcal{W}_k : \hat{s}(v, W^i) \geq \hat{s}(v, W^j)\}$. By the definition of these sets, we infer that $f(A) = \hat{g}(v(A)) = \{W^i \in \mathcal{W}_k : v(A) \in R_i^f\} \subseteq \{W^i \in \mathcal{W}_k : v(A) \in \bar{R}_i^f\}$. Finally, continuity shows that the subset relation is an equality and independence of losers that $s$ is non-decreasing. Thus, $f$ is a Thiele rule. □

**Remark 1.** Based on Theorem 1, it is simple to prove full characterizations of specific Thiele rules. For example, it is known that `AV` is the only non-trivial Thiele rule that satisfies committee monotonicity (the winning committees of size $k$ are derived from the winning committees of size $k-1$ by only adding candidates) and based on Theorem 1, it is simple to formalize this observation. Another example is a characterization of `CCAV` by Delemazure et al. [11] within the class of Thiele rules based on mild proportionality and strategyproofness conditions, which can be turned into a full characterization based on Theorem 1. Both of these results fail within the class of ABC scoring rules and can thus not be obtained from the results of Lackner and Skowron [24].

**Remark 2.** All axioms are required for Theorem 1. If we omit independence of losers, `SAV` satisfies all remaining axioms. If we omit continuity, we can define composed Thiele rules analogous to the composed scoring rules of Young [37]: these rules refine Thiele rules by applying another Thiele rule as tie-breaker in case of multiple chosen committees. If we only omit consistency, sequential Thiele rules satisfy all given axioms. These rules compute the winning committees iteratively by always adding the candidate to a winning committee which increases the score the most. If we omit neutrality or anonymity, biased Thiele rules that double the points of every committee that contains a specific candidate or the points assigned by specific voters to the committees satisfy all other axioms.

## 3.3 BSWAV rules

Next, we discuss the characterization of BSWAV rules: these are the only ABC voting rules that satisfy anonymity, neutrality, consistency, continuity, choice set convexity, and weak efficiency. The central axiom for this characterization (aside of consistency) is choice set convexity as it enforces that candidates become exchangeable.

**Theorem 2.** *An ABC voting rule is a BSWAV rule if and only if it satisfies anonymity, neutrality, consistency, continuity, choice set convexity, and weak efficiency.*

*Proof Sketch.* First, assume that $f$ is a BSWAV rule and let $\alpha = (\alpha_1, \dots, \alpha_m)$ denote its weight vector. It is simple to verify that $f$ is neutral, anonymous, continuous, and consistent. Moreover, $f$ is weakly efficient as the weights $\alpha_i$ are all non-negative. Finally, we show that $f$ is choice set convex. For this, we consider a profile $A$ and two committees $W, W' \in f(A)$ with $|W \setminus W'| = t > 0$. Moreover, we choose two candidates $a \in W \setminus W'$ and $b \in W' \setminus W$ and let $W'' = (W \setminus \{a\}) \cup \{b\}$. The central observation is now that BSWAV scores are additive, i.e., $\hat{s}(A, W) = \sum_{x \in W} \hat{s}(A, x)$ for $\hat{s}(A, x) = \sum_{i \in N_A : x \in A_i} \alpha_{|A_i|}$. Hence, $0 \leq \hat{s}(A, W) - \hat{s}(A, W'') = \hat{s}(A, a) - \hat{s}(A, b)$ as $W \in f(A)$. By applying this argument also to $W'$ and $W''' = (W' \setminus \{b\}) \cup \{a\}$, we obtain $0 \leq \hat{s}(A, b) - \hat{s}(A, a)$, so $\hat{s}(A, a) = \hat{s}(A, b)$ and

$\hat{s}(A, W) = \hat{s}(A, W'')$. This proves that $W'' \in f(A)$ and by repeating the argument, we infer that $\bar{W} \in f(A)$ for all $\bar{W}$ with $W \cap W' \subseteq \bar{W} \subseteq W \cup W'$.

For the converse direction, we give again only a rough proof sketch and note that the outline of this proof is essentially the same as for Theorem 1 as only the technical details differ. Now, suppose that $f$ is an ABC voting rule which satisfies all given axioms. If $f$ is the trivial rule, it is the BSWAV rule defined by $\alpha_i = 0$ for $i \in \{1, \dots, m\}$. On the other hand, if $f$ is non-trivial, we show that it is non-imposing and then apply Lemmas 1 and 2. If $k = 1$ or $k = m-1$, the theorem follows from Proposition 1 as choice set convexity becomes trivial and the set of BSWAV rules coincides with the set of ABC scoring rules. Hence, suppose that $1 < k < m-1$. We again prove that the vectors $\hat{u}^{i,j}$ for $|W^i \setminus W^j| > 1$ can be represented as scaled sum of vectors $\hat{u}^{j_{x+1}, j_x}$ for committees $W^{j_{x-1}}, W^{j_x}$ with $|W^{j_{x-1}} \setminus W^{j_x}| = 1$. For showing this, we infer from choice set convexity that there are $\alpha_x$ such that $\hat{u}^{i,j}_\ell = \alpha_x$ for all committees $W^i, W^j$ and ballots $B(\ell)$ with $|W^i \setminus W^j| = 1$, $|W^i \cap B(\ell)| > |W^j \cap B(\ell)|$, and $|B(\ell)| = x$. Based on this, we derive the score function $s$ with an analogous approach as for Thiele rules. As last step, we then show $f$ is the BSWAV rule described by the score function that $s(|W \cap B(\ell)|, |B(\ell)|) = \alpha_{|B(\ell)|}|W \cap B(\ell)|$ for all committees $W$ and ballots $B(\ell)$. □

**Remark 3.** All axioms are required for Theorem 2. For anonymity, neutrality, and continuity, we can define examples similar to the ones given for Thiele rules. When omitting consistency, the "convex hull" of Phragmen's rule satisfies all remaining axioms. Theorem 2 of Peters and Skowron [29] then shows that this rule cannot be represented as ABC scoring rule and therefore also not as BSWAV rule. When only omitting weak efficiency, "inverse" `AV`, which chooses the committees with minimal approval scores, satisfies all given axioms. Finally, every Thiele rule other than `AV` only fails choice set convexity.

# 4 Characterizations of `AV`, `PAV`, and `SAV`

Finally, we use Theorems 1 and 2 to characterize three specific ABC voting rules, namely `AV`, `SAV`, and `PAV`. While `AV` and `PAV` can also be characterized by combining Theorem 1 with results from the literature, we prefer to give own characterizations of these rules. For `AV`, we do so as the characterization follows naturally from our results and for `PAV` since our characterization highlights a new aspect of this rule. To keep the theorems short, we characterize these rules only within the class of Thiele rules or BSWAV rules; Theorems 1 and 2 then generalize them to full characterizations. Due to space restrictions we defer all proofs to the appendix and give proof sketches instead.

For the characterization of `AV`, we note that this rule and the trivial rule are the only ABC voting rules that are both Thiele rules and BSWAV rules if $k < m-1$. On the other hand, this claim fails if $k = m-1$ because choice set convexity then becomes trivial and all Thiele rules satisfy the given axioms. These insights entail the following theorem.

**Theorem 3.** *Assume $k \leq m-2$. AV is the only non-trivial Thiele rule that satisfies choice set convexity and the only non-trivial BSWAV rule that satisfies independence of losers.*

*Proof Sketch.* First, we note that `AV` is both a Thiele rule and a BSWAV rule and the direction from left to right thus follows from Theorems 1 and 2. For the converse direction, we observe that both claims of the theorem are equivalent because of these theorems. Hence, suppose that $f$ is a non-trivial BSWAV rule satisfying independence of losers and let $\alpha$ denote its scoring vector. By non-triviality, there is $\ell \in \{1, \dots, m-1\}$ such that $\alpha_\ell > 0$. Next, we consider two committees $W, W'$ with $|W \setminus W'| = 1$ and construct a profile $\bar{A}$ such that $f(\bar{A}) = \{W, W'\}$. Furthermore, we let $A_i$ denote a ballot such that $|A_i| = \ell$, $W \setminus W' \subseteq A_i$, $(W' \setminus W) \cap A_i = \emptyset$ and there is $z \in A_i \setminus (W \cup W')$. Moreover, let $A_j = (A_i \setminus (W \setminus W')) \cup (W' \setminus W)$. By continuity, anonymity, neutrality and consistency, we show that there is a $\lambda \in \mathbb{N}$ such that $f(\lambda\bar{A} + A_i + A_j) = \{W, W'\}$. Next, independence of losers entails that $\{W, W'\} \subseteq f(\lambda\bar{A} + A_i \setminus \{z\} + A_j)$ as $z \notin W \cup W'$. From this, we infer that $\alpha_{\ell-1} = \alpha_\ell$ for all $\ell \in \{2, \dots, m-1\}$, which shows that $f$ is AV. □

Next, we turn to the characterizations of SAV and PAV, for which we focus on party-list profiles. In a party-list profile $A$, the candidates are partitioned into parties $\mathcal{C} = P_1 \cup \dots \cup P_\ell$ and every voter $i \in N_A$ supports a single party by approving all of its members, i.e., for all $i \in N_A$, there is a party $P_j$ such that $A_i = P_j$. For such profiles, it is a natural question whether it make sense for individual candidates to form a party or to compete by themselves, and the answer to this clearly depends on the voting rule at hand. For instance, consider the profiles $A^1$, $A^2$, and $A^3$ shown below and assume that $k = 3$. Moreover, we assume that the candidates in $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3\}$, respectively, present rather similar positions. In $A^1$, where all candidates compete by themselves, most voting rules will elect the committee $\{a_1, a_2, a_3\}$. In contrast, AV chooses the committee $\{b_1, b_2, b_3\}$ for $A^2$ and it thus makes sense for the candidates $\{b_1, b_2, b_3\}$ to form a party. On the other hand, CCAV will choose every committee $W$ with $|W \cap \{a_1, a_2, a_3\}| = 1$ and $|W \cap \{b_1, b_2, b_3\}| = 2$ for $A^3$ and it thus makes sense for the candidates in $A$ to compete by themselves.

| | | | | | | |
|---|---|---|---|---|---|---|
| $A^1$: | 2: $\{a_1\}$ | 2: $\{a_2\}$ | 2: $\{a_3\}$ | 1: $\{b_1\}$ | 1: $\{b_2\}$ | 1: $\{b_3\}$ |
| $A^2$: | 2: $\{a_1\}$ | 2: $\{a_2\}$ | 2: $\{a_3\}$ | | 3: $\{b_1, b_2, b_3\}$ | |
| $A^3$: | | 6: $\{a_1, a_2, a_3\}$ | | 1: $\{b_1\}$ | 1: $\{b_2\}$ | 1: $\{b_3\}$ |

We believe that such strategic considerations of candidates about whether to compete as a group or individually are undesirable. Hence, we introduce next the concept of split/merge-proofness which aims to prohibit this behavior. Informally, this axiom requires that it should not matter whether there are $j$ candidates that are each approved by $\ell$ voters or a party of $j$ candidates that is approved by $j\ell$ voters. To formalize this idea, we define the profile $A^X$ given a party-list profile $A$ and a set of parties $X = \{P_1, \dots, P_j\}$ as follows: $N_{A^X} = N_A$ and $A^X_i = A_i$ if $A_i \notin X$ and $A^X_i = \bigcup X$ if $A_i \in X$ for all $i \in N_A$.

**Split/Merge-proofness.** An ABC voting rule $f$ is *split/merge-proof* if $f(A) = f(A^X)$ for all party-list profiles $A \in \mathcal{A}^*$ with parties $\mathcal{P} = \{P_1, \dots, P_\ell\}$ and all sets $X \subseteq \mathcal{P}$ such that $|P_j| = |P_{j'}| = 1$ and $|\{i \in N_A : A_i = P_j\}| = |\{j \in N_A : A_i = P_{j'}\}|$ for all $P_j, P_j' \in X$.

We note that similar conditions have been studied for the closely related model of apportionment (where voters vote for a single party which is large enough to fill up the full committee) [4]. Moreover, this condition is also connected to the study of clones in elections as we can view large parties also as sets of clones [15, 27].

We next show that this axiom characterizes SAV within the class of BSWAV rules.

**Theorem 4.** *SAV is the only non-trivial BSWAV rule that satisfies split/merge-proofness.*

*Proof Sketch.* First, we note that SAV is split/merge-proof because a candidate receives $\ell$ points both if it is uniquely approved by $\ell$ voters and if it is approved by $j\ell$ voters who approve $j$ candidates. Hence, merging candidates into a party does not change their scores and thus also not the scores of the committees and the final outcome. For the other direction, we consider an arbitrary non-trivial and split/merge-proof BSWAV rule $f$ and its weight vector $\alpha = (\alpha_1, \dots, \alpha_m)$. Since $f$ is non-trivial, there is $\ell \in \{1, \dots, m-1\}$ such that $\alpha_\ell > 0$. Based on split/merge-proofness, it is easy to show that also $\alpha_1 > 0$. Since $f$ is invariant under scaling $\alpha$, we assume next that $\alpha_1 = 1$. Now, suppose for contradiction that $f$ is not SAV, i.e., that there is $\ell \in \{1, \dots, m-1\}$ with $\alpha_\ell \neq \frac{1}{\ell}$. We derive a contradiction to this by constructing a profile in which split/merge-proofness is violated. For instance, if $\alpha_\ell < \frac{1}{\ell}$, we define $\Delta = \frac{1}{\ell} - \alpha_\ell$, $B = \{c_1, \dots, c_\ell\}$, and choose $t \geq 2$ such that $t\ell\Delta > 1$. Then, we consider the profile $A$ in which each candidate $c_i \in B$ is uniquely approved by $t$ voters and all candidates $c_j \in \mathcal{C} \setminus B$ are uniquely approved by $t-1$ voters. In this profile, $f$ chooses the committees $W$ that maximize $|W \cap B|$, but if the voters $i \in N_A$ with $A_i \subseteq B$ change their ballot to $B$, this is no longer true and split/merge-proofness is violated. □

Finally, for the characterization of PAV, we observe that no Thiele rule satisfies split/merge-proofness. We thus consider a weakening of this axiom.

**Weak Split/Merge-proofness.** An ABC voting rule $f$ is *weakly split/merge-proof* if $f(A) = f(A^X)$ for all party-list profiles $A$ with parties $\mathcal{P} = \{P_i, \dots, P_\ell\}$ and sets of parties $X \subseteq \mathcal{P}$ such that $|P_j| = |P_{j'}| = 1$ and $|\{i \in N_A \colon A_j = P_j\}| = |\{i \in N_A \colon A_j = P_{j'}\}|$ for all $P_j, P_{j'} \in X$, and $\bigcup X \subseteq W$ for all $W \in f(A)$ or $\bigcup X \subseteq W$ for all $W \in f(A^X)$. Less formally, weak split/merge-proofness weakens full split/merge-proofness as it only applies if all candidates in the split/merged parties are guaranteed to be elected in one of the profiles.

We next characterize PAV within the class of non-trivial Thiele rules based on this axiom.

**Theorem 5.** *PAV is the only non-trivial Thiele rule that satisfies weak split/merge-proofness.*

*Proof Sketch.* First, for showing that PAV satisfies weak split/ merge-proofness, consider a party-list profile $A$ with parties $\mathcal{P} = \{P_1, \dots, P_j\}$ and let $X \subseteq \mathcal{P}$ denote a set of singleton parties with $|\{i \in N_A \colon A_i = P_{j_1}\}| = |\{i \in N_A \colon A_i = P_{j_2}\}| = c$ for all $P_{j_1}, P_{j_2} \in X$. Moreover, let $\hat{X} = \bigcup_{P_i \in X} P_i$ and consider the profile $A^X$ derived from $A$ by merging the parties in $X$. The key point of our argument is that a party $P_i \in X$ contributes $c$ points to an committee in $A$ and the last member of the party $\hat{X}$ contributes $c|\hat{X}|/|\hat{X}| = c$ points to a committee in $A^X$. From this insight, we then infer that $\hat{X} \subseteq W$ for all $W \in f(A)$ if and only if $\hat{X} \subseteq W$ for all $W \in f(A^X)$. Since the scores of the remaining parties is the same in $A$ and $A^X$, PAV is therefore weakly split/merge-proof. For the converse direction, we suppose for contradiction that $f$ is a non-trivial Thiele rule other than PAV that satisfies weak split/merge-proofness. Similar to the proof of Theorem 4, we can then construct profiles in which weak split/merge-proofness is violated as there is a minimal index $\ell$ such that $s(\ell) \neq \sum_{x=1}^{\ell} \frac{1}{x}$. Hence, if we merge $\ell$ parties of size $j$, the contribution of the last member in the new party changes, which can be used to derive a contradiction. □

# 5 Conclusion

In this paper, we axiomatically characterize two important classes of approval-based committee (ABC) voting rules, namely Thiele rules and BSWAV rules. Thiele rules choose the committees that maximize the total score according to a score function that only depends on the intersection size of the considered committee and the ballots of the voters. On the other hand, BSWAV rules are a new generalization of multi-winner approval voting which weight voters depending on the size of their ballot. For both of our characterizations, the central axiom is consistency which has famously been used by Young [37] for a characterization of single-winner scoring rules or by Lackner and Skowron [23] for a characterization of ABC scoring rules in the context of committee ranking rules. In particular, our results allow for simple characterizations of all important ABC scoring rules as all such rules belong to one of our classes. We also demonstrate this point by characterizing the well-known ABC voting rules AV, SAV, and PAV. In particular, the result for SAV is, to the best of our knowledge, the first full characterization of this rule. Figure 1 shows a more detailed overview of our results.

Finally, our paper offers several directions for future work. Firstly, characterizations of many ABC voting rules (e.g., Phragmén's rule and the method of equal shares) are still missing and some of our ideas might be helpful to derive such results. Secondly, even though all relevant ABC scoring rules belong to one of our classes, we would find a full characterization of the set of ABC scoring rules still interesting.

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