Characterizations of Sequential Valuation Rules

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Abstract

Approval-based committee (ABC) voting rules elect a fixed size subset of the candidates, a so-called committee, based on the voters' approval ballots over the candidates. While these rules have recently attracted significant attention, axiomatic characterizations are largely missing so far. We address this problem by characterizing ABC voting rules within the broad and intuitive class of sequential valuation rules. These rules compute the winning committees by sequentially adding candidates that increase the score of the chosen committee the most. In more detail, we first characterize almost the full class of sequential valuation rules based on mild standard conditions and a new axiom called consistent committee monotonicity. This axiom postulates that the winning committees of size k can be derived from those of size $k-1$ by only adding candidates and that these new candidates are chosen consistently. By requiring additional conditions, we derive from this result also a characterization of the prominent class of sequential Thiele rules. Finally, we refine our results to characterize three well-known ABC voting rules, namely sequential approval voting, sequential proportional approval voting, and sequential Chamberlin-Courant approval voting.

1 Introduction

Whether it is choosing dishes for a shared lunch, shortlisting candidates for interviews, or electing a parliament of a country—all these problems require us to elect a fixed size subset of the available candidates based on the voters' preferences. This problem, commonly studied under the term *approval-based committee (ABC) voting*, has recently attracted significant attention within the field of social choice theory because of its versatile applications [11, 12, 19]. In more detail, the study objects for this problem are ABC voting rules which choose a subset of the candidates of predefined size, a so-called committee, based on the voters' approval ballots, i.e., each voter reports the set of candidates she finds acceptable.

Due to the large amount of work on ABC voting, there is a wide variety of ABC voting rules, e.g., Thiele methods, sequential Thiele methods, Phragmen's rules, the method of equal shares, and many more (we refer to [19] for an overview). For deciding which rule to use in a given situation, social choice theorists commonly reason about their properties: if a voting rule satisfies desirable properties, it seems to be a good choice for the election at hand. However, such reasoning does not rule out the existence of an even more attractive voting rule satisfying the required properties. For narrowing down the choice to a single ABC voting rule, a characterization of this rule is required, i.e., one needs to show that the rule is the unique method that satisfies a set of properties. Unfortunately, such characterizations are largely missing in the literature on ABC voting rules and it is therefore an important open problem to derive such results (see, e.g., [19, Q1]).

The goal of this paper thus is to provide such characterizations for ABC voting rules within the new but broad and intuitive class of sequential valuation rules. For computing the winning committees, these rules rely on a valuation function which assigns a score to each pair of ballot and committee. A simple example of such a function is $v(A_i, W) = |A_i \cap W|$, where A_i is an arbitrary ballot and W is a committee. Based on a valuation function, a sequential valuation rule proceeds in rounds and, in each round, it extends the previously chosen committees with the candidates that increase the total score by the most. Clearly, the prominent class of sequential Thiele rules, which only rely on the size of the intersection of the given ballot and committee to compute the score, forms a subset of the class of sequential valuation rules. However, our class is much more general as it contains, for instance, step-dependent sequential scoring rules, whose valuation functions depend on the sizes of the ballot, the committee, and the intersection of these two.

Our Contribution. As our main contribution, we characterize the class of sequential valuation rules that satisfy mild standard conditions based on a new axiom called consistent committee monotonicity. This property combines the well-known notions of committee monotonicity [e.g., 2, 15, 11] and consistency [e.g., 27, 13, 18]. Roughly, committee monotonicity requires that the winning committees of size $k+1$ can be derived from those of size k by simply adding candidates. On the other hand, the idea of consistency is that whenever two disjoint electorates separately elect the same candidates, these candidates should be the winners when we consider both electorates simultaneously. Consistent committee monotonicity combines these two axioms by requiring that the candidates that extend the committees of size k are chosen consistently: if some common candidates extend a committee W in two disjoint elections, these candidates should also extend W in the combined election. Or, to put it more simply, consistent committee monotonicity restricts committee monotonicity by requiring that the newly added candidates are chosen in a reasonable way.

With this axiom, we characterize the class of sequential valuation rules that satisfy anonymity, neutrality, non-imposition, and continuity (Theorem 1). These four conditions are mild standard axioms. Since they are satisfied by almost all ABC voting rules considered in the literature, we summarize them by the term *proper*. In more detail, we first show that every proper sequential valuation rule is a step-dependent sequential scoring rule, i.e., its valuation function only depends on the sizes of the ballots, the committees, and the intersections of these two. As second step, we then characterize step-dependent sequential scoring rules as the only proper and consistently committee monotone ABC voting rules. Or, put differently, when the winning committees should be computed sequentially and the newly added candidates are chosen in a consistent way, we naturally arrive at the class of step-dependent sequential scoring rules.

Based on this characterization, we also infer characterizations of more restricted classes of voting rules by requiring additional axioms. In particular, we present such results for sequential Thiele rules (whose valuation functions only depend on the size of the intersection of the ballot and the committee). Hence, we derive a hierarchy of characterizations based on our first theorem and, in particular, provide a full characterization of the prominent class of sequential Thiele rules. Finally, we leverage these results to characterize three commonly studied ABC voting rules, namely sequential approval voting, sequential proportional approval voting, and sequential Chamberlin-Courant approval voting, by investigating how they treat clones. An overview of our results can also be found in Figure 1.

Related Work. The study of committee monotone ABC voting rules has a long tradition as already Thiele [26] suggested the class of functions nowadays known as sequential Thiele rules. In particular, for a number of applications such as choosing finalists for a competition or shortlisting candidates for an interview, it is frequently reasoned that committee monotonicity is a desirable property [2, 15, 11]. More generally, Faliszewski et al. [12] view committee monotonicity as the fundamental property when choosing candidates only based on their quality because in such settings, there is no reason why a candidate that is elected for a committee of size k should not be elected for a committee of size $k+1$.

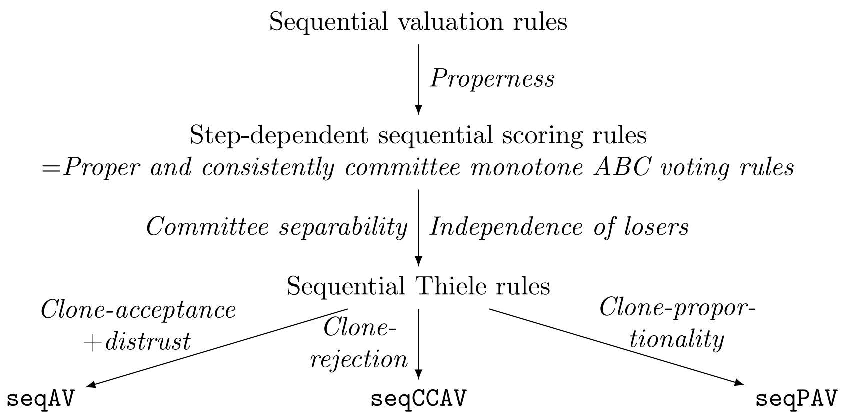

Figure 1: Overview of our results. An arrow from class X to class Y with label Z means an ABC voting rule in the class X is in the class Y if and only if it satisfies properties Z.

Another important advantage of such sequential ABC voting rules is that they are easy to compute, whereas rules that directly optimize the score (e.g., Thiele rules) are usually NP-hard to compute [23]. Indeed, sequential ABC voting rules have even been interpreted as approximation algorithms for these optimizing rules [20, 23]. On the other hand, committee monotonicity conflicts with other desirable properties. For instance, Barberà and Coelho [2] show that this axiom is incompatible with a variant of Condorcet-consistency when voters report strict rankings over the candidates, and it has been repeatedly observed that committee monotone ABC voting rules are less proportional than other rules [11, 19, 24].

Even more work has focused on specific committee monotone ABC voting rules [e.g., 1, 8, 17, 9]. For instance, Delemazure et al. [9] show that all sequential Thiele rules but sequential approval voting fail strategyproofness, and Brill et al. [8] investigate these rules with respect to proportionality axioms. An interesting observation in this context is that Phragmen's sequential rule is committee monotone and satisfies strong proportionality conditions [7, 22]; unfortunately, this rule fails our consistency criterion.

From a conceptual standpoint our results are also related to theorems for different models as consistency led to numerous important characterizations. In particular, based on this axiom, Young [27] characterizes scoring rules for single winner elections, Fishburn [13] characterizes approval voting for single winner elections with dichotomous preferences, Young and Levenglick [28] characterize a method called Kemeny's rule in a model where the outcome is a set of rankings over the candidates, and Brandl et al. [5] characterize a voting rule called maximal lotteries in a randomized setting. Furthermore, Freeman et al. [14] characterize runoff scoring rules for single winner elections based on a consistency notion similar to ours. More recently, Lackner and Skowron [18] characterized ABC scoring rules based on a consistency condition for committees instead of single candidates in a model where the output is a ranking over committees. To the best of our knowledge, this result is the only full characterization in the realm of ABC elections. Similarly, Skowron et al. [25] characterized committee scoring rules in a model where individual preferences are linear.

2 The Model

Let $\mathbb{N} = \{1, 2, 3, \dots\}$ denote an infinite set of voters and let $\mathcal{C} = \{a_1, \dots, a_m\}$ denote a fixed set of m candidates. We define $\mathcal{F}(\mathbb{N})$ as the set of finite and non-empty subsets of $\mathbb{N}$. Intuitively, an element $N \in \mathcal{F}(\mathbb{N})$ represents a concrete electorate, whereas $\mathbb{N}$ is the set of all possible voters. Given an electorate $N \in \mathcal{F}(\mathbb{N})$, we assume that every voter $i \in N$ has dichotomous preferences over the candidates, i.e., she partitions the candidates into approved and disapproved ones. Thus, voters report *approval ballots* A_i which are non-empty subsets of $\mathcal{C}$. Let $\mathcal{A}$ denote the set of all possible approval ballots. An *approval profile* A for an electorate N is an element of $\mathcal{A}^N$, i.e., a function that maps every voter $i \in N$ to her approval ballot A_i. We define $\mathcal{A}^* = \bigcup_{N \in \mathcal{F}(\mathbb{N})} \mathcal{A}^N$ as the set of all possible approval profiles. Given a profile $A \in \mathcal{A}^*$, we let N_A indicate the set of voters who report a ballot in the profile A and we say that two profiles A, A' are disjoint if $N_A \cap N_{A'} = \emptyset$. Moreover, for two disjoint profiles A and A', we define $A + A'$ as the profile with $N_{A+A'} = N_A \cup N_{A'}$, $(A + A')_i = A_i$ for all $i \in N_A$, $(A + A')_i = A'_i$ for all $i \in N_{A'}$.

Given an approval profile, the goal is to choose a committee. Formally, a *committee* is a subset of the candidates with a specific size. We denote by $\mathcal{W}_k$ the set of all committees of size k and by $\mathcal{W} = \bigcup_{k=0}^{m} \mathcal{W}_k$ the set of all committees. For selecting the winning committees for an approval profile A, we use *approval-based committee (ABC) voting rules*. These rules are functions which take an arbitrary approval profile $A \in \mathcal{A}^*$ and target committee size $k \in \{0, \dots, m\}$ as input and return a non-empty subset of $\mathcal{W}_k$. Intuitively, the chosen set contains the winning committees and we allow for sets of committees as output to indicate that multiple committees are tied for the win. Furthermore, note that ABC voting rules are also defined for committees of size 0: $f(A, 0) = \{\emptyset\}$ for all profiles A since the empty set is the only committee of size 0. This definition is only used for notational convenience.

In this paper, we will restrict our attention to *proper* ABC voting rules which satisfy the following four conditions. Note that almost all commonly studied ABC voting rules are proper voting rules as the subsequent axioms are extremely mild.

- Anonymity: An ABC voting rule f is *anonymous* if $f(A, k) = f(\pi(A), k)$ for all $A \in \mathcal{A}^*$, $k \in \{0, \dots, m\}$, and permutations $\pi : \mathbb{N} \to \mathbb{N}$. Here, $A' = \pi(A)$ denotes the profile such that $N_{A'} = \pi(N_A)$ and $A'_{\pi(i)} = A_i$ for all $i \in N_A$.

- Neutrality: An ABC voting rule f is *neutral* if $f(\tau(A), k) = \{\tau(W) \colon W \in f(A, k)\}$ for all $A \in \mathcal{A}^*$, $k \in \{0, \dots, m\}$, and permutations $\tau : \mathcal{C} \to \mathcal{C}$. $A' = \tau(A)$ denotes here the profile such that $N_{A'} = N_A$ and $A'_i = \tau(A_i)$ for all $i \in N_A$.

- Continuity: An ABC voting rule f is *continuous* if for all disjoint profiles $A, A' \in \mathcal{A}^*$ and committee sizes $k \in \{0, \dots, m\}$ such that $|f(A, k)| = 1$, there is an integer $j \in \mathbb{N}$ such that $f(jA + A', k) = f(A, k)$. Here, jA denotes a profile consisting of j disjoint copies of A; the identities of the voters are irrelevant for proper rules due to anonymity.

- Non-imposition: An ABC voting rule f is *non-imposing* if for every committee $W \in \mathcal{W}$, there is a profile $A \in \mathcal{A}^*$ such that $f(A, |W|) = \{W\}$.

Anonymity and neutrality are common fairness conditions which require that voters and candidates, respectively, are treated equally. Continuity, also known as overwhelming majority axiom [21], requires that a sufficiently large group can force the voting rule to choose their desired committee. Finally, non-imposition states that each committee has a chance to be uniquely chosen.

Aside from these standard conditions, we will use two new axioms in our analysis: independence of losers and committee separability. The idea of independence of losers is that a chosen committee $W \in f(A, k)$ should still be chosen if some voters change their preferences by disapproving candidates $c \notin W$ because, intuitively, this does not affect the quality of W. Formally, we say an ABC voting rule f is *independent of losers* if $W \in f(A, |W|)$ implies that $W \in f(A', |W|)$ for all profiles $A, A' \in \mathcal{A}^*$ and committees $W \in \mathcal{W}_k$ with $N_A = N_{A'}$, $W \cap A_i = W \cap A'_i$, and $A'_i \subseteq A_i$ for all $i \in N_A$. Note that this axiom is well-known in classic settings [e.g., 6, 4], but not in multiwinner voting. We find it intuitive and it is satisfied by all commonly considered ABC voting rules which do not depend on the ballot size (e.g., Thiele rules, sequential Thiele rules, Phragmen's rule). Contrary, satisfaction approval voting depends on it and fails independence of losers (see [19] for definitions of the rules).

Our second non-standard axiom is committee separability. The rough intuition of this axiom is that if there are two disjoint profiles A and B such that no voters $i \in N_A$, $j \in N_B$ approve a common candidate, we can decompose every chosen committee W into two subcommittees which are chosen for A and B separately. For formally defining this axiom, let $C_A = \bigcup_{i\in N_A} A_i$ denote the set of candidates that are approved by the voters in a profile A. Then, an ABC voting rule f is *committee separable* if $W \in f(A + B, |W|)$ implies that $W \cap C_A \in f(A, |W \cap C_A|)$ and $W \cap C_B \in f(B, |W \cap C_B|)$ for all disjoint profiles A, B with $C_B = \mathcal{C} \setminus C_A$ and committees $W \in \mathcal{W}$. Indeed, since $C_A \cap C_B = \emptyset$, it seems reasonable that the choice of candidates from C_A (resp. C_B) only depends on A (resp. B). All proper rules named in this paper satisfy committee separability.

2.1 Consistent Committee Monotonicity

The key axiom for our results is consistent committee monotonicity, which is a strengthening of the well-known committee monotonicity. The idea of the latter property is that the winning committees of size k are derived by adding candidates to those of size $k-1$. While this is simple to define for ABC voting rules that always choose a single winning committee, it becomes less clear how to formalize committee monotonicity when allowing for multiple tied winning committees. We use the definition of Elkind et al. [11] in this paper which requires that every winning committee of size k is derived from a winning committee of size $k-1$ and every winning committee of size $k-1$ is extended to a winning committee of size k.

Definition 1. An ABC voting rule f is *committee monotone* if for every profile $A \in \mathcal{A}^*$ and $k \in \{1, \dots, m\}$, it holds that:

(1) $W \in f(A, k)$ implies that there is $W' \in f(A, k-1)$ with $W' \subseteq W$.

(2) $W \in f(A, k-1)$ implies that there is $W' \in f(A, k)$ with $W \subseteq W'$.

Committee monotone ABC voting rules are closely connected to *generator functions* g, which take a profile A and a committee $W \neq \mathcal{C}$ as input and output a possibly empty subset $g(A, W)$ of $\mathcal{C} \setminus W$. In particular, generator functions induce committee monotone ABC voting rules in a natural way: a generator function g *generates* an ABC voting rule f if $W \in f(A, k-1)$ implies $g(A, W) \neq \emptyset$ and $f(A, k) = \{W \cup \{x\} \colon W \in f(A, k-1), x \in g(A, W)\}$ for all $k \in \{1, \dots, m\}$ and $A \in \mathcal{A}^*$. Since $f(A, 0) = \{\emptyset\}$, this recursion is well-defined. As we show next, committee monotonicity is equivalent to the existence of a generator function.

Proposition 1. *An ABC voting rule f is committee monotone if and only if it is generated by a generator function g.*

Proof. Consider any f and g, s.t. $f(A, k) = \{W \cup \{x\} \colon W \in f(A, k-1), x \in g(A, W)\}$ for all profiles A and committee sizes k. Now, fix some $A \in \mathcal{A}^*$ and $k \in \{1, \dots, m\}$. If $W \in f(A, k)$, there is $W' \in f(A, k-1)$ and $x \in g(A, W')$ such that $W = W' \cup \{x\}$ because g generates f. Conversely, if $W' \in f(A, k-1)$, then $g(A, W')$ cannot be empty and there is a candidate $x \in \mathcal{C} \setminus W'$ such that $W \cup \{x\} \in f(A, k)$. Thus f is committee monotone.

Next, let f be committee monotone. Define g as follows: if $W \notin f(A, |W|)$, set $g(A, W) = \emptyset$. On the other hand, if $W \in f(A, |W|)$ and $W \neq \mathcal{C}$, there is a committee $W' \in f(A, |W|+1)$ with $W \subseteq W'$ due to the committee monotonicity of f. Set $g(A, W) = \{x \in \mathcal{C} \setminus W \colon W \cup \{x\} \in f(A, |W| + 1)\}$ if $W \in f(A, |W|)$. Let f_g denote the ABC voting rule defined by

$f_g(A,0) = \{\emptyset\}$ and $f_g(A,k) = \{W \cup \{x\}\colon W \in f_g(A,k-1), x \in g(A,W)\}$ for all $k > 0$. We prove inductively that $f_g(A,k) = f(A,k)$ for all profiles A and $k \in \{0,\dots,m\}$, which implies that f_g is well-defined and that g generates f. The induction basis $k=0$ is true since $f_g(A,0) = \{\emptyset\} = f(A,0)$ for all profiles A. Hence, consider a fixed $k \in \{0,\dots,m-1\}$ and $A \in \mathcal{A}^*$ and suppose that $f_g(A,k) = f(A,k)$. First, let $W \in f(A,k+1)$. Due to committee monotonicity, there is $W' \in \mathcal{W}_k$ and $x \in W \setminus W'$ such that $W' \in f(A,k) = f_g(A,k)$ and $W' \cup \{x\} = W$. This implies that $x \in g(A,W')$ and hence $W \in f_g(A,k+1)$. For the other direction, let $W \in f_g(A,k+1)$, which means that there are $W' \in f_g(A,k) = f(A,k)$ and $x \in g(A,W')$ such that $W = W' \cup \{x\}$. Hence, $f(A,k+1) = f_g(A,k+1)$ and we infer inductively that g generates f. $\square$

We now introduce axioms for generator functions. Our main condition is *consistency*, which is concerned with the behavior when combining two disjoint profiles. In more detail, suppose that the choice of the generator g intersects for two disjoint profiles A and A' and a committee W. Intuitively, the best candidates in the combined profile $A + A'$ should be exactly those that win for both individual electorates. Hence, consistency requires for such situations that, if $g(A+A',W) \neq \emptyset$, it contains precisely the elements in the intersection of $g(A,W)$ and $g(A',W)$. Note that such consistency axioms have already led to several prominent results [e.g., 27, 13, 5, 18].

Definition 2. A generator function g is *consistent* if $g(A,W) \cap g(A',W) \neq \emptyset$ and $g(A + A',W) \neq \emptyset$ imply that $g(A+A',W) = g(A,W) \cap g(A',W)$ for all disjoint profiles $A, A' \in \mathcal{A}^*$ and committees $W \in \mathcal{W} \setminus \{\mathcal{C}\}$. An ABC voting rule f is *consistently committee monotone* if it is generated by a consistent generator function.

Furthermore, analogous to ABC voting rules, we call a generator function g *proper* if it satisfies the following conditions:

- *anonymous*: $g(A,W) = g(\pi(A),W)$ for all $A \in \mathcal{A}^*$, $W \in \mathcal{W} \setminus \{\mathcal{C}\}$, and permutations $\pi : \mathbb{N} \to \mathbb{N}$,
- *neutral*: $g(\tau(A), \tau(W)) = \tau(g(A,W))$ for all $A \in \mathcal{A}^*$, $W \in \mathcal{W} \setminus \{\mathcal{C}\}$, and permutations $\tau : \mathcal{C} \to \mathcal{C}$,
- *continuous*: for all $A, A' \in \mathcal{A}^*$ and $W \in \mathcal{W} \setminus \{\mathcal{C}\}$ with $|g(A,W)| = 1$ and $g(A',W) \neq \emptyset$, there is $j \in \mathbb{N}$ such that $g(jA + A',W) = g(A,W)$, and
- *non-imposing*: for every $W \in \mathcal{W} \setminus \{\mathcal{C}\}$ and $x \in \mathcal{C} \setminus W$, there is $A \in \mathcal{A}^*$ such that $g(A,W) = \{x\}$.

Just as for ABC voting rules, all these axioms are very mild. Finally, we say that a generator function g is *complete* if $g(A,W) \neq \emptyset$ for all profiles $A \in \mathcal{A}^*$ and committees $W \in \mathcal{W}$.

2.2 Sequential Valuation Rules

The main goal of this paper is to characterize the class of sequential valuation rules. These rules rely on *valuation functions* v, which are mappings of the type $v : \mathcal{A} \times \mathcal{W} \to \mathbb{R}$, to compute the outcome. Less formally, a valuation function specifies for every ballot A_i and committee W the number of points that a voter with ballot A_i assigns to the committee W. The score of a committee W in a profile A is defined as $s_v(A,W) = \sum_{i \in N_A} v(A_i,W)$. Now, a *sequential valuation rule* f works as follows: $f(A,0) = \{\emptyset\}$ and for $k \geq 1$, $f(A,k) = \{W \cup \{x\}\colon W \in f(A,k-1) \wedge \forall y \in \mathcal{C} \setminus W\colon s_v(A, W \cup \{x\}) \geq s_v(A, W \cup \{y\})\}$, i.e., f extends in each step the currently chosen committees with the candidates that increase the score by the most.[1]

Note that our definition of sequential valuation rules is so general that it includes even non-proper ABC voting rules. For instance, if v is constant, the corresponding sequential valuation rule is trivial and thus fails non-imposition. Nevertheless, we will focus only on proper sequential valuation rules and in particular on the following subclasses.

- *Sequential Thiele rules* rely on a Thiele counting function to compute the outcome. A Thiele counting function is a mapping $h(x) : \{0, \dots, m\} \to \mathbb{R}$ which is non-negative, non-decreasing, and satisfies $h(1) > h(0)$. Then, the valuation function of a sequential Thiele rule is $v(A_i, W) = h(|A_i \cap W_i|)$. In other words, every voter values a committee only based on how many of its members she approves.[2]

- *Step-dependent sequential scoring rules* compute the winner based on a step-dependent counting function. A step-dependent counting function is a mapping $h(x, y, z) : \{0, \dots, m\} \times \{1, \dots, m\} \times \{1, \dots, m\} \to \mathbb{R}$ such that for every $y \in \{1, \dots, m-1\}$, there is $x \in \{1, \dots, y\}$ and $z \in \{x, \dots, m-1-(y-x)\}$ with $h(x, y, z) \neq h(x-1, y, z)$. Then, the valuation function of a step-dependent sequential scoring rule is $v(A_i, W) = h(|A_i \cap W|, |W|, |A_i|)$.

The class of sequential Thiele rules contains many prominent ABC voting rules, such as *sequential approval voting* (`seqAV`) defined by $h(x) = x$, *sequential proportional approval voting* (`seqPAV`) defined by $h(0) = 0$ and $h(x) = \sum_{i=1}^{x} \frac{1}{i}$ for $x > 0$, and *sequential Chamberlin-Courant approval voting* (`seqCCAV`) defined by $h(0) = 0$ and $h(x) = 1$ for $x > 0$. Finally, *sequential satisfaction approval voting* (`seqSAV`), defined by $h(x, y, z) = \frac{x}{z}$, is an example of a step-dependent sequential scoring rule (which, like `seqAV`, is equivalent to its global version). Every sequential valuation rule f is consistently committee monotone, as clearly its score-based generator function is by definition consistent. Further, all step-dependent sequential scoring rules are proper ABC voting rules. Here, the technical condition on h is necessary to ensure non-imposition. Finally, note that every sequential Thiele rule is a step-dependent sequential scoring rule. Consequently, both subclasses of sequential valuation rules only contain proper ABC voting rules. We can even make the relation between these types of rules precise as shown in the next, axiomatically sharp proposition.

Proposition 2. *The following equivalences hold:*

(1) A sequential valuation rule is a step-dependent sequential scoring rule if and only if it is proper.

(2) A step-dependent sequential scoring rule is a sequential Thiele rule if and only if it is independent of losers and committee separable.

Proof Sketch. The "only if" part of the claims is easy to prove. Hence, we focus on the "if" part. The key insight for (1) is that the valuation function v of a proper sequential valuation rule is neutral. Since $|A_i| = |\tau(A_i)|$, $|W| = |\tau(W)|$, and $|A_i \cap W| = |\tau(A_i \cap W)|$, for all ballots A_i, committees W, and permutations τ, the corresponding sequential valuation rule is a step-dependent sequential scoring rule. For (2), the "if" part intuitively holds because independence of losers excludes the possibility that the step-dependent counting function

[1] It is also possible to choose the committees that maximize the score for a given valuation function. These rules are proper and satisfy a consistency property for chosen committees (see [18]). However, they fail consistent committee monotonicity and it is not clear why they should be more desirable than their sequential variants.

[2] There are multiple different definitions of Thiele counting functions in the literature (e.g., [19, 9]). Our definition agrees with the one of Aziz et al. [1].

$h(x, y, z)$ depends on the size of the ballot z. By formalizing this insight, we can infer that for each fixed step y (and all z), $h(\cdot, y) = h(\cdot, y, z)$ is induced by a Thiele counting function. To remove the step-dependency, committee separability can connect the different steps of the rule: we can construct the disjoint profiles A, B with $f(A + B, |C_A|) = \{C_A\}$ such that committee separability forces all following steps to be equal to the choice for B. Formalizing this argument rules out that $h(x, y)$ depends on y and we obtain a sequential Thiele rule. $\square$

3 Characterizations of Sequential Valuation Rules

We are now ready to discuss our main result, a characterization of step-dependent sequential scoring rules: an ABC voting rule is a step-dependent sequential scoring rule if and only if it is proper and consistently committee monotone. Combined with Proposition 2, we infer as corollary also a characterization of sequential Thiele rules. This proposition also emphasizes the generality of our result since characterizing step-dependent sequential scoring rules is equivalent to characterizing all proper sequential valuation rules. Due to space constraints, we defer the proofs of all auxiliary propositions to the full version [10] and discuss proof sketches instead.

While it is easily shown that every step-dependent sequential scoring rule is proper and consistently committee monotone, the converse claim is much more involved. Our main idea for proving this direction is to investigate the generator function of consistently committee monotone and proper ABC voting rules. Hence, we first verify the conjecture that attractive committee monotone ABC voting rules are generated by well-behaved generator functions.

Proposition 3. *An ABC voting rule is proper and consistently committee monotone if and only if it is generated by a proper, consistent, and complete generator function.*

Proof Sketch. If an ABC voting rule f is generated by a proper, consistent, and complete generator function, it is fairly straightforward that it is consistently committee monotone and proper. Thus, we focus on the inverse direction and suppose that f is a proper and consistently committee monotone ABC voting rule. The key insight for this direction is that non-imposition and continuity can be generalized to sequences of committees $W_1, \dots, W_\ell$ with $|W_k| = k$ and $W_{k-1} \subseteq W_k$ for all $k \in \{1, \dots, \ell\}$ (we assume subsequently that $W_0 = \emptyset$):

(1) If $\ell < m$, there is a profile A such that $f(A, k) = \{W_k\}$ for all $k \in \{1, \dots, \ell\}$ and $f(A, \ell + 1) = \{W_\ell \cup \{x\} \colon x \in \mathcal{C} \setminus W_\ell\}$.

(2) For any two profiles A, A' such that $f(A, k) = \{W_k\}$ for all $k \in \{1, \dots, \ell\}$, there is an integer j such that $f(jA + A', k) = \{W_k\}$ for all $k \in \{1, \dots, \ell\}$.

We prove (1) by induction over ℓ: by non-imposition, there is a profile A^1 for every committee $W_{\ell+1} \in \mathcal{W}_{\ell+1}$ such that $f(A^1, \ell+1) = \{W_{\ell+1}\}$. Committee monotonicity implies then that there is a sequence of committees $W_1, \dots, W_\ell$ such that $W_k \in f(A^1, k)$ and $W_{k+1} \setminus W_k \subseteq g(A, W_k)$ for all $k \in \{1, \dots, \ell\}$, where g is a consistent generator function of f. By the induction hypothesis, there is a profile A^2 such that $f(A^2, k) = \{W_k\}$ for all $k \in \{1, \dots, \ell\}$ and $f(A^2, \ell + 1) = \{W_\ell \cup \{x\} \colon x \in \mathcal{C} \setminus W_\ell\}$. We can now use the consistency of g to infer that $f(A^1 + A^2, k) = \{W_k\}$ for all $k \in \{1, \dots, \ell + 1\}$. Finally, we can further add copies to ensure that $W_{\ell+1}$ is extended by all remaining candidates due to anonymity and neutrality. Continuity instantly implies (2) by choosing j large enough.

Now, we will extend the consistent generator function g of f to make it complete. Consider for this a sequence of committees $W_1, \dots, W_\ell$ with $|W_k| = k$ and $W_{k-1} \subseteq W_k$ for all $k \in \{1, \dots, \ell\}$. Due to (1), there is a profile A^{W_ℓ} with $f(A^{W_\ell}, k) = \{W_k\}$ for all $k \in \{1, \dots, \ell\}$ and $f(A^{W_\ell}, \ell + 1) = \{W_\ell \cup \{x\} \colon x \in \mathcal{C} \setminus W_\ell\}$. By (2), we can define the function $\hat{g}(A, W_\ell) = g(A + jA^{W_\ell}, W_\ell)$, where j is the smallest integer such that

$f(A + jA^{W_\ell}, k) = \{W_k\}$ for all $k \in \{1, \dots, \ell\}$. $\hat{g}$ generates f since $\hat{g}(A, W) = g(A, W)$ for all $A \in \mathcal{A}^*$ and $W \in f(A, |W|)$. This follows from consistent committee monotonicity as $g(jA^W, W) = \mathcal{C} \setminus W$, $g(A, W) \neq \emptyset$, and $g(A + jA^W, W) \neq \emptyset$. If $\hat{g}$ would fail anonymity, neutrality, non-imposition, or continuity, f would fail it too as g generates it. $\square$

As second step, we characterize the class of proper, consistent, and complete generator functions. In particular, we show that for every committee W, $g(A, W)$ can be described by a weighted variant of single winner approval voting. Formally, let $v : \{0, \dots, m\} \times \{1, \dots, m\} \to \mathbb{R}$ be a weight function. Then, v-weighted approval voting is defined as the generator function $AV_v(A, W) = \{c \in \mathcal{C} \setminus W : \forall d \in \mathcal{C} \setminus W : \sum_{i \in N_A : c \in A_i} v(|W \cap A_i|, |A_i|) \geq \sum_{i \in N_A : d \in A_i} v(|W \cap A_i|, |A_i|)\}$.

Proposition 4. *Let g denote a proper, consistent, and complete generator function. For every committee $W \neq \mathcal{C}$, there is a weight function v^W such that $g(A, W) = AV_{v^W}(A, W)$ for all profiles $A \in \mathcal{A}^*$.*

Proof Sketch. Let g denote a proper, consistent, and complete generator function and fix a committee $W \neq \mathcal{C}$. We show the proposition by applying a separating hyperplane argument analogous to how Young [27] derives his characterization of scoring rules.

For doing so, we first transform the domain of $g(\cdot, W)$ from preference profiles to a numerical space and we show thus that $g(\cdot, W)$ can be computed only based on the values $n(c, A, W, k, \ell) = |\{i \in N_A : c \in A_i \wedge |A_i \cap W| = k \wedge |A_i| = \ell\}|$ for $c \in \mathcal{C} \setminus W$, $k \in \{0, \dots, |W|\}$, and $\ell \in \{k+1, \dots, m-1-|W|+k\}$. For proving this, we first show that if $A_i \cap W = A_j \cap W$ and $|A_i| = |A_j|$ for all $i, j \in N_A$ and all candidates $x \in \mathcal{C} \setminus W$ are approved by the same number of voters, then $g(A, W) = \mathcal{C} \setminus W$. Once this restricted claim is proven, we can use our axioms to generalize it; e.g., consistency, neutrality, and anonymity then entail that $g(A^{k,\ell}, W) = \mathcal{C} \setminus W$ for all k, ℓ and profiles $A^{k,\ell}$ in which $|A_i^{k,\ell} \cap W| = k$ and $|A_i^{k,\ell}| = \ell$ for all $i \in N_A$. Finally, this means that if there are constants $c_{k,\ell}$ such that $n(x, A, W, k, \ell) = c_{k,\ell}$ for all candidates $c \in \mathcal{C} \setminus W$ and indices k and ℓ, then $g(A, W) = \mathcal{C} \setminus W$ as we can decompose A with respect to k and ℓ into these profiles $A^{k,\ell}$. Together with consistency, we infer from this observation that $g(\cdot, W)$ can indeed be computed based on the matrix $N(A, W)$ that contains all the values $n(c, A, W, k, \ell)$.

As next step, we use standard constructions to extend the domain of g further from integer matrices $N(A, W)$ to rational matrices. To this end, let Q_2 be the matrix that corresponds to the profile in which each ballot is reported once and note that $g(Q_2, W) = \mathcal{C} \setminus W$ due to anonymity and neutrality. Based on this matrix, we extend g to negative numbers by defining $g(Q_1, W) = g(Q_1 + jQ_2, W)$ (where $j \in \mathbb{N}$ is a scalar such that $Q_1 + jQ_2$ contains only positive integers) and as second step extend g to rational numbers by defining $g(Q_1, W) = g(jQ_1, W)$ (where j is the smallest integer such that jQ_1 only contains integers). For both steps, consistency ensures that g remains well-defined. Moreover, the extension of $g(\cdot, W)$ to rational numbers preserves all desirable properties of g.

Finally, we group the feasible input matrices Q into sets $R_c = \{Q : c \in g(Q, W)\}$ for $c \in \mathcal{C} \setminus W$. These sets are convex (with respect to $\mathbb{Q}$) and symmetric since g is consistent, anonymous, and neutral. Moreover, the interior of R_c and R_d is disjoint for $c, d \in \mathcal{C} \setminus W$ with $c \neq d$ and we can thus derive a separating hyperplane between these sets (see, e.g., [3]). As last step, we infer from these hyperplanes the weight function v^W. $\square$

Based on Proposition 4, we finally prove our axiomatically sharp main result.

Theorem 1. *An ABC voting rule is a step-dependent sequential scoring rule if and only if it is proper and consistently committee monotone.*

Proof. By Proposition 2, every step-dependent sequential scoring rule f is proper. For consistent committee monotonicity, let h denote its step-dependent counting function and let $W^x = W \cup \{x\}$ for all W and $x \in \mathcal{C} \setminus W$. By definition, $f(A, 0) = \emptyset$ and $f(A, k) = \{W^c : W \in f(A, k-1), c \in \mathcal{C} \setminus W : \forall d \in \mathcal{C} \setminus W : s_h(A, W^c) \geq s_h(A, W^d)\}$. Thus, $g(A, W) = \{c \in \mathcal{C} \setminus W : \forall d \in \mathcal{C} \setminus W : s_h(A, W^c) \geq s_h(A, W^d)\}$ is complete and generates f. Further, $s_h(A + A', W) = s_h(A, W) + s_h(A', W)$ for all profiles A, A' and committees W. Hence, g is consistent.

For the other direction, consider a proper and consistently committee monotone ABC voting rule f. By Proposition 3, f is generated by a proper, consistent, and complete generator function g and by Proposition 4, for every $W \neq \mathcal{C}$ there is some v^W such that $g(A, W) = AV_{v^W}(A, W)$ for all $A \in \mathcal{A}^*$. Now, consider W and W' with $|W| = |W'| < m$ and weight functions v^W and $v^{W'}$. We first show that $AV_{v^W}(A', W') = AV_{v^{W'}}(A', W')$ for every profile A'. For this, let $c' \in AV_{v^{W'}}(A', W')$ which is the case if and only if $\sum_{i \in N_{A'} : c' \in A'_i} v^{W'}(|W' \cap A'_i|, |A'_i|) \geq \sum_{i \in N_{A'} : d' \in A'_i} v^{W'}(|W' \cap A'_i|, |A'_i|)$ for all $d' \in \mathcal{C} \setminus W'$. Next, let $\tau : \mathcal{C} \to \mathcal{C}$ denote a permutation such that $\tau(W) = W'$, and let $A \in \mathcal{A}^*$ and $c \in \mathcal{C}$ such that $\tau(A) = A'$ and $\tau(c) = c'$. Because of $g(A, W) = AV_{v^W}(A, W)$, $g(A', W') = AV_{v^{W'}}(A', W')$, and the neutrality of g, it holds that $c' \in AV_{v^{W'}}(A', W')$ if and only if $c \in AV_{v^W}(A, W)$. By the definition of AV_{v^W}, the last claim is true if and only if $\sum_{i \in N_A : c \in A_i} v^W(|W \cap A_i|, |A_i|) \geq \sum_{i \in N_A : d \in A_i} v^W(|W \cap A_i|, |A_i|)$ for all $d \in \mathcal{C} \setminus W$. Finally, observe that $x \in A_i$ if and only if $\tau(x) \in A'_i$, $|A_i| = |A'_i|$, and $|W \cap A_i| = |W' \cap A'_i|$ for all candidates $x \in \mathcal{C} \setminus W$ and voters $i \in N_A$. Hence, we conclude that $\sum_{i \in N_A : c \in A_i} v^W(|W \cap A_i|, |A_i|) \geq \sum_{i \in N_A : d \in A_i} v^W(|W \cap A_i|, |A_i|)$ if and only if $\sum_{i \in N_A : c' \in A'_i} v^W(|W' \cap A'_i|, |A'_i|) \geq \sum_{i \in N_A : \tau(d) \in A'_i} v^W(|W' \cap A'_i|, |A'_i|)$ for all $d \in \mathcal{C} \setminus W$. This proves the claim.

Next, let $W_0, \dots, W_{m-1}$ denote committees such that $|W_i| = i$ and let $v^i = v^{W_i}$. We define the function $v(x, y, z) : \{0, \dots, m\} \times \{0, \dots, m-1\} \times \{1, \dots, m\} \to \mathbb{R}$ by $v(x, y, z) = v^y(x, z)$. By our previous reasoning, it holds that $g(A, W) = AV_{v^{|W|}}(A, W) = \{c \in \mathcal{C} \setminus W : \forall d \in \mathcal{C} \setminus W : \sum_{i \in N_A : c \in A_i} v(|A_i \cap W|, |W|, |A_i|) \geq \sum_{i \in N_A : d \in A_i} v(|A_i \cap W|, |W|, |A_i|)\}$. Our next goal is to derive a valuation function from v. For doing so, define the function $h(x, y, z) : \{0, \dots, m\} \times \{1, \dots, m\} \times \{1, \dots, m\} \to \mathbb{R}$ as follows: $h(0, y, z) = 0$ for all $y, z \in \{1, \dots, m\}$ and $h(x, y, z) = h(x-1, y, z) + v(x-1, y-1, z)$ for all $x, y, z \in \{1, \dots, m\}$. We claim that f is the sequential valuation rule induced by the valuation function $w(A_i, W) = h(|A_i \cap W|, |W|, |A_i|)$. For this, let $g_w(A, W) = \{c \in \mathcal{C} \setminus W : \forall d \in \mathcal{C} \setminus W : \sum_{i \in N_A} w(A_i, W \cup \{c\}) \geq \sum_{i \in N_A} w(A_i, W \cup \{d\})\}$. We will show that $g_w(A, W) = g(A, W)$ for all profiles $A \in \mathcal{A}^*$ and committees $W \neq \mathcal{C}$. Note for this that for all profiles A, committees W, and candidates $c \in \mathcal{C} \setminus W$, we can partition the voters $i \in N_A$ into ones approving or not approving c, which yields $\sum_{i \in N_A} h(|W^c \cap A_i|, |W^c|, |A_i|) - h(|W \cap A_i|, |W^c|, |A_i|) = \sum_{i \in N_A : c \in A_i} v(|W \cap A_i|, |W|, |A_i|)$. Now, define $C(A, W) = \sum_{i \in N_A} h(|W \cap A_i|, |W|+1, |A_i|)$. Then $s_w(A, W^c) \geq s_w(A, W^d)$ if and only if $s_w(A, W^c) - C(A, W) \geq s_w(A, W^d) - C(A, W)$ if and only if $\sum_{i \in N_A : c \in A_i} v(|W \cap A_i|, |W|, |A_i|) \geq \sum_{i \in N_A : d \in A_i} v(|W \cap A_i|, |W|, |A_i|)$. Hence, $g_w(A, W) = g(A, W)$ for all A and W, i.e., f is the sequential valuation rule generated by g. Since f is proper, an application of Proposition 2 concludes the proof. □

Due to Proposition 2, we obtain a characterization of sequential Thiele rules.

Corollary 1. *An ABC voting rule is a sequential Thiele rule if and only if it is consistently committee monotone, independent of losers, committee separable, and proper.*

4 Characterizations of Specific ABC Voting Rules

Finally, we leverage our results to derive characterizations of specific voting rules. First, we note that our characterizations can be combined with known results that single out rules within the class of, e.g., sequential Thiele rules, to derive full characterizations [e.g., 16, 18].

Nevertheless, we prefer to present our own characterizations for `seqCCAV`, `seqAV`, and `seqPAV` to highlight new aspects of these rules. We state our results restricted to sequential Thiele rules; Corollary 1 turns them into full characterizations by adding the axioms. In this section, we assume $m \geq 3$ since every sequential Thiele rule coincides with `seqAV` if $m = 2$.

The main idea for our characterizations is to study how ABC voting rules treat clones. To this end, we say that two candidates c, d are *clones* in a profile A if $c \in A_i$ if and only if $d \in A_i$ for all voters $i \in N_A$. Depending on the goal of the election, clones should be treated differently. For instance, if our goal is to choose a committee that is as diverse as possible, there is no point in choosing both clones. We formalize this new condition as follows: an ABC voting rule f is *clone-rejecting* if $f(A, |W|) = \{W\}$ implies that $\{c, d\} \not\subseteq W$ for all profiles A with clones c, d and committees $W \neq \mathcal{C}$. This axiom characterizes `seqCCAV`.

Theorem 2. `seqCCAV` *is the only sequential Thiele rule that satisfies clone-rejection.*

Proof. Since `seqCCAV` clearly satisfies clone-rejection, we focus on the inverse direction. Consider a sequential Thiele rule f other than `seqCCAV` and let h denote its Thiele counting function, where we can suppose w.l.o.g. that $h(0) = 0$ and $h(1) = 1$. Moreover, because f is not `seqCCAV`, there is $x \in \{2, \dots, m-1\}$ such that $h(x) > 1$ and $h(x') = 1$ for all $x' \in \{1, \dots, x-1\}$. Now, let $\Delta = h(x) - 1$ and $\ell \in \mathbb{N}$ such that $\ell\Delta > 1$. We consider the following profile A: there are ℓ voters who approve the candidates $c_1, \dots, c_x$, x voters who approve c_1 and c_2, and for all $i \in \{3, \dots, x+1\}$ there are $x + 2 - i$ voters who approve only c_i. Now, due to the minimality of x, f agrees in the first $x - 1$ rounds with `seqCCAV` and we thus have that $f(A, x-1) = \{\{c_1, c_3, \dots, c_x\}, \{c_2, c_3, \dots, c_x\}\}$. On the other hand, it holds that $s_h(A, \{c_1, \dots, c_x\}) \geq s_h(A, \{c_1, c_3, \dots, c_x\}) + \ell\Delta > s_h(A, \{c_1, c_3, \dots, c_x\}) + 1$ and $s_h(A, \{c_1, c_3, \dots, c_x, c_{x+1}\}) = s_h(A, \{c_2, c_3, \dots, c_x, c_{x+1}\}) = s_h(A, \{c_1, c_3, \dots, c_x\}) + 1$. Thus, $f(A, x) = \{\{c_1, \dots, c_x\}\}$ and f fails clone-rejection as c_1 and c_2 are contained. $\square$

The polar opposite to diverse committees are quality-based ones, where the goal is to find the k best candidates regardless of how well they represent the voters. Here, clones should be treated as equal as possible and we thus propose the following notion: an ABC voting rule f is *clone-accepting* if for all profiles A with clones c, d and committees $W \subseteq \mathcal{C} \setminus \{c, d\}$, it holds that $W \cup \{c\} \in f(A, |W \cup \{c\}|)$ implies that $W \cup \{c, d\} \in f(A, |W \cup \{c, d\}|)$. Or, in words, the only reason that a winning committee does not contain both clones is if this conflicts with the committee size. Perhaps surprisingly, clone-acceptance does not characterize `seqAV` as, e.g., the sequential Thiele rule defined by $h(0) = 0$, $h(1) = 1$, and $h(x) = 2x + 1$ for $x \geq 2$ satisfies this axiom, too. However, this rule prefers to choose candidates that are approved by voters who already approve a chosen candidate. This behavior can be interpreted as trust in a voter's recommendation and can be reasonable for quality-based elections. Nevertheless, to single out `seqAV`, we use a mild condition prohibiting this behavior: an ABC voting rule f is *distrusting* if for all profiles A, committees $W \neq \mathcal{C}$ with $f(A, |W|) = \{W\}$, and candidates b, c, it holds that $b \in W$ implies $c \in W$ if more voters in A report the ballot $\{c\}$ than there are voters who approve b. This leads to the following result.

Theorem 3. `seqAV` *is the only sequential Thiele rule that is clone-accepting and distrusting.*

Proof Sketch. We focus on the direction from right to left and consider a sequential Thiele rule $f \neq$ `seqAV`. Let h denote its Thiele counting function with $h(0) = 0$ and $h(1) = 1$. Since f is not `seqAV`, there is an integer $x \in \{2, \dots, m-1\}$ such that $h(x) \neq x$ but $h(x') = x'$ for $x' < x$. Now, let $\Delta = |h(x) - x|$ and $\ell \in \mathbb{N}$ such that $\ell\Delta > 1$. If $h(x) > x$, f fails distrust in the following profile A, where W is a committee of size $x - 1 \leq m - 2$ and $c, d \in \mathcal{C} \setminus W$: ℓ voters approve $W \cup \{c\}$, $\ell + 1$ voters approve d, and two voters approve W. It can be checked that $f(A, x) = \{W \cup \{c\}\}$ but distrust requires that d is not chosen after c. If $h(x) < x$, f fails clone-acceptance in the following profile A, where we choose W with

$|W| = x - 2 \leq m - 3$ and $b, c, d \in \mathcal{C} \setminus W$: ℓ voters report $W \cup \{c, d\}$ and $\ell - 1$ voters report b. As desired, $f(A, x-1) = \{W \cup \{c\}, W \cup \{d\}\}$ but $f(A, x) = \{W \cup \{b, c\}, W \cup \{b, d\}\}$. □

Finally, a large stream of research on ABC voting rules tries to find proportional committees, i.e., the chosen committee should proportionally reflect the voters' preferences. For defining this concept, we rely on heavily restricted profiles A in which n_1 voters report the same ballot A_1 and n_2 voters approve a single candidate $c \notin A_1$. In such a profile, each clone $d \in A_1$ that is in the elected committee W represents on average $\frac{n_1}{|A_1 \cap W|}$ voters, whereas the candidate c represents n_2 voters. Following the idea of proportionality, we should choose a subset of A_1 for a committee size k if $\frac{n_1}{k} > n_2$ as every candidate $d \in A_1$ represents on average more voters than c. Conversely, if $\frac{n_1}{k} < n_2$, the chosen committee should contain c. Thus, we say an ABC voting rule is *clone-proportional* if for all such profiles A, committee sizes $k \leq |A_1|$, and committees $W \in f(A, k)$, it holds that $c \notin W$ if $\frac{n_1}{k} > n_2$ and $c \in W$ if $\frac{n_1}{k} < n_2$. Note that clone-proportionality is closely related to D'Hondt proportionality [8, 18] with two parties. Next, we show that this axiom characterizes `seqPAV`.

Theorem 4. `seqPAV` *is the only sequential Thiele rule that satisfies clone-proportionality.*

Proof Sketch. We only show that no other sequential Thiele rule f but `seqPAV` satisfies clone-proportionality. For this, let h denote the Thiele counting function of f and normalize h such that $h(0) = 0$ and $h(1) = 1$. Since f is not `seqPAV`, there is a minimal integer $x \in \{2, \dots, m-1\}$ such that $h(x) \neq \sum_{i=1}^{x} \frac{1}{i}$. As in the proofs of Theorems 2 and 3, we can now construct a profile in which f fails clone-proportionality. For instance, if $h(x) > \sum_{i=1}^{x} \frac{1}{i}$, let $\Delta = h(x) - \sum_{i=1}^{x} \frac{1}{i}$ and $\ell \in \mathbb{N}$ such that $\ell x \cdot \Delta > 1$ and consider the following profile A: ℓx voters report $\{c_1, \dots, c_x\}$ and $\ell + 1$ voters approve a single candidate $c \notin \{c_1, \dots, c_x\}$. It can be checked that $f(A, x) = \{\{c_1, \dots, c_x\}\}$ but clone-proportionality requires that $c \in W$ for $W \in f(A, x)$ as $\ell + 1 > \frac{\ell x}{x}$. A similar counter example can be constructed if $h(x) < \sum_{i=1}^{x} \frac{1}{i}$ and thus, `seqPAV` is the only sequential Thiele rule that satisfies this axiom. □

5 Conclusion

In this paper, we provide axiomatic characterizations for the new class of sequential valuation rules. These rules are based on valuation functions, which assign each pair of ballot and committee a score and compute the winning committees greedily by extending the current winning committees with the candidates that increase the score by the most. Clearly, sequential valuation rules generalize the prominent class of sequential Thiele rules whose valuation function only depends on the size of the intersection between the given ballot and committee. Our main result characterizes the class of proper (i.e., anonymous, neutral, continuous, and non-imposing) sequential valuation rules based on a new axiom called consistent committee monotonicity. This axiom combines the well-known notions of committee monotonicity and consistency by requiring that the winning committees of size k are derived from those of size $k - 1$ by only adding new candidates, and that these newly added candidates are chosen in a consistent way across the profiles. By adding additional conditions, we also derive characterizations of important subclasses such as sequential Thiele rules and of prominent ABC voting rules such as sequential proportional approval voting. For a full overview of our results, we refer to Figure 1.

Our theorems address one of the major open problems in the field of ABC voting: while there is an enormous number of different voting rules, there are almost no characterizations. Such characterizations are crucial for reasoning about which rule to use because without a characterization, there is always the possibility that a more attractive rule exists. Moreover, many ideas of our results seem rather universal and it might be possible to reuse them to characterize other rules such as Phragmen's rule or Thiele rules.

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