Toward Fair and Strategyproof Tournament Rules for Tournaments with Partially Transferable Utilities

David M. Pennock, Ariel Schvartzman, Eric Xue[1]

Abstract

A tournament on n agents is a complete oriented graph with the agents as vertices and edges that describe the win-loss outcomes of the $\binom{n}{2}$ matches played between each pair of agents. The winner of a tournament is determined by a *tournament rule* that maps tournaments to probability distributions over the agents. We want these rules to be fair (i.e., choose a high-quality agent) and robust to strategic manipulation. Prior work has shown that under minimally fair rules, manipulations between two agents can be prevented when utility is nontransferable but not when utility is completely transferable. We introduce a partially transferable utility model that interpolates between these two extremes using an selfishness parameter λ. Our model is that an agent may be willing to lose on purpose, sacrificing some of her own chance of winning, but only if the colluding pair's joint gain is more than λ times the individual's sacrifice.

We prove that no fair tournament rule can prevent manipulations when $\lambda < 1$. We computationally solve for fair and manipulation-resistant tournament rules for $\lambda = 1$ for up to 6 agents. We conjecture and leave as a major open problem that such a tournament rule exists for all n. We analyze the trade-offs between relative and absolute approximate strategyproofness for all rules previously studied in related settings and derive as a corollary that all of these rules require $\lambda = \Omega(n)$ to be robust to manipulation. We show that for stronger notions of fairness, non-manipulable tournament rules are closely related to tournament rules that witness decreasing gains from manipulation as the number of agents increases.

1 Introduction

A tournament on n agents is a complete oriented graph in which the agents are vertices and an edge from agent i to agent j means "agent i defeats agent j". These structures frequently arise in sports as the outcome of $\binom{n}{2}$ pairwise matches between n agents or teams. However, tournaments can arise whenever the performance of every two agents is comparable (e.g., agents are candidates in an election and edges are pairwise majority votes).

A tournament rule maps a tournament to a probability distribution over the agents. These probabilities encode the likelihood that each agent is declared the tournament winner, or prescribe how to divide up a monetary reward [7]. While a tournament rule should be fair in that it chooses some qualified agent who beats many other agents, it also should not reward manipulations: for example, losing a match on purpose should not improve an agent's or their co-conspirator's chances of winning the tournament. If the rule is manipulable, then agents may act in ways that undermine the primary goal of choosing a highly qualified winner. In fact, instances of these actions are not unheard of in sports. At the London 2012 Olympic Games, four women's doubles teams were disqualified for attempting to throw their final matches in the round-robin group stage in order to earn a more favorable seed in the knockout stage of the tournament.

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[1] Authors listed in alphabetical order.

Unfortunately, prior work has shown that fairness and non-manipulability are largely incompatible. A prevailing notion of fairness studied by prior work [1, 2, 6, 9, 10] is Condorcet consistency. A tournament rule is Condorcet consistent if, whenever one agent beats all other agents, the undefeated agent wins the tournament with certainty.

Altman et al. [2] showed that any deterministic rule that satisfies this notion is susceptible to pairwise manipulations. In other words, for any Condorcet-consistent rule, there exist tournaments in which two agents can influence the choice of winner by colluding to reverse the outcome of their match.

Altman and Kleinberg [1] extended this work to randomized rules that map tournaments to probability distributions over agents. They showed that there exist Condorcet consistent and pairwise non-manipulable rules when two agents collude only if one of them can strictly improve her probability of winning at no cost to the other. Rules that are pairwise non-manipulable under this assumption are said to be 2-Pareto non-manipulable (2-PNM). However, no Condorcet consistent rule exists when utility is completely transferable—that is, when two agents only care about the probability that at least one of them wins the tournament. Instead, Altman and Kleinberg [1] demonstrated rules that are approximately Condorcet consistent and pairwise non-manipulable in this setting, which the authors term 2-strongly non-manipulable (2-SNM). Another line of work [6, 9, 10] sought rules that were fair and approximately 2-SNM.

Motivated by the fact that collusion and the deliberate throwing of matches in sports occur less frequently than the negative results of prior work imply, we extend prior work to the setting in which utility is partially transferable. These settings are natural: agents value their own probability of winning *and* their collusion partner's probability of winning in some ratio. Agents are not completely altruistic to their partner (fully transferable utility) nor completely selfish (non-transferable utility). Agents care about winning themselves and will only sacrifice their own probability if it achieves a significant proportional gain for their partner. We extend prior notions of non-manipulability by introducing a term that accounts for the range of selfishness of agents. More specifically, we say a rule is 2-NM_λ if no agent can collude with another to improve her probability of winning by at least a $\lambda + 1$ factor of the decrease in probability witnessed by her collusion partner. In other words, in a model where an agent won't sacrifice herself unless her partner gains at least $\lambda+1$ times the amount she loses, no pairwise collusion would occur under a 2-NM_λ rule.

We show that this model connects the notions of Pareto and strong non-manipulability by varying λ. Moreover, we conjecture that there exists a tournament rule that is monotone, Condorcet consistent, and 2-NM_1, implying that it is possible to prevent deliberate loss and collusion, as long as each agent weighs her own probability of winning twice as much as her opponents'. However, we show that none of the rules proposed in five previous papers [1, 3, 6, 9, 10] satisfy this combination of conditions by demonstrating how these rules trade-off between λ, our notion of relative approximate strategyproofness, and the established notion of absolute approximate strategyproofness [9].

In a separate direction, we introduce another notion of fairness, termed dominant sub-tournament consistency (DSTC), and show that several natural rules satisfy this condition. Intuitively, a rule is DSTC if the addition of an agent that loses to the original agents does not affect their probabilities. A closely related notion is top cycle consistency (TCC), which requires the winner to come from the top cycle with certainty. We show that within these notions of fairness, the problem of finding a rule that is 2-NM_λ reduces to the problem of finding a rule that witnesses gains from manipulation that vanishes as the number of agents increases.

1.1 Related Work

For a broad discussion of recent developments on tournaments in computational social choice, see Suksompong's excellent survey [11]. We discuss work closely related to ours.

Altman and Kleinberg [1] and Altman, Procaccia, and Tennenholtz [2] were the first to consider the question of strategic manipulations of tournaments by agents. Their main conclusion is that Condorcet consistency and strong non-manipulability are directly at odds: no tournament rule, even randomized ones, can satisfy both properties. Later, Schneider, Schvartzman, and Weinberg [9] considered a relaxation of the problem: they sought tournament rules that are Condorcet consistent and are minimally manipulable. Their main result is that the Randomized Single Bracket Elimination (RSEB) rule is 2-SNM-1/3, meaning that the most probability that any pair can gain is 1/3, and this is optimal among all Condorcet-consistent rules. This result was later strengthened to show that the Randomized King of the Hill (RKotH) rule is also 2-SNM-1/3 and cover consistent, a notion strictly stronger than Condorcet consistent [10]. Recent discoveries include a rule that is 3-SNM-31/60, meaning that the most probability that any coalition of three agents can gain is 31/60, the first explicit rule that is 3-SNM-α for $\alpha < 1$ [5], and a different rule that is 3-SNM-1/2 [8]. Parallel lines of work have considered variations on this problem, including probabilistic tournaments [6] and tournaments with prize vectors for multiple places rather than only one prize for the winner [4].

2 Preliminaries

Definition 2.1 (Tournament)**.** A tournament is a pair $T = (A, \succ_T)$ where A is a finite set of agents and $\succ_T$ is a collection of ordered pairs of agents such that T constitutes a complete oriented graph with vertices in A and edges in $\succ_T$. Intuitively, T describes the outcomes of the $\binom{|A|}{2}$ matches played between each pair of distinct agents. We write $i \succ_T j$ and say that i defeats j in T if $(i, j) \in \succ_T$. Let $\mathcal{T}_n$ denote the set of tournaments where $[n]$ is the set of agents.

Definition 2.2 (Tournament rule)**.** A tournament rule on n agents $r^{(n)} : \mathcal{T}_n \to \Delta^n$ maps a tournament to a probability distribution over the agents. A tournament rule r is a family of tournament rules on n agents $\{r^{(n)}\}_{n=1}^{\infty}$. For $T \in \mathcal{T}_n$, let $r(T) := r^{(n)}(T)$ and for $i \in [n]$, let $r_i(T) := \mathbb{P}[r(T) = i]$ denote the probability that i wins T under r.

2.1 Fairness Properties

A desirable tournament rule should choose the most qualified agent as the winner of a tournament. In line with this reasoning, we want a tournament rule to choose an undefeated agent with probability 1 since this agent is clearly better than the rest of her opponents.

Definition 2.3 (Condorcet consistency)**.** A tournament rule r is Condorcet consistent (CC) if for all $n \in \mathbb{N}$ and $T \in \mathcal{T}_n$, $r_i(T) = 1$ whenever there exists $i \in [n]$ such that $i \succ_T j$ for all $j \in [n] \setminus \{i\}$.

Note that Condorcet consistency is quite a minimal notion of fairness since it is binding only when there is an agent that is clearly superior than the others. Unfortunately, it is often the case that no such agent exists. The following notions of fairness seek to restrict the subset of agents that should be named the winner in such cases by eliminating those who are in some sense clearly worse than her opponents.

Definition 2.4 (Dominant sub-tournament)**.** For $S \subseteq [n]$ and a tournament $T \in \mathcal{T}_n$, the sub-tournament $T[S]$ is the subgraph induced by S. $T[S]$ is a dominant sub-tournament if $i \succ_T j$ for all $i \in S, j \in [n] \setminus S$.

Definition 2.5 (Top cycle consistency)**.** Given a tournament T, the top cycle $TC(T)$ is the minimal dominant sub-tournament of T. The top cycle of a tournament always exists and is unique. A tournament rule r is top cycle consistent (TCC) if $r_i(T) = 0$ for all $n \in \mathbb{N}$, $T \in \mathcal{T}_n$, and $i \in [n] \setminus TC(T)$.

Top cycle consistency extends Condorcet consistency quite naturally: Condorcet consistency requires that an undefeated agent be declared the winner, while top cycle consistency requires this winner to come from the smallest undefeated subset. Moreover, since the agents in the top cycle are undefeated by those outside of the top cycle, they are in some sense better. On the other hand, no agent in the top cycle is clearly superior than the others since every agent in the top cycle is defeated by another in the top cycle.

Definition 2.6 (Cover consistency)**.** For $i, j \in [n]$, we say i covers j if (1) $i \succ_T j$ and (2) $j \succ_T k \implies i \succ_T k$ for all $k \in [n] \setminus \{i, j\}$. Moreover, we say j is covered if there exists i such that i covers j. A tournament rule r is cover consistent (CovC) if for all $n \in \mathbb{N}$ and $T \in \mathcal{T}_n$, $r_j(T) = 0$ whenever j is covered.

Cover consistency refines top cycle consistency by further restricting the set of potential winners. If i covers j, then not only did i defeat j, but i also defeated everyone that j defeated. Thus, covered agents are worse than the agents that cover them in some sense.

Definition 2.7 (Dominant sub-tournament consistency)**.** A tournament rule r is dominant sub-tournament consistent (DSTC) if $r_i(T[S]) = r_i(T)$ for all $n \in \mathbb{N}$, $T \in \mathcal{T}_n$, $S \subseteq [n]$ such that $T[S]$ is a dominant sub-tournament of T, and $i \in S$.

Dominant sub-tournament consistency strengthens top cycle consistency in a different direction than cover consistency. Rather than narrow down the set of potential winners, dominant sub-tournament consistency requires that the probability of choosing a certain member of the top cycle as the winner is the same as the probability of choosing her if the agents outside the top cycle were removed. To the best of our knowledge, dominant sub-tournament consistency has not been considered before in the tournament literature.

The following result formalizes the hierarchy of fairness conditions.

Proposition 2.8 (Fairness hierarchy)**.** *Any tournament rule that satisfies either cover consistency or dominant sub-tournament consistency also satisfies top cycle consistency. Moreover, any top cycle consistent tournament rule is also Condorcet consistent.*

Proof. Let $T \in \mathcal{T}_n$ and let r be a cover consistent tournament rule. Let $j \notin TC(T)$ and consider any $i \in TC(T)$. We show that i covers j. Since $i \in TC(T)$ and $j \notin TC(T)$, $i \succ_T j$. Moreover, since $j \notin TC(T)$, for all $k \in [n]$ such that $j \succ_T k$, $k \notin TC(T)$. Thus, for all such k, since $i \in TC(T)$ and $k \notin TC(T)$, $i \succ_T k$, so i covers j. Since r is cover consistent, $r_j(T) = 0$. Our choice of $j \notin TC(T)$ was arbitrary, so $r_j(T) = 0$ for all $j \notin TC(T)$ and r is top cycle consistent.

If r is dominant sub-tournament consistent, then since $TC(T)$ is the smallest dominant sub-tournament of T, $r_i(T) = 0$ for all $i \notin TC(T)$. Thus, r is top cycle consistent.

Now, let r be top cycle consistent. Note that whenever some agent i is undefeated in T, $TC(T) = \{i\}$. Certainly, i dominates every $j \in [n] \setminus \{i\}$ and no proper subset of $\{i\}$ satisfies this property. Thus, $r_i(T) = 1$ and r is Condorcet consistent. □

2.2 Non-Manipulability Properties

In addition to satisfying some notion of fairness, tournament rules should be robust to manipulation. In this work, we consider manipulations where a single agent purposefully loses her match against one of her opponents and manipulations where two agents collude to reverse the outcome of their match.

Definition 2.9 (S-adjacent)**.** Given $S \subseteq [n]$, tournaments $T, T' \in \mathcal{T}_n$ are S-adjacent if $i \succ_T j \iff i \succ_{T'} j$ for $i \neq j \in [n] \setminus S$. In other words, T and T' are S-adjacent if they coincide on every match except possibly those between agents in S.

When utilities are nontransferable, two agents are willing to collude only if one of them can strictly improve her probability of winning at no cost to the other. Formally, distinct agents $i, j \in [n]$ collude from T to T' only if $\max\{r_i(T') - r_i(T), r_j(T') - r_j(T)\} > 0$ and $\min\{r_i(T') - r_i(T), r_j(T') - r_j(T)\} \geq 0$. Thus, to incentivize agents against such manipulations, a tournament rule must satisfy the following notion of non-manipulability.

Definition 2.10 (Pareto non-manipulability)**.** A tournament rule r is 2-Pareto non-manipulable (2-PNM) if for all $i \neq j \in [n]$ and $\{i, j\}$-adjacent $T, T' \in \mathcal{T}_n$, either (1) $\min\{r_i(T') - r_i(T), r_j(T') - r_j(T)\} < 0$ or (2) $\max\{r_i(T') - r_i(T), r_j(T') - r_j(T)\} \leq 0$.

Altman and Kleinberg [1] give rules that are monotone, top cycle consistent, and 2-PNM. The barrier to pairwise manipulation is much lower when utilities are completely transferable since two agents only care about the probability that at least one of them wins the tournament. In other words, i and j collude from T to T' only if $r_i(T') + r_j(T') > r_i(T) + r_j(T)$. Under this utility model, an agent may be willing to sacrifice and shift a significant portion of her probability to her partner in crime. Thus, tournament rules must satisfy a stronger notion of non-manipulability in this setting.

Definition 2.11 (Strong non-manipulability)**.** A tournament rule r is 2-strongly non-manipulable (2-SNM) if $r_i(T') + r_j(T') \leq r_i(T) + r_j(T)$ for all $i \neq j \in [n]$ and $\{i, j\}$-adjacent $T, T' \in \mathcal{T}_n$.

Prior work has shown that no Condorcet consistent tournament rule is 2-SNM. However, despite this strong impossibility result, instances of collusion are relatively infrequent in the real world, suggesting that settings in which utilities are completely transferable are uncommon. On the other hand, instances of collusion are not unheard of, suggesting that utility is neither always nontransferable.

In this paper, we consider a third utility model in which utilities are partially transferable: distinct agents i and j collude from T to T' only if $r_i(T') + r_j(T') > r_i(T) + r_j(T) + \lambda \max\{r_i(T) - r_i(T'), r_j(T) - r_j(T')\}$. One interpretation of this condition is that agents would rather win the tournament themselves but are still willing to collude if the gain in probability is significantly larger than each agent's loss. Here, λ is a parameter that measures how transferable utility is, with smaller λ corresponding to greater transferability. We will later see how λ can be interpreted as agents' level of selfishness. We now define a notion of non-manipulability for this model.

Definition 2.12 (Non-manipulability for λ)**.** A tournament rule r is 2-non-manipulable for $\lambda \geq 0$ (2-NM$_\lambda$) if $r_i(T') + r_j(T') \leq r_i(T) + r_j(T) + \lambda \max\{r_i(T) - r_i(T'), r_j(T) - r_j(T')\}$ for all $i \neq j \in [n]$ and $\{i, j\}$-adjacent $T, T' \in \mathcal{T}_n$. We say r is 2-NM$_\infty$ if it is 2-NM$_\lambda$ as $\lambda \to \infty$.

When $\lambda = 0$, our notion of non-manipulability coincides with strong non-manipulability. We show that our notion coincides with Pareto non-manipulability when $\lambda = +\infty$. Thus, our notion of non-manipulability generalizes strong and Pareto non-manipulability while connecting the two.

Proposition 2.13. *A tournament rule is 2-*PNM *if and only if it is 2-*NM$_\infty$.

As in previous work [9, 10], we are interested in approximately non-manipulable tournament rules; that is, rules under which no two agents can collude to gain in joint probability more than α more than each agent's loss (weighted by λ).

Definition 2.14 (2-non-manipulability up to α for λ)**.** A tournament rule r is 2-non-manipulable up to α for $\lambda \geq 0$ (2-NM$_\lambda$-α) if $r_i(T') + r_j(T') \leq r_i(T) + r_j(T) + \lambda \max\{r_i(T) - r_i(T'), r_j(T) - r_j(T')\} + \alpha$ for all $i \neq j \in [n]$ and $\{i,j\}$-adjacent $T, T' \in \mathcal{T}_n$.

In addition to being robust against pairwise manipulations, a tournament rule should be robust to the intentional throwing of matches.

Definition 2.15 (Monotonicity)**.** A tournament rule is monotone if $r_i(T) \geq r_i(T')$ for all $i \neq j \in [n]$ and $\{i,j\}$-adjacent tournaments $T \neq T' \in \mathcal{T}_n$ such that $i \succ_T j$.

Intuitively, monotonicity says that no agent should be able to improve her chances of winning by deliberately losing one of her matches. Thus, agents have an incentive to win each of their matches under monotone rules. Violations of this property should be seen as quite severe.

Proposition 2.16. *Let r be a 2-*NM$_\lambda$ *tournament rule for some $\lambda > 0$, then the following two statements are equivalent.*

1. *r is monotone*
2. *For all $i \neq j \in [n]$ and $\{i,j\}$-adjacent $T \neq T' \in \mathcal{T}_n$ such that $i \prec_T j$, $r_i(T') - r_i(T) \leq (\lambda + 1)(r_j(T) - r_j(T'))$*

Proposition 2.16 offers a natural interpretation of the parameter λ and the 2-NM$_\lambda$ property for monotone tournament rules: λ is how much each agent weighs her own probability of winning over her opponents' and a tournament rule is 2-NM$_\lambda$ if switching the outcome of a match does not increase the probability of winning for the new winner by more than a $\lambda + 1$ factor the loss of the new loser. Note that Proposition 2.16 does not hold for $\lambda = 0$. Indeed, monotonicity and 2-SNM are independent properties: neither implies the other.

3 Trade-off Between α and λ: Lower Bounds

3.1 A Universal Lower Bound

Theorem 3.1. *No Condorcet consistent tournament rule is 2-*NM$_\lambda$-*α for $\lambda < 1 - 3\alpha$.*

Schneider et al. [9] showed a similar result that said no Condorcet consistent tournament rule is 2-NM$_0$-α for $\alpha < 1/3$. The same lower bound construction yields Theorem 3.1. We prove the theorem for monotone rules, but with the ideas presented here and some additional casework, one can extend the result to non-monotone rules.

Proof. Suppose tournament rule r is CC and 2-NM$_\lambda$-α, and consider any tournament T on $[n]$ in which 1 dominates 2, 2 dominates 3, and 3 in turn dominates 1. Note that any two agents among $\{1,2,3\}$ can collude so that one of them becomes undefeated. Since r is monotone, Condorcet consistent, and 2-NM$_\lambda$-α,

$$1 - r_2(T) \leq (\lambda + 1)r_1(T) + \alpha$$
$$1 - r_3(T) \leq (\lambda + 1)r_2(T) + \alpha$$
$$1 - r_1(T) \leq (\lambda + 1)r_3(T) + \alpha$$

Adding these three inequalities together and isolating λ yields

$$\tfrac{3(1-\alpha)}{r_1(T)+r_2(T)+r_3(T)} - 2 \leq \lambda$$

Since $r_1(T) + r_2(T) + r_3(T) \leq 1$, this inequality imply that $\lambda \geq 1 - 3\alpha$. $\square$

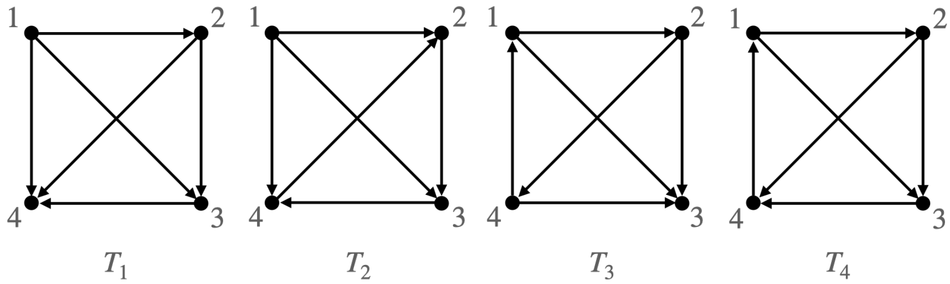

Figure 1: All non-isomorphic tournaments on 4 agents. The following conditions are necessary and sufficient for a tournament rule on 4 agents r to be Condorcet-consistent and 2-NM$_1$. In T_1 and T_2, r chooses 1 as the winner with probability 1. In T_3, r chooses the winner uniformly at random among 1, 2, and 4. In T_4, r chooses the winner according to a distribution that is a convex combination of $\left(\frac{4}{9}, \frac{2}{9}, 0, \frac{3}{9}\right)$, $\left(\frac{5}{9}, \frac{1}{9}, 0, \frac{3}{9}\right)$, $\left(\frac{13}{33}, \frac{8}{33}, \frac{2}{33}, \frac{10}{33}\right)$, $\left(\frac{5}{12}, \frac{13}{48}, \frac{1}{48}, \frac{7}{24}\right)$, $\left(\frac{11}{21}, \frac{4}{21}, \frac{1}{21}, \frac{5}{21}\right)$, and $\left(\frac{17}{39}, \frac{7}{39}, \frac{4}{39}, \frac{11}{39}\right)$.

Corollary 3.2. *No Condorcet consistent tournament rule is 2-*NM$_\lambda$ *for* $\lambda < 1$.

In fact, we believe this lower bound is tight. That is, we believe that there exists a monotone and Condorcet consistent tournament rule that is 2-NM$_1$. Thus, pairwise collusion can be prevented without sacrificing fairness as long as agents prefer not to collude if their sacrifice in probability is greater than the joint gain. Figure 1 shows such a rule for 4 agents. Expressing and computationally solving the problem as a feasibility linear program show that such rules exist for tournaments of up to 6 agents. Unfortunately, as the number of tournaments on n agents grows exponentially with n, it became computationally difficult to check whether such rules exist for tournaments of larger size.

Conjecture 3.3. *There exists a monotone, Condorcet consistent, 2-*NM$_1$ *tournament rule.*

3.2 Lower Bounds For Specific Tournament Rules

We now consider several tournament rules, analyze their fairness properties, and examine their trade-off between α and λ. Table 1 provides a summary of our findings. For rules that have been previously analyzed in related settings, we cite the relevant references and list what was known about them prior to this work (to the best of our knowledge). Interestingly, the following tournament (and its variants) identified by Schneider et al. [9] is responsible for all our trade-off lower bounds, suggesting that it is especially problematic.

Definition 3.4 (Superman-kryptonite tournament)**.** The superman-kryptonite tournament on $[n]$ has $i \succ j$ whenever $i < j$ except $n \succ 1$. In other words, agent 1, the superman, defeats all agents but agent n, the kryptonite, and agent n loses to all agents except agent 1.

Iterative Condorcet Rule [1]

The Iterative Condorcet Rule (ICR) chooses the undefeated agent if one exists. Otherwise, eliminate an agent uniformly at random and repeat. Equivalently, ICR chooses an ordering of the agents uniformly at random and eliminates agents in order until an agent undefeated by the remaining agents exists.

Theorem 3.5. *ICR is monotone, dominant sub-tournament consistent, and* 2-PNM. *ICR is not cover consistent. If ICR satisfies* 2-NM$_\lambda$-α *for some* $\lambda \geq 0$*, then* $\lambda \geq (\frac{1}{2} - \alpha)\Omega(n^2)$.

Table 1: Performance summary

Rule	Monotone?	Fairness	2-NM$_\lambda$-α	2-NM$_\lambda$-$f(n)$
ICR [1]	Yes [1]	DSTC	$\lambda \geq \left(\frac{1}{2} - \alpha\right)\Omega(n^2)$	$f(n) \geq 1/2$
RVC [1]	Yes [1]	DSTC	$\lambda \geq \left(\frac{1}{2} - \alpha\right)\Omega(n)$	$f(n) \geq 1/2$
TCR [1]	Yes [1]	DSTC	$\lambda \geq (1-\alpha)\Omega(n)$	$f(n) \geq 1$
RSEB [9]	Yes [9]	TCC	$\alpha \geq 1/n$	$f(n) \geq \varepsilon(\lambda) > 0$
RKotH [10]	Yes [10]	CovC [10], DSTC	$\alpha \geq 1/10$	$f(n) \geq 1/10$
RDM [6]	Yes	DSTC	$\lambda \geq \left(1 - \frac{n\alpha}{2}\right)\Omega(n)$	$f(n) \geq \frac{\lambda-1}{\lambda(2\lambda+1)}$
PR [3]	?	DSTC	$\alpha \geq 1/13$	$f(n) \geq 1/13$
PRSL [3]	?	DSTC	$\lambda \geq (1-\alpha)\Omega(n)$	$f(n) \geq 1$

Proof. [1] proved that the Iterative Condorcet Rule is monotone, top cycle consistent, and 2-PNM. ICR is DSTC because inserting a dummy agent who loses to the existing agents into an ordering does not change the winner, so each agent wins the same proportion of orderings they did before. ICR is not CovC since 2 covers 3 in the superman-kryptonite tournament on $\{1, 2, 3, 4\}$ (see T_4 in Figure 1) yet 3 wins if the chosen permutation is $(2, 1, 4, 3)$.

Let T denote the superman-kryptonite tournament on $[n]$. Under ICR, the superman wins if and only if the kryptonite is chosen before her in the first $n-2$ rounds, so $r_1(T) = \frac{1}{2}\left(1 - \frac{2}{n(n-1)}\right) = \frac{1}{2} - \frac{1}{n(n-1)}$. Meanwhile, the kryptonite wins if and only if neither her nor the superman are chosen in the first $n-2$ rounds, so $r_n(T) = \frac{2}{n(n-1)}$. Thus, if ICR satisfies 2-NM$_\lambda$-α, then $\lambda \geq \frac{1-r_1(T)-\alpha}{r_n(T)} - 1 = \left(\frac{1}{4} - \frac{\alpha}{2}\right) n(n-1) - 1/2$. □

Randomized Voting Caterpillar [1]

The Randomized Voting Caterpillar rule (RVC) begins by choosing an ordering of the agents uniformly at random. In the first iteration, RVC eliminates the loser between the first and second agents in the ordering. In each subsequent iteration until only one agent remains, RVC eliminates the loser between the previous winner and the next agent in the ordering.

Theorem 3.6. *RVC is monotone, dominant sub-tournament consistent, and* 2-PNM. *RVC is not cover consistent. If RVC satisfies* 2-NM$_\lambda$-α *for some* $\lambda \geq 0$*, then* $\lambda \geq (\frac{1}{2} - \alpha)\Omega(n)$.

Proof. [1] proved that RVC is monotone, top cycle consistent, and 2-PNM. RVC is DSTC but not CovC for the same reason as ICR. Now, let T denote the superman-kryptonite tournament on $[n]$. Under RVC, the superman wins if and only if she comes after the kryptonite in the chosen permutation and they are not the first two agents. Thus, $r_1(T) = \frac{1}{2} - \frac{1}{n(n-1)}$. The kryptonite on the other hand wins if and only if she comes last in the chosen permutation, so $r_n(T) = 1/n$. Thus, $\lambda \geq \frac{1-r_1(T)-\alpha}{r_n(T)} - 1 = \left(\frac{1}{2} - \alpha\right) n - \frac{n-2}{n-1}$. □

Top Cycle Rule [1]

The Top Cycle Rule (TCR) declares a uniformly random agent from the top cycle the victor.

Theorem 3.7. *TCR is monotone, dominant sub-tournament consistent, and* 2-PNM. *TCR is not cover consistent. If TCR satisfies* 2-NM$_\lambda$-α *for some* $\lambda \geq 0$*, then* $\lambda \geq (1-\alpha)\Omega(n)$.

Proof. [1] proved that TCR is monotone, top cycle consistent, and 2-PNM. TCR is DSTC because the addition of an agent that loses to all existing agents does not change the top cycle. TCR is not CovC since 2 covers 3 in the superman-kryptonite tournament on $\{1, 2, 3, 4\}$ (see T_4 in Figure 1) but 3 is in the top cycle, so she wins with positive probability.

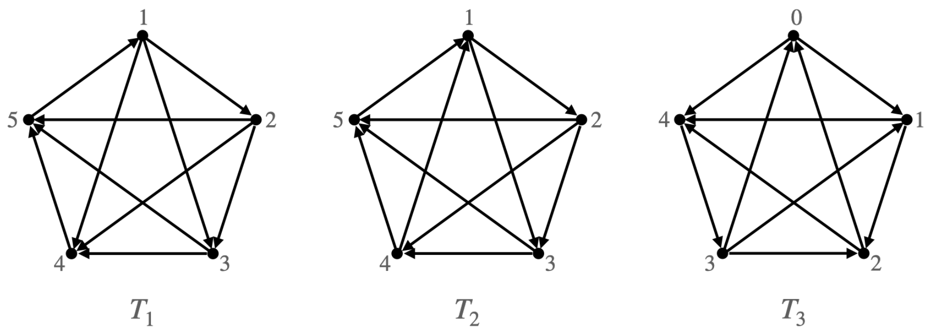

Figure 2: The tournaments that lead to our lower bounds for specific tournament rules. T_1 is the superman-kryptonite tournament (here, on 5 players) that leads to most of our lower bounds. T_2 is the tournament that leads to the $\alpha \geq 1/10$ lower bound for RKotH. T_3 is the tournament (here, on 5 players) that leads to our lower bound for PRSL.

Let T denote the superman-kryptonite tournament on $[n]$. Under TCR, both the superman and kryptonite win with probability $\frac{1}{n}$ since all agents are in the top cycle. Thus, $\lambda \geq \frac{1-r_1(T)-\alpha}{r_n(T)} - 1 = (1-\alpha)n - 2$. $\square$

Random Single Elimination Bracket [9]

A single elimination bracket is a complete binary tree whose leaves are labeled by a permutation of the agents. Each node is labeled by the winner of the match between its two children. The winner of the bracket is the agent labeling the root node. The Randomized Single Elimination Bracket rule (RSEB) introduces $2^{\lceil \log n \rceil} - n$ dummy agents who lose to the existing agents, chooses a bracket uniformly at random, and declares the winner of this bracket the winner of the tournament.

Theorem 3.8. *RSEB is monotone, top cycle consistent, and* 2-NM$_0$-1/3*. RSEB is neither dominant sub-tournament nor cover consistent. If RSEB satisfies* 2-NM$_\lambda$-α *for some* $\lambda \geq 0$*, then* $\alpha \geq 1/n$*. That is, RSEB is always pairwise manipulable regardless of* λ*.*

Proof. [9] proved that RSEB is monotone, Condorcet consistent, and 2-NM$_0$-1/3. RSEB satisfies TCC since for any agent to win, she must eventually face an agent in the top cycle. To see how RSEB violates DSTC and CovC, consider the 8-agent tournament T in which

$$1 \succ_T 2, 2 \succ_T 3, 3 \succ_T 4, 4 \succ_T 1, 1 \succ_T 3, 2 \succ_T 4$$

and $i \succ_T j$ for all $i \in \{1,2,3,4\}, j \in \{5,6,7,8\}$. Note that 2 covers 3 in T yet 3 can win the bracket whose leaves are labeled by the permutation $(1,2,4,5,3,6,7,8)$. Moreover, the sub-tournament induced by the first four agents is a dominant sub-tournament. Observe that 4 never wins a bracket in the sub-tournament $T[\{1,2,3,4\}]$, but in T, 4 can win the bracket whose leaves are labeled by the permutation $(1,2,3,5,4,6,7,8)$.

Let T denote the superman kryptonite tournament on $[n]$ where $n \geq 4$. Observe that $r_1(T) = 1 - 1/n$ since the superman loses if and only if she is paired with the kryptonite in the first round of the bracket. Moreover, $r_n(T) = 0$ since the winner of a bracket must win at least $\lceil \log_2 n \rceil$ matches. If the superman and kryptonite collude to make the superman the Condorcet winner, then the superman gains $1/n$ in probability, while the kryptonite's probability of winning remains the same, so $\alpha \geq 1/n$. $\square$

Randomized King of the Hill [10]

The Randomized King of the Hill rule (RKotH) chooses the undefeated agent if one exists. Otherwise, choose an agent uniformly at random, eliminate her and the agents she dominates, and repeat. Equivalently, RKotH chooses an ordering of the agents uniformly at random and in each round until an agent undefeated by the remaining agents exists, eliminates the next non-eliminated agent and the agents she dominates.

Theorem 3.9. *RKotH is monotone, both dominant sub-tournament and cover consistent, and* 2-NM$_0$-1/3. *If RKotH satisfies* 2-NM$_\lambda$-α *for some* $\lambda \geq 0$*, then* $\alpha \geq 1/10$. *That is, RKotH is always pairwise manipulable regardless of* λ.

Proof. [10] proved that RKotH is monotone, CovC, and 2-NM$_0$-1/3. RKotH is DSTC for the same reason as ICR and RVC: inserting a dummy agent who loses to the existing agents into an ordering does not change the winner, so each agent wins the same proportion of orderings they did before.

Now, consider the tournament $T \in \mathcal{T}_5$ in which $i \succ_T j$ whenever $i < j$, except both $4, 5 \succ_T 1$. Note that since RKotH is cover consistent and 4 covers 5, we have that $r_5(T) = 0$. Meanwhile, $r_1(T) = 2/5$ since 1 wins if and only if the agent who is chosen first is not among 1, 4, and 5. If 1 and 5 reverse the outcome of their match, then in the resulting tournament T', 1 will win with probability 1/2 since by DSTC, we can restrict our attention to the sub-tournament $T'[\{1, 2, 3, 4\}]$ and 1 wins in this sub-tournament if and only if the agent who is chosen first is not among 1 and 4. Meanwhile 5 remains covered in T'. Thus, the superman gains 1/10 in probability, while the kryptonite's probability of winning remains the same, so $\alpha \geq 1/10$. Since RKotH is DSTC, this problematic tournament on five agents remains problematic when there are more agents. □

Randomized Death Match [6]

The Randomized Death Match rule (RDM) chooses a pair of agents uniformly at random, eliminates the loser, and repeats.

Theorem 3.10. *RDM is monotone, dominant sub-tournament consistent, and* 2-NM$_0$-1/3. *RDM is not cover consistent. If RDM satisfies* 2-NM$_\lambda$-α *for some* $\lambda \geq 0$*, then* $\lambda \geq (1 - n\alpha/2)\Omega(n)$.

Proof. [6] proved that RDM is Condorcet consistent and 2-NM$_0$-1/3. RDM is monotone since for any deterministic sequence of matches, an agent gets at least as far as she did in the original tournament if she wins an additional match and is DSTC because inserting a match involving a dummy agent who loses to the existing agents into a sequence of matches does not change the winner of the tournament. RDM is not CovC since 2 covers 3 in the superman-kryptonite tournament on $\{1, 2, 3, 4\}$ (see T_4 in Figure 1) but 3 wins if $(1, 2)$ is the first pair chosen, $(1, 4)$ is the second, and $(3, 4)$ is the last.

Let T denote the superman-kryptonite tournament on $[n]$. Under RDM, the kryptonite wins if and only if she is not chosen in the first $n - 2$ rounds, so $r_n(T) = \frac{2}{n(n-1)}$. Note that the superman loses if and only if she is paired with the kryptonite in some round. The probability that this event occurs if $2/n$, so $r_1(T) = 1 - 2/n$. Therefore, $\lambda \geq \frac{1 - r_1(T) - \alpha}{r_n(T)} - 1 = \left(1 - \frac{n\alpha}{2}\right)(n - 1) - 1$. □

PageRank [3]

The PageRank rule randomly chooses an agent from the top cycle according to the solution of the following system of linear equations:

$$\forall\, i \in TC(T),\ r_i(T) = \textstyle\sum_{j \in TC(T): i \succ_T j} \frac{1}{|\{k: j \prec_T k\}|} r_j(T)$$
$$\textstyle\sum_{i \in TC(T)} r_i(T) = 1$$

Note that PR is well-defined since the top cycle is strongly connected, so the stationary distribution is indeed unique. PageRank's recursive definition is natural for tournaments: an agent has high PageRank if she beats many other agents with high PageRank.

Theorem 3.11. *PR is dominant sub-tournament but not cover consistent. If PR satisfies* 2-NM$_\lambda$-α*, then* $\alpha \geq 1/13$*. That is, PR is always pairwise manipulable regardless of* λ*.*

Proof. PR is DSTC by definition. It is not CovC for the same reason as TCR: it assigns positive probability to all members of the top cycle. Now, consider the superman-kryptonite tournament T on $\{1, 2, 3, 4\}$. The associated system of linear equations is

$$\begin{aligned} r_1(T) &= r_2(T) + \tfrac{1}{2} r_3(T) \\ r_2(T) &= \tfrac{1}{2} r_3(T) + \tfrac{1}{2} r_4(T) \\ r_3(T) &= \tfrac{1}{2} r_4(T) \\ r_4(T) &= r_1(T) \\ r_1(T) + r_2(T) + r_3(T) + r_4(T) &= 1 \end{aligned}$$

The solution to this system is

$$r_1(T) = 4/13, r_2(T) = 3/13, r_3(T) = 2/13, r_4(T) = 4/13$$

Consider the manipulation between teams 1 and 3. This manipulation simply "rotates" the original tournament clockwise. Thus, letting T' denote the resulting tournament,

$$r_1(T') = 4/13, r_2(T') = 4/13, r_3(T') = 3/13, r_4(T') = 2/13$$

Note that 3 gains 1/13 in probability, while 1's probability of winning remains the same, so $\alpha \geq 1/13$. Since PR is DSTC, the superman-kryptonite tournament on $\{1, 2, 3, 4\}$ remains problematic when there are more agents. □

PageRank with Self Loops [3]

Another natural instantiation of PageRank as a tournament rule is the PageRank with Self-Loops rule (PRSL), which randomly chooses an agent from the top cycle according to the solution of the following system of linear equations:

$$\forall\, i \in TC(T),\ r_i(T) = \textstyle\sum_{j \in TC(T): i \succeq_T j} \frac{1}{|\{k: j \preceq_T k\}|} r_j(T)$$
$$\textstyle\sum_{i \in TC(T)} r_i(T) = 1$$

Theorem 3.12. *PRSL is dominant sub-tournament but not cover consistent. If PRSL satisfies* 2-NM$_\lambda$-α*, then* $\lambda \geq (1 - \alpha)\Omega(n)$*.*

Proof. PRSL is DSTC and is not CovC for the same reasons as PR. Now, consider the following tournament on $n = 2k + 1$ agents, denoted $0, 1, \ldots, n - 1$. Have agent $n - 1$ defeat agent $n - 2$. Have agents $0, 1, \ldots, n - 3$ lose to agent $n - 2$ and defeat agent $n - 1$. For

$i = 0, \ldots, n-3$, have agent i defeat agents $(i+1) \bmod (n-2), \ldots, (i+k-1) \bmod (n-2)$ and lose to agents $(i-1) \bmod (n-2), \ldots, (i-k+1) \bmod (n-2)$. Since agents $0, \ldots, n-3$ are indistinguishable from each other (in particular, they will have the same mean return time), $r_0(T) = \cdots = r_{n-3}(T)$. Thus, it suffices to consider the following reduced system:

$$\begin{aligned} r_{n-2}(T) &= \tfrac{1}{2} r_{n-2}(T) + \tfrac{2}{n+1} \textstyle\sum_{i=0}^{n-3} r_i(T) = \tfrac{1}{2} r_{n-2}(T) + \tfrac{2(n-2)}{n+1} r_0(T) \\ r_{n-1}(T) &= \tfrac{1}{2} r_{n-2}(T) + \tfrac{1}{n-1} r_{n-1}(T) \\ (n-2) r_0(T) + r_{n-2}(T) + r_{n-1}(T) &= \textstyle\sum_{i=0}^{n-1} r_i(T) = 1 \end{aligned}$$

The solution is

$$\begin{aligned} r_0(T) = \cdots = r_{n-3}(T) &= \tfrac{n+1}{2(n-1)} \left(\tfrac{2(n-2)}{n-1} + \tfrac{(n-2)(n+1)}{2(n-1)} + 1 \right)^{-1} \\ r_{n-2}(T) &= \tfrac{2(n-2)}{n-1} \left(\tfrac{2(n-2)}{n-1} + \tfrac{(n-2)(n+1)}{2(n-1)} + 1 \right)^{-1} \\ r_{n-1}(T) &= \left(\tfrac{2(n-2)}{n-1} + \tfrac{(n-2)(n+1)}{2(n-1)} + 1 \right)^{-1} \end{aligned}$$

To conclude, note that if agents $n-2$ and $n-1$ were to manipulate, then agent $n-2$ would become the Condorcet winner in the resulting tournament T'. Thus, if PR-SL is 2-NM$_\lambda$, then

$$\lambda \geq \tfrac{1 - r_{n-2}(T) - \alpha}{r_{n-1}(T)} - 1 \geq \tfrac{(1-\alpha)(n-2)(n+1)}{2(n-1)} - 3\alpha$$

□

4 Reductions

In a separate direction, we consider fair tournament rules that for fixed λ become increasingly non-manipulable with the number of agents; that is, rules that satisfy 2-NM$_\lambda$-$f(n)$ where n is the number of agents and f is a non-negative, non-increasing function. Under stronger notions of fairness, it turns out this problem is just as hard as finding rules that satisfy 2-NM$_\lambda$-$\lim_{n\to\infty} f(n)$.

Theorem 4.1. *Let $\lambda \geq 0$ and $f \geq 0$ be a non-increasing function such that $f(n) \overset{n\to\infty}{\to} \alpha$. A DSTC tournament rule is 2-NM$_\lambda$-$f(n)$ if and only if it is 2-NM$_\lambda$-α.*

The idea behind the proof is as follows: by DSTC, the gains from manipulation in a tournament T on n agents are exactly the same as the gains from manipulation among these agents in a larger tournament on $n' > n$ agents in which T is a dominant subtournament. Thus, the gains from manipulation in T are in fact at most $f(n')$ for all $n' > n$ and hence, at most $\lim_{n'\to\infty} f(n')$. A similar but weaker result holds for top cycle consistent rules.

Theorem 4.2. *Let $\lambda \geq 1$ and $f \geq 0$ be a non-increasing function such that $f(n) \overset{n\to\infty}{\to} \alpha$. There exist a TCC tournament rule satisfying 2-NM$_\lambda$-$f(n)$ if and only if there exists a TCC tournament rule satisfying 2-NM$_\lambda$-α.*

The proof is similar to that of Theorem 4.1. However, because we do not have DSTC, we cannot directly relate the gains from manipulation in tournaments on n agents to those in tournaments on $n' > n$ agents. Nonetheless, we can define a tournament rule on n agents as the limit point of a sequence of tournament rules on n' agents for all $n' > n$. The gains from manipulation under this limit point will then be at most $\lim_{n'\to\infty} f(n')$.

Theorem 4.3. *If ICR, RVC, TCR, RKotH, PR, or PRSL satisfy 2-NM$_\lambda$-$f(n)$ for some fixed $\lambda \geq 1$ and some non-increasing function $f \geq 0$, then $f \geq \Omega(1)$. If RDM satisfies this property, then $f \geq \Omega(1/\lambda)$. If RSEB satisfies this property, then $f \geq \varepsilon(\lambda)$ where $\varepsilon(\lambda)$ is some strictly positive function of λ.*

5 Acknowledgements

This work was carried out while one of the authors, Eric Xue, was a participant in the 2021 DIMACS REU program at Rutgers University, supported by NSF grant CCF-1852215, under the supervision of Ariel Schvartzman (who at the time was affiliated with DIMACS) and David Pennock.

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David Pennock
Rutgers University
New Brunswick, NJ, USA
Email: `david.pennock@rutgers.edu`

Ariel Schvartzman
Google Research
Mountain View, CA, USA
Email: `aschvartzman@google.com`

Eric Xue
Princeton University
Princeton, NJ, USA
Email: `ex3782@princeton.edu`

A Omitted Proofs

Proof of Proposition 2.13. Suppose tournament rule r is not 2-PNM. There exist distinct agents $i, j \in [n]$ and a pair of $\{i, j\}$-adjacent tournaments $T \neq T' \in \mathcal{T}_n$ such that $r_i(T') - r_i(T) > 0$ and $r_j(T') - r_j(T) \geq 0$. Thus,

$$r_i(T') + r_j(T') - r_i(T) - r_j(T) > 0 \geq \lim_{\lambda \to \infty} \lambda \max\{r_i(T) - r_i(T'), r_j(T) - r_j(T')\}$$

so r is not 2-NM$_\infty$.

Conversely, take r to be 2-PNM, so for all distinct agents $i, j \in [n]$ and $\{i, j\}$-adjacent $T \neq T' \in \mathcal{T}_n$, either (1) $\min\{r_i(T') - r_i(T), r_j(T') - r_j(T)\} < 0$, in which case

$$r_i(T') + r_j(T') - r_i(T) - r_j(T) < +\infty = \lim_{\lambda \to \infty} \lambda \max\{r_i(T) - r_i(T'), r_j(T) - r_j(T')\}$$

or (2) $\max\{r_i(T') - r_i(T), r_j(T') - r_j(T)\} \leq 0$, in which case

$$r_i(T') + r_j(T') - r_i(T) - r_j(T) \leq 0 \leq \lim_{\lambda\to\infty} \lambda \max\{r_i(T) - r_i(T'), r_j(T) - r_j(T')\}$$

Thus, r is 2-NM$_\infty$. □

Proof of Proposition 2.16. Let $T \neq T' \in \mathcal{T}_n$ be $\{i,j\}$-adjacent tournaments such that $i \prec_T j$, and suppose r is monotone. Then, by monotonicity, $r_i(T) - r_i(T') \leq 0 \leq r_j(T) - r_j(T')$, so

$$r_i(T') - r_i(T) \leq r_j(T) - r_j(T') + \lambda \max\{r_i(T) - r_i(T'), r_j(T) - r_j(T')\} = (\lambda+1)(r_j(T) - r_j(T'))$$

where the inequality follows from the fact that r is 2-NM$_\lambda$.

Now, suppose the second statement holds. Applying the second statement twice yields

$$\begin{aligned} r_i(T') - r_i(T) &\leq (\lambda + 1)(r_j(T) - r_j(T')) \\ r_j(T) - r_j(T') &\leq (\lambda + 1)(r_i(T') - r_i(T)) \end{aligned}$$

Multiplying the second inequality by $\lambda + 1$, adding the result to the first inequality, and simplifying yields

$$\lambda(\lambda + 2)(r_i(T') - r_i(T)) \geq 0$$

Since $\lambda > 0$, this inequality implies $r_i(T') \geq r_i(T)$. □

Proof of Theorem 4.1. The backward implication is trivial, so we focus on the forward implication. Suppose a DSTC tournament rule r is not 2-NM$_\lambda$-α so that there exist $\varepsilon > 0$, $m \in \mathbb{N}$, distinct agents $i, j \in [m]$, and a pair of $\{i,j\}$-adjacent tournaments $T \neq T' \in \mathcal{T}_m$ such that

$$r_i(T') + r_j(T') - r_i(T) - r_j(T) - \lambda \max\{r_i(T) - r_i(T'), r_j(T) - r_j(T')\} \geq \alpha + \varepsilon$$

Since $f(n) \overset{n\to\infty}{\to} \alpha$, there exists $n > m$ such that $f(n) < \alpha + \varepsilon$. Now, consider a tournament $T_n \in \mathcal{T}_n$ in which T is a dominant sub-tournament. Let T'_n denote the tournament $\{i,j\}$-adjacent to T_n such that $T'_n \neq T_n$. Since r satisfies DSTC,

$$\begin{aligned} r_i\left(T'_n\right) + r_j\left(T'_n\right) - r_i\left(T_n\right) - r_j\left(T_n\right) - \lambda \max\left\{r_i\left(T_n\right) - r_i\left(T'_n\right), r_j\left(T_n\right) - r_j\left(T'_n\right)\right\} &\geq \alpha + \varepsilon \\ &> f(n) \end{aligned}$$

so r is not 2-NM$_\lambda$-$f(n)$. □

Proof of Theorem 4.2. Again, the backward implication is trivial, so we focus on the forward implication. Suppose a tournament rule r satisfies TCC and 2-NM$_\lambda$-$f(n)$. Fix the number of agents m. We will construct a tournament rule on m agents $s^{(m)}$ that is TCC and 2-NM$_\lambda$-α.

For each $n \in \mathbb{N}$, define a tournament rule on m agents $w^{m,n}$ as follows. For all $i \in [m], T \in \mathcal{T}_m$, define

$$w_i^{m,n}(T) := r_i\left(T_n\right)$$

where $T_n \in \mathcal{T}_n$ is a tournament in which T is a dominant sub-tournament. Note that $TC(T_n) = TC(T) \subseteq T$, so $w^{m,n}(T) \in \Delta^m$ and since r is TCC and 2-NM$_\lambda$-$f(n)$, $w^{m,n}$ is as well.

Now, since $[m] \times \mathcal{T}_m$ is a finite set and each coordinate of $((w^{m,n}(T))_{T\in\mathcal{T}_m})_{n=1}^{\infty}$ is bounded, there exists a convergent subsequence $((w^{m,n_k}(T))_{T\in\mathcal{T}_m})_{k=1}^{\infty}$. Define $s^{(m)}(T) = \lim_{k\to\infty} w^{n_k}(T)$ for all $T \in \mathcal{T}_m$. By construction, $s^{(m)}(T) \in \Delta^m$ for all $T \in \mathcal{T}_m$ and $s^{(m)}$ is

TCC. Moreover, $s^{(m)}$ satisfies 2-NM$_\lambda$-α: for any distinct agents $i, j \in [m]$ and $\{i, j\}$-adjacent tournaments $T, T' \in \mathcal{T}_m$,

$$\begin{aligned}
s_i^{(m)}(T') + s_j^{(m)}(T') - s_i^{(m)}(T) - s_j^{(m)}(T) - \lambda \max\{s_i^{(m)}(T) - s_i^{(m)}(T'), s_j^{(m)}(T) - s_j^{(m)}(T')\} \\
&= \lim_{k\to\infty} (w_i^{m,n}(T') + w_j^{m,n}(T') - w_i^{m,n}(T) - w_j^{m,n}(T) \\
&\qquad - \lambda \max\{w_i^{m,n}(T) - w_i^{m,n}(T'), w_j^{m,n}(T) - w_j^{m,n}(T')\}) \\
&\le \lim_{k\to\infty} f(n_k) = \alpha
\end{aligned}$$

Carrying out this procedure for all $m \in \mathbb{N}$ yields a TCC and 2-NM$_\lambda$-α tournament rule $s := \{s^{(m)}\}_{m=1}^{\infty}$.

□

Proof of Theorem 4.3. The results for all rules except RDM and RSEB follow as direct consequences of Theorems 3.5, 3.6, 3.7, 3.9, 3.11, and 3.12. To see the result for RDM, consider the superman kryptonite tournament T on $[2\lambda + 1]$ and the distinct $\{1, n\}$-adjacent tournament T' (in which the superman is now the Condorcet winner). Using the probabilities computed in Theorem 3.10, $f(2\lambda+1) \ge 1 - r_1(T) - (\lambda+1)r_j(T) = \frac{2}{2\lambda+1} - \frac{\lambda+1}{\lambda(2\lambda+1)} = \frac{\lambda-1}{\lambda(2\lambda+1)}$. Since RDM is DSTC, this problematic tournament remains problematic when $n \ge 2\lambda + 1$.

The idea behind the proof of the result for RSEB is to assume by way of contradiction that RSEB satisfies 2-NM$_\lambda$-$f(n)$ for some fixed $\lambda \ge 0$ and some $f(n) \overset{n\to\infty}{\to} 0$ and then show that the tournament rule that results from applying the limiting process in Theorem 4.2 to RSEB actually requires that λ grow with n in order to be 2-NM$_\lambda$, which contradicts Theorem 4.2. We give the details in Section A.1 of the appendix. □

A.1 Gains From Manipulation Under RSEB Do Not Vanish

In what follows, let r denote the RSEB tournament rule and for a given tournament T, let $T_n \in \mathcal{T}_n$ denote a tournament in which T is a dominant sub-tournament. We prove the following result.

Theorem A.1. *If RSEB is 2-*NM$_\lambda$*-$f(n)$ for some fixed $\lambda \ge 1$ and some non-increasing function $f \ge 0$, then $f \ge \varepsilon(\lambda)$ where $\varepsilon(\lambda)$ is some strictly positive function of λ.*

Suppose by way of contradiction that RSEB is 2-NM$_\lambda$-$f(n)$ for some fixed $\lambda \ge 0$ and some non-increasing function $f \ge 0$ such that $f(n) \to 0$ as $n \to \infty$. Fix $n = 2^h$. Since RSEB is TCC and assumed to be 2-NM$_\lambda$-$f(n)$, following the logic in Theorem 4.2, we can find a convergent subsequence $((r_1(T_{2^{m_j}}), \ldots, r_n(T_{2^{m_j}})))_{j=1}^{\infty}$ such that the limit point $s^{(n)}$ is 2-NM$_\lambda$.

Now, let $T \in \mathcal{T}_n$ be the superman kryptonite tournament on $n = 2^h$ agents and let T' denote the distinct tournament that is $\{1, n\}$-adjacent to T. Since $s^{(n)}$ is 2-NM$_\lambda$,

$$\begin{aligned}
0 \ge 1 - s_1^{(n)}(T) - \lambda s_n^{(n)}(T) &\ge \lim_{j\to\infty} (1 - r_1(T_{2^{m_j}}) - \lambda r_n(T_{2^{m_j}})) \\
&\ge \frac{1}{3n} - \frac{\lambda}{2^{n-1} - 1} \qquad \text{(Lemmas A.3 and A.5)}
\end{aligned}$$

Thus,

$$\lambda \ge \frac{2^{n-1} - 1}{3n}$$

Since our choice of convergent subsequence was arbitrary, this inequality holds for all limit points $s^{(n)}$. Carrying out this analysis for all $n = 2^h$, we get that any tournament rule

s that arises from carrying out the limiting procedure from the proof of Theorem 4.2 on RSEB requires that λ grows exponentially in the number of agents in order to be 2-NM$_\lambda$, contradicting Theorem 4.2. Thus, RSEB is not 2NM$_\lambda$-$f(n)$ for a fixed $\lambda \geq 0$ and $f(n) \to 0$ as $n \to \infty$.

Lemma A.2. *Let $n = 2^h$ and let $T \in \mathcal{T}_n$ be the superman-kryptonite tournament on n agents. For all $m \geq h+1$,*

$$r_n(T_{2^m}) = \frac{2}{\binom{2^m}{2^{m-1}}} \left[\binom{2^m - n}{2^{m-1} - 1} + \binom{2^m - n}{2^{m-1} - n} r_n(T_{2^{m-1}}) \right]$$

where $T_{2^m} \in \mathcal{T}_{2^m}$ is some tournament in which T is a dominant subtournament.

Proof. For any tournament S, consider the following approach for computing $r_i(S)$. For a given partition of $[n]$ into two equal sets A and B in which $i \in A$, the probability that i wins a bracket in which the players on one side are in A and the players on the other are in B is

$$r_i(S|_A) \sum_{k: i \succ_S k} r_k(S|_B)$$

where $S|_A$ is the tournament subgraph of S induced by the players in A. In other words, the probability that i wins such a bracket is the probability that she wins her side of the bracket times the probability that someone she can beat the winner of the other side. Randomizing over all such partitions,

$$r_i(S) = \frac{2}{\binom{n}{n/2}} \sum_{A,B} \left(r_i(S|_A) \sum_{k: i \succ_S k} r_k(S|_B) \right)$$

Now, to prove the recurrence relation, we consider several cases of partitions. If $n \in A$, while $[n-1] \subseteq B$, then $r_n(T_{2^m}|_A) = r_1(T_{2^m}|_B) = 1$ since n can defeat all the dummy players on her side of the bracket, and 2 only loses to n, who is on the other side of the bracket. If $[n] \subseteq A$, then $r_n(T_{2^m}|_A) = r_n(T_{2^{m-1}})$ and $\sum_{k: n \succ_T k} r_k(T|_B) = 1$ since n can defeat all the dummy players on the other side. If $n \in A, 1 \in B$, and there exists $i \in A \cap [n-1]$, then $r_n(T|_A) = 0$ since n will have to face someone she loses to before the finals. Otherwise, $1, n \in A$ and there exists $i \in B \cap [n-1]$. In this case, $\sum_{k: n \succ_T k} r_k(T|_B) = 0$ since the winner of the B side of the bracket will be someone n loses to. Thus,

$$r_n(T_{2^m}) = \frac{2}{\binom{2^m}{2^{m-1}}} \left(\sum_{n \in A, [n-1] \subseteq B} 1 + \sum_{[n] \in A} r_n(T_{2^{m-1}}) \right)$$

The number of partitions up to symmetry such that $n \in A, [n-1] \subseteq B$ is $\binom{2^m - n}{2^{m-1} - 1}$ (fix $n \in A$ and choose the remaining $2^{m-1} - 1$ players from outside of $[n]$), and the number of partitions such that $[n] \subseteq A$ is $\binom{2^m - n}{2^{m-1}}$ (fix $[n] \subseteq A$ and choose the remaining $2^{m-1} - n$ players from outside of $[n]$). The recurrence relation now follows. $\square$

Lemma A.3. *Let $n = 2^h$ and let $T \in \mathcal{T}_n$ be the superman-kryptonite tournament on n agents. For all $m \geq h+1$,*

$$r_n(T_{2^m}) \leq \frac{1}{2^{n-1} - 1}$$

where $T_{2^m} \in \mathcal{T}_{2^m}$ is some tournament in which T is a dominant subtournament.

Proof. We show by induction that $r_n(T_{2^m}) \le \sum_{k=1}^{m-h} \frac{1}{2^{k(n-1)}}$. For our base case, consider $m = h+1$, i.e., $2^m = 2n$. Note that the kryptonite n cannot win in the superman kryptonite tournament of size n, so $r_n(T_n) = 0$. Thus,

$$\begin{aligned}
r_n(T_{2n}) &= \frac{2}{\binom{2n}{n}} \binom{n}{n-1} \\
&= \frac{n!n!}{(2n-1)!} \\
&= \frac{(n-1)!}{\prod_{k=1}^{n-1}(2n-k)} \\
&= \frac{(n-1)!}{2^{n-1}\prod_{k=1}^{n-1}(n-k/2)} \\
&= \frac{1}{2^{n-1}} \prod_{k=1}^{n-1} \frac{n-k}{n-k/2} \le \frac{1}{2^{n-1}}
\end{aligned}$$

as desired.

Now, suppose that $r_n(T_{2^{m-1}}) \le \sum_{k=1}^{m-h-1} \frac{1}{2^{k(n-1)}}$, and consider $r_n(T_{2^m})$. For notational convenience, let $N := 2^{m-1}$. Observe that

$$\begin{aligned}
\frac{2}{\binom{2N}{N}} \binom{2N-n}{N-1} &= \frac{2N!N!}{(2N)!} \cdot \frac{(2N-n)!}{(N-1)!(N-n+1)!} \\
&= \frac{\prod_{k=0}^{n-2}(N-k)}{\prod_{k=1}^{n-1}(2N-k)} \\
&= \frac{1}{2^{n-1}} \prod_{k=1}^{n-1} \frac{N-k+1}{N-k/2} \le \frac{1}{2^{n-1}}
\end{aligned}$$

The last line comes from the fact that $\frac{N}{N-1/2} \cdot \frac{N-3}{N-2} \le 1$, and the rest of the terms in the product are ≤ 1. Moreover,

$$\begin{aligned}
\frac{2}{\binom{2N}{N}} \binom{2N-n}{N-n} &= \frac{2N!N!}{(2N)!} \cdot \frac{(2N-n)!}{(N-n)!N!} \\
&= \frac{\prod_{k=1}^{n-1}(N-k)}{\prod_{k=1}^{n-1}(2N-k)} \\
&= \frac{1}{2^{n-1}} \prod_{k=1}^{n-1} \frac{N-k}{N-k/2} \le \frac{1}{2^{n-1}}
\end{aligned}$$

Thus,

$$\begin{aligned}
r_n(T_{2^m}) &= \frac{2}{\binom{2^m}{2^{m-1}}} \left[\binom{2^m-n}{2^{m-1}-1} + \binom{2^m-n}{2^{m-1}-n} r_n(T_{2^{m-1}}) \right] \\
&\le \frac{1}{2^{n-1}} + \frac{r_n(T_{2^{m-1}})}{2^{n-1}} \\
&= \frac{1}{2^{n-1}} + \sum_{k=1}^{m-h-1} \frac{1}{2^{(k+1)(n-1)}} && \text{(inductive hypothesis)} \\
&= \sum_{k=1}^{m-h} \frac{1}{2^{k(n-1)}}
\end{aligned}$$

as desired. We conclude by remarking that $\sum_{k=1}^{\infty} \frac{1}{2^{k(n-1)}} = \frac{1}{2^{n-1}-1}$. $\square$

Lemma A.4. *Let $n = 2^h$. There exists ℓ such that $\sum_{k=1}^{\ell}(1-\frac{1}{2^k})^{n-2}\frac{1}{2^k} \geq \frac{1}{3n}$.*

Proof. Consider the series $\sum_{k=1}^{\infty}(1-\frac{1}{2^k})^{n-1}\frac{1}{2^k}$. The partial sums in this series are increasing and upper bounded by $\sum_{k=1}^{\infty}\frac{1}{2^k} = 1$, so the series converges. Define $f(x) := (1-\frac{1}{2^x})^{n-1}\frac{1}{2^x}$.

$$f'(x) = \frac{\ln(2)(1-1/2^x)^n(n-2^x)}{(2^x-1)^2}$$

Thus, f is strictly increasing on $[0,h]$ and strictly decreasing on $[h,\infty]$. Moreover, f achieves a local maximum at h. Therefore,

$$\begin{aligned}\left(1-\frac{1}{2^h}\right)^{n-1}\frac{1}{2^h} + \sum_{k=1}^{\infty}\left(1-\frac{1}{2^k}\right)^{n-1}\frac{1}{2^k} &\geq \int_0^{\infty} f(x)\,\mathrm{d}x \\ &= \left[\frac{(1-1/2^x)^n}{n\ln(2)}\right]_0^{\infty} \\ &= \frac{1}{n\ln(2)}\end{aligned}$$

so

$$\sum_{k=1}^{\infty}\left(1-\frac{1}{2^k}\right)^{n-1}\frac{1}{2^k} \geq \frac{1}{n\ln(2)} - \left(1-\frac{1}{n}\right)^{n-1}\frac{1}{n} \geq \frac{1}{n\ln(2)} - \frac{1}{n}$$

Since $\frac{1-\ln(2)}{n\ln(2)} > \frac{1}{3n}$, there exists ℓ such that $\sum_{k=1}^{\ell}(1-\frac{1}{2^k})^{n-2}\frac{1}{2^k} > \sum_{k=1}^{\ell}(1-\frac{1}{2^k})^{n-1}\frac{1}{2^k} \geq \frac{1}{3n}$. $\square$

Lemma A.5. *Let $n = 2^h$. Let $T \in \mathcal{T}_n$ be the superman-kryptonite tournament on n agents and let $T_{2^m} \in \mathcal{T}_{2^m}$ be some tournament in which T is a dominant subtournament. If $(r_1(T_{2^{m_i}}))_{i=1}^{\infty}$ is a convergent (sub)sequence, then*

$$\lim_{i\to\infty} 1 - r_1(T_{2^{m_i}}) \geq \frac{1}{3n}$$

Proof. Note that $1 - r_1(T_{2^m})$ is the probability that the superman (in T) loses T_{2^m}. Since the superman loses if and only if she encounters the kryptonite in some round, for $m \geq h+1$,

$$1 - r_1(T_{2^m}) = \sum_{k=0}^{m-1}\frac{\binom{2^m-2^h}{2^k-1}}{\binom{2^m-1}{2^k}}$$

To see why this expression is correct, observe that

$$\frac{\binom{2^m-2^h}{2^k-1}}{\binom{2^m-1}{2^k}}$$

is the probability that the superman encounters the kryptonite in round $k+1$. Given the seed position of the superman, there is exactly one subtree of height $k+1$ that the kryptonite must be seeded into in order to have a chance of facing the superman in round $k+1$. Moreover, the kryptonite faces the superman in the desired round if and only if the remaining $2^k - 1$ players in this subtree are dummy players. Thus, the probability that the kryptonite reaches round k is the probability that the 2^k players in this subtree consist of the kryptonite and $2^k - 1$ dummy players. Now, re-index to get

$$1 - r_1(T_{2^m}) = \sum_{k=1}^{m} \frac{\binom{2^m-2^h}{2^{m-k}-1}}{\binom{2^m-1}{2^{m-k}}}$$

Now, consider a term in the sum.

$$\begin{aligned}
\frac{\binom{2^m-2^h}{2^{m-k}-1}}{\binom{2^m-1}{2^{m-k}}} &= \frac{(2^m-2^h)!}{(\frac{2^m}{2^k}-1)!(\frac{2^m(2^k-1)}{2^k}-2^h+1)!} \cdot \frac{(\frac{2^m}{2^k})!(\frac{2^m(2^k-1)}{2^k}-1)!}{(2^m-1)!} \\
&= \prod_{j=1}^{2^h-2} \frac{\frac{2^m(2^k-1)}{2^k}-j}{2^m-j} \cdot \frac{\frac{2^m}{2^k}}{2^m-2^h+1} \\
&= \left(\frac{2^k-1}{2^k}\right)^{2^h-2} \frac{1}{2^k} \prod_{j=1}^{2^h-2} \frac{2^m - \frac{2^k j}{(2^k-1)}}{2^m-j} \cdot \frac{2^m}{2^m-2^h+1}
\end{aligned}$$

Thus,

$$\lim_{m\to\infty} \frac{\binom{2^m-2^h}{2^{m-k}-1}}{\binom{2^m-1}{2^{m-k}}} = \left(\frac{2^k-1}{2^k}\right)^{n-2} \frac{1}{2^k} = \left(1-\frac{1}{2^k}\right)^{n-2} \frac{1}{2^k} \qquad (n = 2^h)$$

Now, let $\varepsilon > 0$ be given. By Lemma A.4, there exists ℓ such that

$$\sum_{k=1}^{\ell} (1-\frac{1}{2^k})^{n-2} \frac{1}{2^k} \geq \frac{1}{3n}$$

Fix ℓ. Since for each $1 \leq k \leq \ell$,

$$\lim_{m\to\infty} \frac{\binom{2^m-2^h}{2^{m-k}-1}}{\binom{2^m-1}{2^{m-k}}} = \left(1-\frac{1}{2^k}\right)^{n-2} \frac{1}{2^k}$$

there exists $M_k \geq \ell$ such that for all $m \geq M_k$,

$$\left| \frac{\binom{2^m-2^h}{2^{m-k}-1}}{\binom{2^m-1}{2^{m-k}}} - (1-\frac{1}{2^k})^{n-2}\frac{1}{2^k} \right| < \frac{\varepsilon}{\ell}$$

Let $M := \max_{1\leq k\leq \ell} M_k$, so that for all $m \geq M$,

$$\begin{aligned}
\left| \sum_{k=1}^{\ell} \frac{\binom{2^m-2^h}{2^{m-k}-1}}{\binom{2^m-1}{2^{m-k}}} - \sum_{k=1}^{\ell} (1-\frac{1}{2^k})^{n-2}\frac{1}{2^k} \right| &\leq \sum_{k=1}^{\ell} \left| \frac{\binom{2^m-2^h}{2^{m-k}-1}}{\binom{2^m-1}{2^{m-k}}} - (1-\frac{1}{2^k})^{n-2}\frac{1}{2^k} \right| \\
&< \sum_{k=1}^{\ell} \frac{\varepsilon}{\ell} = \varepsilon
\end{aligned}$$

Thus,

$$\lim_{i\to\infty} 1 - r_1(T_{2^{m_i}}) = \lim_{i\to\infty} \sum_{k=1}^{m_i} \frac{\binom{2^{m_i}-2^h}{2^{m_i-k}-1}}{\binom{2^{m_i}-1}{2^{m_i-k}}} \geq \lim_{m\to\infty} \sum_{k=1}^{\ell} \frac{\binom{2^m-2^h}{2^{m-k}-1}}{\binom{2^m-1}{2^{m-k}}} = \sum_{k=1}^{\ell} (1-\frac{1}{2^k})^{n-2}\frac{1}{2^k} \geq \frac{1}{3n}$$

□