Market-Based Explanations of Collective Decisions

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Abstract

We consider approval-based committee elections, in which a size-k subset of available candidates must be selected given approval sets for each voter, indicating the candidates approved by the voter. A number of axioms capturing ideas of fairness and proportionality have been proposed for this framework. We argue that even the strongest of them, such as priceability and the core, only rule out certain undesirable committees, but fail to ensure that the selected committee is fair in all cases. We propose two new solution concepts, *stable priceability* and *balanced stable priceability*, and show that they select arguably fair committees. Our solution concepts come with a non-trivial-to-construct but easy-to-understand market-based explanation for why the chosen committee is fair. We show that stable priceability is closely related to the notion of Lindahl equilibrium from economics.

1 Introduction

A committee election is a scenario where a group of individuals–called voters—collectively selects a size-k subset of available candidates, for a given k. The model of committee elections describes real-life situations such as selecting political representatives for a group of voters, selecting finalists or laureates in a contest (where voters correspond to judges or experts who collectively select a subset of contestants), deciding on locations of public facilities [Farahani and Hekmatfar, 2009, Skowron et al., 2016], and selecting validators in the blockchain protocol [Amoussou-Guenou et al., 2020a,b].

A committee election rule is a function specifying how voters' preferences map to the collective decision on which candidates should be selected. We focus on the model of approval preferences, in which each voter approves a subset of the candidates. A number of different committee election rules have been proposed for this model [Faliszewski et al., 2017, Lackner and Skowron, 2020]. In order to choose the right rule for a given scenario, one needs to be able to reason about these rules in a principled way. Various approaches have been proposed for this purpose. A compelling one is the axiomatic approach, in which one formulates desirable mathematical properties and asks which voting rules satisfy them.

For approval-based committee elections, axioms that capture how well minorities of voters with common interests are represented have received considerable attention in recent years [Aziz et al., 2017, Brill et al., 2017, Sánchez-Fernández et al., 2017, Lackner and Skowron, 2018, Peters and Skowron, 2020]. The starting point of our discussion is that even the strongest of these axioms fail to ensure that the selected committee is intuitively fair or proportionally representative on all instances. We explain this via the example of *the core* [Aziz et al., 2017, Fain et al., 2018, Cheng et al., 2019, Jiang et al., 2019, Peters and Skowron, 2020].

Outcomes in the Core do not Have to be Fair

Assume that there are n voters. Each voter i specified a subset of candidates A_i that she approves, and the goal is to select a committee of exactly k candidates. The high-level idea behind the core is that a group of voters S should be able to decide on at least $\lfloor {}^{|S|}/_{n} \cdot k \rfloor$

candidates in the elected committee. Formally, we say that a committee W is in the core if there exists no group of voters S and no subset of candidates T such that $|T| \leqslant |S|/n \cdot k$ and each voter from S prefers T over W (i.e. approves more candidates in T than in W). The core is a very strong concept. It implies a number of weaker properties, such as extended justified representation (EJR) [Aziz et al., 2017], proportional justified representation (PJR) [Sánchez-Fernández et al., 2017], and justified representation (JR) [Aziz et al., 2017]. In fact, the core is so strong that, for the time being, it is not known whether there always exists a committee in the core for approval-based elections. However, even this strong property can sometimes allow dramatically unfair committees, as the example below illustrates.

Example 1. Fix an integer L. Consider the following instance with $n = kL$ voters. Voters $1, \dots L$ approve candidates $c_1, \dots, c_k$. For each $i \in \{1, 2, \dots, k-1\}$ voters $iL+1, \dots, iL+L-1$ approve candidate c_{k+i}. The remaining $k-1$ voters approve candidates $c_{2k}, \dots, c_{3k-2}$, each voter approving a different candidate.

For this instance, the committee $W_1 = \{c_1, \dots, c_k\}$ is in the core. In fact this committee would be uniquely selected by the following natural rule: "among all committees in the core (assuming it is non-empty), select the one that maximizes the total number of approvals". This committee gives zero satisfaction to a majority of the voters. One can argue that a committee that consists of at most one candidate per group, e.g., $W_2 = \{c_1, c_{k+1}, \dots, c_{2k-1}\}$, is a much fairer choice. □

Similar observations have been made by Bredereck et al. [2019] for the axiom of extended justified representation (EJR). They observe that in practice many very different committees satisfy EJR, and concluded that EJR on its own does not guarantee a sensible selection of committees.

Priceability and Evidence of Fairness

The problem illustrated in Example 1 is that properties like the core (and also EJR, PJR and JR) prevent specific pathological situations, but beyond their definitions, do not provide intuitive justifications for why a committee they allow should be selected. In this paper, we take a different approach and aim for solution concepts that provide explicit and intuitive explanations for why the chosen committee is fair.

In their recent paper, Peters and Skowron [2020] introduced the concept of *priceability*. Intuitively, it decides on a fixed price that it will cost to add a candidate, endows each voter with a fixed amount of virtual money, and allows voters to spend money on buying candidates they like. Candidates are added to the committee when voters collectively pay the price. Priceability seeks committees which can be explained via this process, and under which no group of voters have so much money left over so that they could collectively buy one more candidate. The latter condition ensures that voters were able to spend a large chunk of their money, and may thus already have derived sufficient satisfaction from the candidates they purchased.

This is a step in the right direction: the individual payments that voters make towards buying approved candidates constitute an intuitive explanation and serve as evidence that the chosen committee is fair. However, this explanation is weak: it only requires that voters have limited leftover money, but not that their money is spent wisely. We add a *stability* condition: informally, voters should not want to change how their money is spent. We borrow the idea of making payments stable from the classic economic concept of Lindahl equilibrium for public economies [Foley, 1970], which ensures fair outcomes in a model with divisible goods [Fain et al., 2016]. In fact, we show that one of the two notions we propose is closely related to (a discrete version of) Lindahl equilibrium.

Our Contribution

We introduce two solution concepts: stable priceability (SP) and balanced stable priceability (BSP).

SP strengthens the concept of priceability by Peters and Skowron [2020]. We show that it is a strong fairness notion. It logically implies both the core and priceability, and also guarantees a higher proportionality degree [Skowron, 2018] than both. In contrast to the core, whether a committee satisfies SP can be checked in polynomial time. We also present a compact integer linear program for finding SP committees.

We show that, unfortunately, SP committees do not always exist. However, through a series of extensive experiments, we argue that "almost SP" committees often do (specifically ones whose size is very close to k); see Appendix D for details. Finally, we adapt the notion of Lindahl equilibrium to the committee election context, and show that SP is closely related to it.

One potential source of unfairness under stable priceability is that two voters may be paying different amounts of virtual money for the same candidate that they both approve. Our notion of balanced stable priceability (BSP) addresses this by requiring that any two voters paying for a candidate must pay the same amount. We uniquely characterize BSP committees as those returned by a variant of the recently introduced Rule X [Peters and Skowron, 2020]. Similarly to SP, we show that BSP committees do not always exist, but "almost BSP" committees often do.

Due to space constraints, we must defer almost all proofs to the supplementary material.

2 Preliminaries

For $t \in \mathbb{N}$, let $[t] = \{1, 2, \ldots, t\}$. An *election* is a tuple $(C, N, \{A_i\}_{i \in N}, k)$, where:

1. $C = \{c_1, \ldots, c_m\}$ and $N = [n]$ are sets of m *candidates* and n *voters*, respectively;

2. For each voter $i \in N$, $A_i \subseteq C$ denotes the set of candidates *approved* by i. Conversely, for a candidate $c \in C$, we denote by $N(c)$ the set of voters who approve c: $N(c) = \{i \in N \colon c \in A_i\}$. For clarity of presentation, we define the *utility function* of each voter $i \in N$ as $u_i(T) = |A_i \cap T|$ for all subsets of candidates (also referred to as *committees*) $T \subseteq C$. For simplicity, for each $c \in C$, we write $u_i(c) \in \{0, 1\}$ instead of $u_i(\{c\})$;

3. $k \in [m]$ is the number of candidates to be selected. We say that committee $W \subseteq C$ is *feasible* if $|W| = k$.

2.1 Proportionality of Election Rules

An *election rule*, or in short a *rule*, is a function that for each election returns a nonempty set of feasible committees, called *winning committees*.[1] This paper studies group-fairness of election rules in the approval-based model. One such concept of fairness is the core.

Definition 1 (The Core)**.** Given an election, we say that a committee $W \subseteq C$ is in the *core*, if for each $S \subseteq N$ and $T \subseteq C$ with $|T|/k \leqslant |S|/n$, there exists $i \in S$ such that $u_i(W) \geqslant u_i(T)$. We say that an election rule $\mathcal{R}$ satisfies the *core property* if for every election E, each winning committee $W \in \mathcal{R}(E)$ is in the core.

Two other well-established properties in the literature—the proportionality degree [Sánchez-Fernández et al., 2017, Aziz et al., 2018, Skowron, 2018] and extended justified

[1]Typically, a rule selects a single winning committee; however, we allow the possibility of ties.

representation (EJR) [Aziz et al., 2017, 2018]—provide guarantees for cohesive groups of voters. A group $S \subseteq N$ is *ℓ-cohesive* if it is large enough ($|S| \geqslant \ell \cdot n/k$) and if its members approve at least ℓ common candidates ($|\bigcap_{i\in S} A_i| \geqslant \ell$).

Definition 2 (Extended Justified Representation)**.** A rule $\mathcal{R}$ satisfies *extended justified representation (EJR)* if for every election E, each winning committee $W \in \mathcal{R}(E)$, and each ℓ-cohesive group of voters S, there exists a voter $i \in S$ who approves at least ℓ members of W.

Definition 3 (Proportionality Degree)**.** Let $f\colon \mathbb{N} \to \mathbb{R}$. We say that a rule $\mathcal{R}$ has the proportionality degree of f, if for every election E, each winning committee $W \in \mathcal{R}(E)$, and each ℓ-cohesive group of voters S, the average number of committee members a voter from S approves is at least $f(\ell)$, that is., $(1/|S|) \cdot \sum_{i\in S} u_i(W) \geqslant f(\ell)$.

It is known that EJR implies a proportionality degree of at least $f(\ell) = \frac{\ell-1}{2}$, and that a proportionality degree of more than $f(\ell) = \ell - 1$ implies EJR [Aziz et al., 2018].

2.2 Priceability

Peters and Skowron [2020] define the concept of *price systems* and the related property of *priceablity.* A price system is a pair $\mathrm{ps} = (p, \{p_i\}_{i\in N})$, where $p \in \mathbb{R}_+$ is the *price* of electing one candidate, and for each voter $i \in N$, $p_i\colon C \to [0, p]$ is a *payment function* that specifies the amount of money a particular voter pays for the elected candidates. Formally, a committee W is *supported by a price system* $\mathrm{ps} = (p, \{p_i\}_{i\in N})$ if the following conditions hold:

(C1). A voter only pays for candidates she approves, so that $p_i(c) = 0$ for each $i \in N$ and $c \notin A_i$.

(C2). Each voter has the same initial budget of 1 unit of a virtual currency: $\sum_{c\in C} p_i(c) \leqslant 1$ for each $i \in N$.

(C3). Each elected candidate gathers a total payment of p: $\sum_{i\in N} p_i(c) = p$ for each $c \in W$.

(C4). Voters do not pay for non-elected candidates: $\sum_{i\in N} p_i(c) = 0$ for each $c \notin W$.

(C5). For each unelected candidate, her supporters have an unspent budget of at most p: formally, $\sum_{i\in N(c)} r_i \leqslant p$ for each $c \notin W$, where for each $i \in N$:

$$r_i = 1 - \textstyle\sum_{c'\in W} p_i(c'). \tag{1}$$

Given a payment function p_i, it will be useful to write $p_i(W) = \sum_{c\in W} p_i(c)$ for sets $W \subseteq C$.

A committee W is said to be *priceable* if there exists a price system $\mathrm{ps} = (p, \{p_i\}_{i\in N})$ that supports W (i.e., that satisfies conditions (C1)–(C5)). For each $k \in \mathbb{N}$, a feasible priceable committee always exists; for example, Phragmén's sequential rule always returns one [Peters and Skowron, 2020].

Note that two voters might pay different amounts of money for the same candidate. In Section 4, we consider price systems where this is not allowed.

3 Stable Price Systems

While priceability is an intuitively appealing property, on its own it does not imply other desired fairness-related properties (except for the rather weak PJR property, see Peters and Skowron, 2020). For example, consider what we will call the *Utilitarian Priceable Rule* (UPR) which picks, among priceable committees, those that maximize the utilitarian social welfare, i.e., total number of approvals from voters. UPR fails EJR. In fact, as we show in Proposition 1, the proportionality degree of UPR is at most 2, which means that UPR does not even approximate EJR up to a sublinear factor.[2] Intuitively, this means priceability provides a very weak proportionality guarantee for cohesive groups of voters. Because the core implies EJR, UPR also violates the core.

Proposition 1. *The proportionality degree of Utilitarian Priceable Rule is at most 2.*

Corollary 1. *The Utilitarian Priceable Rule violates EJR; in fact, it does not approximate EJR by a factor better than* $\ell/4$.

The price system constructed in the priceability definition serves as some evidence that the committee selected is fair to groups: no group of voters can use their leftover money to buy a new candidate, and hence that group must already have used most of their money to buy approved candidates. However, this is weak evidence because the definition does not require that the money already spent by the voters is spent wisely. This is why priceability, on its own, does not imply strong fairness guarantees.

In this paper, we enhance the definition of priceability by replacing (C5) with a stronger condition which requires that voters' money be spent wisely. Later, we show that this is strong enough to imply a high proportionality degree and the core (and therefore EJR too).

Let $\succ$ be a linear order over $\mathbb{N} \times \mathbb{R}_+$ defined as follows:

$$(x, p) \succ (y, q) \iff x > y \text{ or } (x = y \text{ and } p < q). \tag{2}$$

We will use $(x, p) \succ (y, q)$ to model a voter who "prefers" to pay p dollars for a committee where she approves x candidates than pay q dollars for a committee with y approved members. Thus, under this linear order, the voter "prefers" to maximize her utility for the committee, and only in case of a tie, prefers to pay less. We note that these are not the true preferences of the voters, but rather an artificial relation that helps us formulate our definition of stable priceability.

We say that a price system $\mathrm{ps} = (p, \{p_i\}_{i\in N})$ is *stable* if it satisfies (C1)–(C4), and:

(S5). Condition for Stability: There exists no coalition of voters $S \subseteq N$, no subset $W' \subseteq C \setminus W$, and no collections $\{p'_i\}_{i\in S}$ ($p'_i \colon W' \to [0, 1]$) and $\{R_i\}_{i\in S}$ (with $R_i \subseteq W$ for all $i \in S$) such that all the following conditions hold:

1. For each $c \in W'$: $\sum_{i\in S} p'_i(S) > p$.
2. For each $i \in S$: $p_i(W \setminus R_i) + p'_i(W') \leqslant 1$.
3. For each $i \in S$:

$$\big(u_i(W \setminus R_i \cup W'), p_i(W \setminus R_i) + p'_i(W')\big) \succeq \big(u_i(W), p_i(W)\big).$$

A committee W is said to be *stable priceable* if there exists a stable price system $\mathrm{ps} = (p, \{p_i\}_{i\in N})$ that supports W (i.e., that satisfies conditions (C1)–(C4) and (S5)).

In order to better understand the condition, assume (S5) is not satisfied. Then, each voter $i \in S$ can find a set R_i of currently approved candidates such that she would "prefer" to stop paying for R_i and to pay for W' instead (i.e. she would be at least as "happy", according to $\succeq$, with $(W \setminus R_i) \cup W'$ and the new payments than with W and the old payments). In

[2]See the work of Skowron [2018] for more on EJR approximation.

addition, the total amount paid by the voters from S to each candidate in W' would *exceed* the price p of a candidate.

Let us explain why we require a strict inequality in the first condition of (S5). This way, our definition is consistent with the standard definition of priceability; this also allows us to deal with tie-breaking issues that lead to nonexistence of stable priceable committees in very small symmetric instances (see Theorem 2). We note that we can use other possible linear orders $\succ$ in the definition of stable priceability; we discuss one such alternative in Appendix B.

3.1 A Simpler Formulation of SP

Condition (S5) can be formulated in a simpler and rather more concise form. Consider the following inequality:

$$\forall c \notin W \sum_{i \in N(c)} \max\left(\max_{a \in W}\left(p_i(a)\right), r_i\right) \leqslant p. \tag{3}$$

Here, r_i is as defined in (1). Condition (3) is similar to (S5), but only prevents a group of voters from paying for a single new candidate. For example, we can easily observe that (S5) for $|W'| = 1$ implies (3). Indeed, assume (S5), take $c \notin W$, let $W' = \{c\}$, and consider $i \in N(c)$. If $r_i \geqslant \max_{a \in W}(p_i(a))$, then set $R_i = \emptyset$ and $p'_i(c) = r_i$; otherwise, let $c' = \operatorname{argmax}_{a \in W}(p_i(a))$, set $R_i = \{c'\}$ and $p'_i(c) = p_i(c')$. In both cases, voter i weakly prefers to replace R_i with W', and i can exchange R_i for W' within her budget. Thus, by (S5) we to have that $p \geqslant \sum_{i \in S} p'_i(c)$:

$$p \geqslant \sum_{i \in S} p'_i(c) \geqslant \sum_{i \in N(c)} \max\left(\max_{a \in W}\left(p_i(a)\right), r_i\right).$$

We show the other implication in the proof of Theorem 1.

At first, it might seem that restricting to $|W'| = 1$ makes the condition weaker. For example, inequality (3) does not imply (C1), as taking $W' = \emptyset$ is no longer possible. However, surprisingly, it turns out that this is the only difference: allowing $|W'| > 1$ does not increase the strength of the condition. We will show that (3) together with (C1) is equivalent to (S5). Thus, every price system satisfying (3) is SP.

Theorem 1. *Inequality* (3) *together with condition (C1) is equivalent to condition (S5).*

An important consequence of Theorem 1 is that (C1)–(C4) and (3) can be formulated as a linear program, and thus, we can efficiently check whether a given committee is SP.

Corollary 2. *Given election and a committee W, it can be checked in polynomial-time whether there exists a stable price system supporting W.*

This is in contrast to many other group fairness properties, which are coNP-hard to check [Aziz et al., 2018]. Further, one can formulate a compact integer linear program for finding SP committees (see Appendix C.1).

The most pressing question is whether SP committees exist for all elections. The answer is negative. The counterexample is provided in Appendix A.

Theorem 2. *There exists an election for which no feasible commitee is supported by a stable price system.*

In Appendix D we describe the results of experiments that we conducted for synthetic distributions of voters' preferences and for real datasets. There, we assumed that we are allowed to return committees that are slightly smaller or slightly larger than k. We found that it is almost always possible to find a committee, large part of which is SP (thus, even if an SP committee does not exist we have means to find a committee which is "almost" SP). Conversly, our experiments suggest it is possible to select an SP committee that exceeds the desired size k only by a small magnitude.

3.2 SP versus Priceability and the Core

Stable priceability obviously implies priceability. The following result shows that it also implies the core, and therefore, in turn, EJR.

Theorem 3. *SP implies the core.*

Corollary 3. *SP implies EJR.*

The core on its own is already a formidable axiom, and not known to be achievable in all elections. Are there any advantages to considering an axiom that further strengthens the core, and also strengthens priceability? We argue that there are several advantages.

As already mentioned, a first advantage that SP has over the core is that whether a committee is SP can be checked in polynomial time (Corollary 2), whereas the same question is known to be difficult for the core (see, e.g., the proof of Theorem 2 of Aziz and Monnot, 2020). Additionally, in elections like Example 1 presented in the introduction, the core allows apparently unfair solutions (such as the committee of all green candidates), while SP rules them out and allows only fairer solutions (such as the committee of all blue candidates and one green candidate). Moreover, an advantage that SP has over priceability is that SP implies the core, and in turn, EJR (Theorem 3), whereas priceability does not even imply EJR (Corollary 1). Finally, one advantage that SP has over both the core and priceability is that SP implies a high proportionality degree, as the following result shows.

Theorem 4. *SP implies a proportionality degree of $\ell - 1$.*

In contrast, it is known that EJR and the core only implies a proportionality degree of $\frac{\ell-1}{2}$ [Skowron, 2018], and priceability does not imply a proportionality degree better than 2 (see Proposition 1).

3.3 SP and Lindahl equilibrium

The concept of (stable) priceability suggests there might exist a relation between our voting model and classic market models for economies with public goods. In this section we explain this relation in more detail, focusing on the most influential equilibrium concept from the literature on public goods—the Lindahl equilibrium, which was formalized by Foley [1970]. The relation that we explain in this section: (i) gives additional insights into the concept of SP, and (ii) explains the key differences that prohibit one to use the concepts from the public economics directly for designing voting systems.

The public economics (PE) model for committee elections (CE), adapted from Foley [1970], is set up as follows. Each voter is endowed with 1 dollar; thus, the total endowment is n dollars. We imagine that there is a *producer* who will set up the committee in exchange for money. The *production function* $\pi\colon 2^C \to \mathbb{R}_+$ assigns to each committee $W \subseteq C$ the cost to the producer of producing W.[3] We assume the cost of producing a candidate is the same for all candidates, so we use $\pi(W) = |W| \cdot p$ for some $p \in \mathbb{R}_+$ and all $W \subseteq C$.

[3]In Foley's model [Foley, 1970], the production function specifies how private goods can be transformed into public goods. In our case, we assume there is only one private good, money (which represents voting

A *PE price system* is a collection $\{\gamma_i\}_{i\in N}$, where each γ_i is a payment function (see the definitions at the beginning of Section 3). PE price systems differ from the price systems as used in Section 3 which we now call CE price systems. In CE price systems, the endowments of the voters are fixed, while in PE price systems they are allowed to vary. To avoid confusion we use different symbols to denote PE and CE price system (γ_i and p_i, respectively).

A committee W is in *Lindahl equilibrium* if there is a price system $\{\gamma_i\}_{i\in N}$ such that the following conditions hold:

(Lin-PM). Profit maximization: For each $W' \subseteq C$ it holds that:

$$\underbrace{\sum_{c\in W}\sum_{i\in N}\gamma_i(c)}_{\substack{\text{total payments for the}\\ \text{produced public goods}}} \quad - \quad \underbrace{\pi(W)}_{\substack{\text{total cost}\\ \text{of production}}} \quad \geqslant \sum_{c\in W'}\sum_{i\in N}\gamma_i(c) - \pi(W').$$

Note that since $\pi(\emptyset) = 0$ the above condition implies a feasibility condition: $\sum_{c\in W}\sum_{i\in N}\gamma_i(c) \geqslant \pi(W)$ (the total payments payed to W are sufficient to produce W).

(Lin-UM). Utility maximization: voters spend their money to maximise their utility. For each voter i we have that:

(a) $\sum_{c\in W}\gamma_i(c) \leqslant 1$ (feasibility), and
(b) there is no committee W' with $\sum_{c\in W'}\gamma_i(c) \leqslant 1$ and:

$$\left(u_i(W'), \sum_{c\in W'}\gamma_i(c)\right) \succ \left(u_i(W), \sum_{c\in W}\gamma_i(c)\right).$$

In the definition above, the relation $\succ$ can be defined arbitrarily—however, we further assume that it is equivalent to the one defined in (2).

In the divisible PE model (where we can elect candidates fractionally) the conditions (Lin-PM) and (Lin-UM) are always satisfiable, and the resulting committee is guaranteed to be in the core [Foley, 1970]. For us, neither is true. We start by providing an example of a profile where a Lindahl equilibrium is not Pareto optimal.

Example 2. There are 3 candidates $C = \{a, b_1, b_2\}$, and 2 voters:

$$A_1 = \{a, b_1\} \quad A_2 = \{a, b_2\}.$$

Assume the price for each candidate is $p = 2/3$ (as each voter has 1 dollar, we can buy at most 3 candidates). Consider the following price system:

$$\begin{aligned} &\gamma_1(a) = 2/3 - 3/1000 && \gamma_2(a) = 2/3 - 3/1000\\ &\gamma_1(b_1) = 2/3 - 2/1000 && \gamma_2(b_1) = 1/1000\\ &\gamma_1(b_2) = 1/1000 && \gamma_2(b_2) = 2/3 - 2/1000. \end{aligned}$$

This price system witnesses that $\{a\}$ is a Lindahl equilibrium. Intuitively, the producer wants to produce a and does not want to produce b_1 nor b_2. Also, each voter prefers to spend her money on a than on b_1 or b_2, and cannot buy both. Yet, $\{a\}$ is Pareto-dominated by $\{a, b_1, b_2\}$. $\square$

power); the candidates are the public goods. Thus, as in Foley's model, the production function describes how private goods can be transformed into public goods. A crucial difference to Foley's model is that we use an indivisible model, where each candidate can be either bought (elected) or not, and there are no intermediate states. Due to indivisibilities, Foley's existence proof does not apply. Further, in our model each candidate is available in a single copy, which can affect decisions of the producers, and thus the prices.

The problem underlying Example 2 is that the producer gets paid less than the cost of b_1 and b_2 if the producer chooses to produce these candidates. In contrast, the producer receives a payment of almost double the cost of a for producing a. Thus, in this equilibrium, the producer is better off at the cost of consumers. In the divisible model this issue never appears: in every Lindahl equilibrium the total payment to the producer for producing a unit of candidate c is always equal to the cost of producing that unit. (Otherwise, the producer would want to produce an unlimited amount of c.) Since this equality is implied in the divisible model, it is natural to add it as an additional property to our definition of Lindahl equilibrium in the indivisible model.

We say that a committee W is a *cost-efficient Lindahl equilibrium (CELE)* if there exists a price system $\{\gamma_i\}_{i \in N}$ that satisfies (Lin-PM), (Lin-UM), and:

$$\textbf{(Lin-CE).}\ \text{Cost-Efficiency:} \quad \textstyle\sum_{c \in W} \sum_{i \in N} \gamma_i(c) \leqslant \pi(W).$$

By (Lin-PM), the condition in (Lin-CE) could also be written as an equality. Further, by (Lin-PM) and (Lin-CE) we can infer a seemingly stronger condition, that for each $c \in W$:

$$\textstyle\sum_{i \in N} \gamma_i(c) = \pi(c).$$

Theorem 5, below, shows a close relationship between stable priceability and Lindahl equilibrium. Let us slightly adapt condition (S5), by making the first inequality weak, and the third inequality strict. We refer to this condition as (S5*) and call the resulting solution concept *strict SP*.

Proposition 2. *Every strict SP committee is SP.*

We chose SP based on (S5) as our official definition, because strict SP does not exist even on very simple instances, as illustrated in Example 3 below.

Example 3. Consider an election with two voters and two candidates, a and b, both approved by one voter. The goal is to select a committee of size $k = 1$. It is straighforward to check that the only strict SP committees are $\emptyset$ and $\{a, b\}$, both of which are not feasible.

As one of our main results, we can prove that strict SP coincides with cost-efficient Lindahl equilibrium.

Theorem 5. *For each $p \in \mathbb{R}$, a committee satisfies cost-efficient Lindahl equilibrium for price p, if and only if it satisfies strict SP for price p.*

Based on this equivalence, we can immediately deduce several other properties of cost-efficient Lindahl equilibria.

Corollary 4. *Cost-efficient Lindahl Equilibria are SP.*

Corollary 5. *Every feasible committee that is in a cost-efficient Lindahl equilibrium is in the core.*

The latter result mirrors Foley's theorem in the classical model [Foley, 1970].

Summarizing, the idea of SP is very close to the idea of Lindahl equilibrium. The key conceptual difference is that in the public economics model, the price of the candidates is a fixed element of the model. In our case, the price is an adjustable part of price systems—the voters do not truly have money, they only have preferences, and money is a virtual concept that we use to ensure that public decisions are fair.

4 Balanced Price Systems

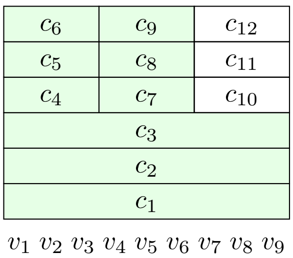
So far we have considered priceability notions where two voters could face significantly different prices for the same candidate. This can seem unnatural—why does one voter need to pay much more for the same thing as another?—and might thereby limit the usefulness of using these price systems as explanations. Here, we will study what happens if we insist that all voters pay the same price.

As before, we assume that in order to be selected, a candidate needs to collect a total payment of some value p that is identical across candidates. Previously, we implicitly assumed that whenever a candidate c is picked, all the voters obtain utility from c's election. Now, we will assume that voters only appreciate candidates when they had to pay for them. More concretely, in this section we consider price systems where for each candidate c there is *one* individual price ρ_c. A voter i, in order to be able to derive utility from the elected candidate c, needs to pay ρ_c dollars.

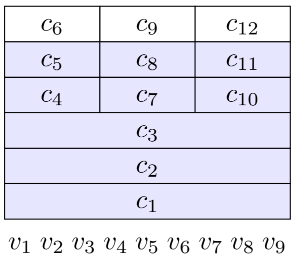
4.1 Motivation

As we have argued in Section 3, there is convincing evidence that stable priceability gives strong fairness guarantees. However, as we just noted, when voters pay different prices for the same candidate, there can be cases where even stable priceable committees can be argued to not be entirely fair.

Example 4. Consider an election with 12 candidates and 9 voters. The voters have the following approval sets. All 9 voters approve candidates c_1, c_2, and c_3. Further, voters v_1, v_2, v_3 approve c_4, c_5, and c_6; voters v_4, v_5, v_6 approve c_7, c_8, and c_9; and voters v_7, v_8, v_9 approve c_{10}, c_{11}, and c_{12}. The committee size is $k = 9$. The election is depicted below.



(a)



(b)

Here, the committee marked green in the left-hand side of the figure is SP. The corresponding price system can be the following: The price is $p = \frac{1}{3}$. Each from the last three voters (v_7, v_8 and v_9) pays $1/3$ for each commonly approved candidate (c_1, c_2 and c_3). The voters v_1, v_2, v_3 pay $1/3$ for candidates c_4, c_5, and c_6; the voters v_4, v_5, v_6 pay $1/3$ for candidates c_7, c_8, and c_9. However, the committee is arguably not fair. A much better choice would be to pick the committee marked blue in the right-hand side part of the figure. $\square$

The reason why the SP solution from Example 4 is not fair is that the candidates who are approved by all the voters (candidates c_1, c_2, and c_3), are paid for by only a small subset of them. Example 4 shows that the properties of committees supported by stable price systems very much depend on the structure of payment functions. Specifically, in Example 4 the payment functions were very unbalanced. Even though all voters approved c_1, only v_7, v_8, and v_9 payed for it. In a way, the mechanism "stole money" from v_7, v_8, and v_9, depriving them the possibility of paying for other candidates.

This example suggests that in an ideally-fair price system, all voters who enjoy the same utility from the same candidate should pay the same amount of money for it. We call such price systems *balanced*.

4.2 Formal Definition

The notion of balanced stable priceability differs from the notion that we considered in Section 3 in two main aspects. First, we require that any two voters, i and j, who decide to pay for a given candidate c must pay the same price, i.e., $p_i(c) = p_j(c)$. Second, we allow a voter not to pay for some elected candidates—but then the voter takes no utility from an approved candidate, even if the candidate is elected. This affects how we represent the committees. Now, a committee is a pair $(W, \{u_i\}_{i \in N})$, where $u_i \colon W \to \{0, 1\}$ is a binary utility function denoting whether voter i can use candidate c. We assume that, for each $i \in N$ and $c \in W$, $c \notin A_i \implies u_i(c) = 0$ (voters are never interested in using candidates they do not approve). For convenience, we extend the utility function to sets: for each $i \in N$ and $X \subseteq W$, we set $u_i(X) = \sum_{c \in X} u_i(c)$.

We say that a committee $(W, \{u_i\}_{i \in N})$ is supported by a *balanced stable price system (BSP)* $\mathrm{ps} = (p, \{p_i\}_{i \in N})$ if ps satisfies conditions (C2)–(C4), and:

(E1). Balanced payments: For each $c \in W$, there exists a value ρ_c such that for each $i \in N$ either $p_i(c) = \rho_c$, or $u_i(c) = 0$. Equivalently, $p_i(c) = u_i(c) \cdot \rho_c$.

(E5). Condition for Stability: There exists no coalition of voters $S \subseteq N$, no committee $(W', \{u'_i\}_{i \in N})$ $(W' \subseteq C \setminus W)$ and no collections $\{p'_i\}_{i \in S}$ $(p'_i \colon W' \to [0, 1])$ and $\{R_i\}_{i \in N}$, (with $R_i \subseteq W$ for each $i \subseteq N$) such that all the following conditions hold:

1. For each $c \in W'$, there exists a value ρ_c such that for each $i \in N$ it holds that $p'_i(c) = u'_i(c) \cdot \rho_c$
2. For each $c \in W'$: $\sum_{i \in S} p'_i(c) > p$.
3. For each $i \in S$: $p_i(W \setminus R_i) + p'_i(W') \leqslant 1$.
4. For each $i \in S$:

$$\big(u_i(W \setminus R_i) + u'_i(W'), p_i(W \setminus R_i) + p'_i(W')\big) \succeq \big(u_i(W), p_i(W)\big).$$

Intuitively, (E1) implies (C1) and requires that all the voters using a candidate c pay the same for c. (E5) is similar to (S5) with the additional requirement that the new payments that witness breaking the stability must also be balanced.

The green committee in Example 4 is not BSP. The price system given in the example violates condition (E5): all the voters would prefer to share the cost of candidate c_1 ($W' = \{c_1\}$ and $\rho'(c_1) = 1/9$). The first three voters would prefer to pay for c_1 instead of c_4 ($W'_i = \{c_1\}$, $R_i = \{c_4\}$), since then the number of their representatives would not change—recall that according to our definition of stability, a voter cannot be represented by a candidate for whom she does not pay—but they would need to pay for them a smaller amount of money (they would need to pay $1/9$ dollars for c_1 versus $1/3$ dollars for c_4). Similarly, voters v_4, v_5, and v_6 would prefer to pay for c_1 instead of c_7 ($W'_i = \{c_1\}$, $R_i = \{c_7\}$). Finally, the last 3 voters would be happy with the change ($W'_i = \{c_1\}$, $R_i = \{c_1\}$) since the individual price they would need to pay for c_1 would be lower ($1/9$ instead of $1/3$).

Besides being an intuitively appealing property, BSP also implies some other well-known fairness properties, like EJR.

Proposition 3. *Every feasible BSP committee satisfies EJR.*

4.3 A Characterization of BSP Committees

Like in the case of SP, imposing $|W'| \leqslant 1$ in the definition of BSP does not reduce the strength of the notion. Indeed, below we present the analogue of Theorem 1 for (E5), (E1) and a suitably modified inequality (3):

$$\forall c \notin W\ \forall S \subseteq N(c)\ |S| \min_{i \in S} \max\left(\max_{c' \in W}\left(p_i(c')\right), r_i\right) \leqslant p \tag{4}$$

Theorem 6. *Inequality* (4) *together with condition (E1) is equivalent to condition (E5).*

This result allows us to prove that BSP committees can be found and verified in polynomial time, as stated in the following theorem:

Theorem 7. *It can be verified in polynomial time whether a given committee is BSP. Besides, for given election instance and price p, a BSP committee can be found in polynomial time.*

We discuss this issue in detail in Appendix A.10—intuitively, we design a voting rule computable in polynomial time characterizing the set of BSP committees. This rule is a slight modification of a rule that was recently proposed by Peters and Skowron [2020] under the name of Rule X.

Basing on the characterization, in Appendix C.2 we describe a polynomial-time heuristic algorithm for finding BSP committees of a specified size k.

4.4 Existence

Like for SP, one could wonder whether BSP committees always exist, that is exist for every size k. The answer again is negative.

Theorem 8. *There exists an election for which no feasible committee is supported by an BSP price system.*

Using heuristic algorithms, we can show that in practice, committees which are "almost" BSP exist. In Appendix D we describe experiments that provide quantive arguments for the viability of this approach.

5 Conclusion

In this paper we have introduced two market-based solution concepts that allow to reason about, explain, and justify fairness of the outcome of an election to voters. We specifically focussed on approval-based committee elections, though our concepts generalize to participatory budgeting with cardinal utilities (we discuss this generalization in Appendix E). We have shown relations between our notions of stable priceability and known concepts of fairness and stability from the literature, such as EJR, core, proportionality degree, and Lindahl equilibrium. We have characterized the stable-priceable outcomes using simpler formulas, which allowed us to obtain more efficient algorithms for finding stable-priceable outcomes. As a consequence, we have characterized a close variant of Rule X as the only rule that returns BSP committees. Although SP/BSP committees do not always exist, our algorithms allow to find committees which are close to being SP/BSP—through extensive experiments we have shown that these algorithms can effectively find stable-priceable committees which are almost feasible.

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A Proofs Omitted From the Main Text

A.1 Proof of Proposition 1

Proposition 1. *The proportionality degree of Utilitarian Priceable Rule is at most 2.*

Proof. Consider the following construction. Let $x \geqslant 2$ be a natural number. We introduce $n = 4x^3$ voters, and $4x^2 + x$ candidates; $k = 2x^2$. The voters are divided into two groups.

1. The first group consists of $2x^3$ voters—we divide these voters into $2x^2$ equal-size subgroups. Each subgroup (of size x) approves a single different candidate. Let A denote the set of candidates approved by these voters; clearly $|A| = 2x^2$.
2. The second group (containing also $2x^3$ voters) is constructed as follows. We divide these voters again into x^2 subgroups, each of size $2x$. Each such a subgroup approves 2 common candidates—let B be the set of candidates approved by these voters; $|B| = 2x^2$. Additionally, from each subgroup we take one voter—let V denote the set of these voters; clearly $|V| = x^2$. The voters from V approve some common x candidates; let C denote the set of these candidates.

First, observe that if the price is equal to x then B would be supported by a system with equal prices, inducing the total utility of $x^2 \cdot 2x \cdot 2 = 4x^3$. The price cannot be lower than x. If it were, all the candidates from A would need to be members of the winning committee, leaving no room for the candidates from B; though the voters have money to pay a higher price for these candidates. The price cannot be higher than x. If it were, then at most half of the candidates from B could be members of the winning committee. The maximum possible utility would be then:

$$\underbrace{x^2 \cdot 2x \cdot 1}_{\text{from } B} + \underbrace{x \cdot x^2}_{\text{from } C} + \underbrace{(x^2 - x) \cdot x}_{\text{from } A} = 2x^3 + x^3 + x^3 - x^2 < 4x^3.$$

Thus, the price must equal to x. Now, observe that if the committee contains at least one candidate from C, then at most half of the candidates from B can get to the committee. By the same reasoning as above we get that the total utility obtained in such a case is lower than $4x^3$. Conseqently, the winning committee must be B. Each voters from V has 2 representatives in this committee. Yet, group V is $(x/2)$-cohesive. Thus, since x can be arbitrarily large, this induces the proportionality degree of 2 (independently of ℓ). $\square$

A.2 Proof of Proposition 2

Proposition 2. *Every strict SP committee is SP.*

Proof. It is clear that strict SP implies (C1). Hence, let W be some strictly SP committee and assume for sake of contradiction that inequality (3) does not hold. Then there exists $c \notin W$ such that:

$$\sum_{i \in N(c)} \max\left(\max_{a \in W}\left(p_i(a)\right), r_i\right) > p.$$

Then, there exists $\varepsilon > 0$ such that:

$$\sum_{i \in N(c)} \max\left(\max_{a \in W}\left(p_i(a)\right) - \varepsilon, r_i\right) > p.$$

However, then $W' = \{c\}$, $S = N(c)$ and collection $\{R_i\}_{i \in S}$ such that $R_i = \emptyset$ if $r_i \geqslant \max_{a \in W}\left(p_i(a)\right)$ and $R_i = \{\mathrm{argmax}_{a \in W}\left(p_i(a)\right)\}$ otherwise, witness violating condition (S5*). $\square$

A.3 Proof of Proposition 3

Proposition 3. *Every feasible BSP committee satisfies EJR.*

Proof. Consider any feasible BSP committee W and any ℓ-cohesive group of voters S. Suppose for the sake of contradiction that every voter in S approves only at most $\ell - 1$ members of W. Since W is feasible, then $p \leqslant n/k$. If $p = n/k$, then for every voter $i \in S$ we have that $r_i = 0$ (as the committee is feasible and there are only n dollars in the system) and:

$$\max\left(\max_{a\in W} p_i(a), r_i\right) = \max_{a\in W} p_i(a) \geqslant \frac{1}{\ell - 1}$$

Then $|S| \cdot 1/\ell{-}1 = \ell/\ell{-}1 \cdot n/k > p$, so S is a witness for violating (4).

If $p < n/k$, then for every voter $i \in S$ we have that:

$$\max\left(\max_{a\in W} p_i(a), r_i\right) \geqslant \frac{1}{\ell}$$

(i has 1 dollar and either has at least $1/\ell$ dollars left, or pays for some of her representatives at least $1/\ell$ dollars). Then $|S| \cdot 1/\ell = n/k > p$, so again S is a witness for violating (4), which completes the proof. □

A.4 Proof of Theorem 1

Theorem 1. *Inequality* (3) *together with condition (C1) is equivalent to condition (S5).*

Proof. The fact that condition (S5) implies the inequality (3) and (C1) was explained in the main text.

For the other direction the proof proceeds as follows. Let W be a committee supported by a price system $\mathrm{ps} = (p, \{p_i\}_{i\in N})$ satisfying (C1) and (3). Assume towards a contradiction that there exist a coalition of voters S, a subset W', a collection of functions $\{p'_i\}_{i\in S}$ and a collection of sets $\{R_i\}_{i\in S}$ with $R_i \subseteq W$ for all $i \in S$, witnessing the violation of (S5).

Since each voter pays only for the candidates she approves (this follows from (C1)), without loss of generality, we can assume that $R_i \subsetneq A_i$ for each voter $i \in S$.

Consider a voter $i \in S$. If $u_i(W') = u_i(R_i)$, then $|W' \cap A_i| = |R_i|$ and:

$$\sum_{c\in W'} p'_i(c) \leqslant \sum_{c\in R_i} p_i(c) \leqslant |R_i| \cdot \max_{a\in W}(p_i(a)) = |W' \cap A_i| \cdot \max_{a\in W}(p_i(a)).$$

On the other hand, if $u_i(W') > u_i(R_i)$, then $|W' \cap A_i| > |R_i|$. If $r_i \geqslant \max_{a\in W}(p_i(a))$, then:

$$\sum_{c\in W'} p'_i(c) \leqslant r_i + \sum_{c\in R_i} p_i(c) \leqslant (|R_i| + 1) \cdot \max\left(\max_{a\in W}(p_i(a)), r_i\right) \leqslant |W' \cap A_i| \cdot \max\left(\max_{a\in W}(p_i(a)), r_i\right).$$

For each $c \in W'$ we have that:

$$p < \sum_{i\in S} p'_i(c),$$

and so:

$$\sum_{c\in W'} p < \sum_{c\in W'} \sum_{i\in S} p'_i(c) = \sum_{i\in S} \sum_{c\in W'} p'_i(c) \leqslant \sum_{i\in S} \left(|W' \cap A_i| \cdot \max\left(\max_{a\in W}(p_i(a)), r_i\right)\right)$$
$$= \sum_{c\in W'} \sum_{i\in N(c)\cap S} \max\left(\max_{a\in W}(p_i(a)), r_i\right) \leqslant \sum_{c\in W'} \sum_{i\in N(c)} \max\left(\max_{a\in W}(p_i(a)), r_i\right).$$

Finally:

$$0 < \sum_{c \in W'} \left(\sum_{i \in N(c)} \max\left(\max_{a \in W}(p_i(a)), r_i \right) - p \right).$$

Thus, there must exist $c \in W'$ such that:

$$\sum_{i \in N(c)} \max\left(\max_{a \in W}(p_i(a)), r_i \right) > p.$$

Which gives a contradiction, and completes the proof. □

A.5 Proof of Theorem 3

Theorem 3. *SP implies the core.*

Proof. Let W be a committee supported by a stable price system $\mathrm{ps} = (p, \{p_i\}_{i \in [n]})$, and assume towards a contradiction that there exists a group of voters S and a set of candidates T such that: (i) $|T| \leqslant k \cdot {|S|}/{n}$, and (ii) $|A_i \cap T| \geqslant |A_i \cap W| + 1$ for each $i \in S$.

Since ps is stable, for each candidate $c \in T \setminus W$ we have that:

$$\sum_{i \in N(c)} \max\left(\max_{c' \in W}\left(p_i(c')\right), r_i \right) \leqslant p \tag{5}$$

Also, for each $c \in W$ (in particular, for each $c \in T \cap W$) we have that:

$$\sum_{i \in N(c)} p_i(c) \leqslant p. \tag{6}$$

Now, let us sum inequalities (5) and (6) over all $c \in T$, using inequality (5) whenever $c \in T \setminus W$, and using inequality (6), for $c \in T \cap W$:

$$\sum_{c \in T \cap W} \sum_{i \in N(c)} p_i(c) + \sum_{c \in T \setminus W} \sum_{i \in N(c)} \max\left(\max_{c' \in W}\left(p_i(c')\right), r_i \right) \leqslant p \cdot |T|.$$

Let us regroup the terms in the left-hand side of the above inequality:

$$\sum_{i \in N} \left(\sum_{c \in A_i \cap T \cap W} p_i(c) + |A_i \cap (T \setminus W)| \cdot \max\left(\max_{c' \in W}\left(p_i(c')\right), r_i \right) \right) \leqslant p \cdot |T|.$$

Since $S \subseteq N$, the above inequality would also hold if we changed the range of summation so that we consider only $i \in S$ Next, note that since $|A_i \cap T| \geqslant |A_i \cap W| + 1$ for each $i \in S$, we also have that $|A_i \cap (T \setminus W)| \geqslant |A_i \cap (W \setminus T)| + 1$. As a result, we get that:

$$\begin{aligned}
p \cdot |T| &\geqslant \sum_{i \in S} \left(\sum_{c \in A_i \cap T \cap W} p_i(c) + (|A_i \cap (W \setminus T)| + 1) \cdot \max\left(\max_{c' \in W}\left(p_i(c')\right), r_i \right) \right) \\
&\geqslant \sum_{i \in S} \left(\sum_{c \in A_i \cap T \cap W} p_i(c) + \sum_{c \in A_i \cap (W \setminus T)} p_i(c) + \max\left(\max_{c' \in W}\left(p_i(c')\right), r_i \right) \right) \\
&= \sum_{i \in S} \left(\sum_{c \in A_i \cap W} p_i(c) + \max\left(\max_{c' \in W}\left(p_i(c')\right), r_i \right) \right).
\end{aligned}$$

From now, on we consider two cases.

Case 1: $p < n/k$**.** Here, we continue as follows:

$$\begin{aligned}\frac{n}{k} \cdot |T| &> \sum_{i \in S} \left(\sum_{c \in A_i \cap W} p_i(c) + \max\left(\max_{c' \in W} \left(p_i(c')\right), r_i \right) \right) \\ &\geqslant \sum_{i \in S} \left(\sum_{c \in A_i \cap W} p_i(c) + r_i \right) = \sum_{i \in S} 1 = |S|.\end{aligned}$$

This gives a contradiction.

Case 2: $p = n/k$**.** In this case, we note that the whole budget of all the voters must have been spent (they, in total have n dollars). Thus, we continue, as follows:

$$\begin{aligned}\frac{n}{k} \cdot |T| &\geqslant \sum_{i \in S} \left(\sum_{c \in A_i \cap W} p_i(c) + \max\left(\max_{c' \in W} \left(p_i(c')\right), r_i \right) \right) \\ &> \sum_{i \in S} \left(\sum_{c \in A_i \cap W} p_i(c) \right) = \sum_{i \in S} 1 = |S|.\end{aligned}$$

This, again, leads to a contradiction, and completes the proof. □

A.6 Proof of Theorem 4

Theorem 4. *SP implies a proportionality degree of* $\ell - 1$*.*

Proof. Fix an election instance, and consider a size-k SP committee W. Let S be an ℓ-cohesive group of voters and let T be a set of ℓ candidates who are approved by all members of S. We will show that an average number of representatives that the voters from S have in W equals at least $\ell - 1$.

Without loss of generality, let us assume that there exists a not-elected candidate $c \notin W$ that is approved by all members of S (as, otherwise, the average number of representatives for voters from S would be at least ℓ).

Note that, by the pigeonhole principle, for each voter $i \in N$ we have that:

$$\max\left(\max_{c' \in W} \left(p_i(c')\right), r_i \right) \geqslant \frac{1}{|A_i \cap W| + 1}.$$

By condition (3) applied to c, we get that:

$$p \geqslant \sum_{i \in S} \max\left(\max_{c' \in W} \left(p_i(c')\right), r_i \right) \geqslant \frac{|S|}{|A_i \cap W| + 1}.$$

By the inequality between the harmonic and arithmetic mean, we get that:

$$\begin{aligned}\frac{\sum_{i \in S}(|A_i \cap W|)}{|S|} &= \frac{\sum_{i \in S}(|A_i \cap W| + 1)}{|S|} - 1 \geqslant \frac{|S|}{\sum_{i \in S} \frac{1}{|A_i \cap W|+1}} - 1 \\ &\geqslant \frac{|S|}{p} - 1 \geqslant \frac{k}{n} \cdot |S| - 1 \geqslant \frac{k}{n} \cdot \frac{n\ell}{k} - 1 = \ell - 1.\end{aligned}$$

This completes the proof. □

A.7 Proof of Theorem 5

Theorem 5. *For each $p \in \mathbb{R}$, a committee satisfies cost-efficient Lindahl equilibrium for price p, if and only if it satisfies strict SP for price p.*

Proof. We first prove that the committees that are in a cost-efficient Lindahl equilibrium are strictly SP. Consider a committee $W \subseteq C$ that is in the cost-efficient Lindahl equilibrium, and let $\{\gamma_i\}_{i \in N}$ be the corresponding price system. From $\{\gamma_i\}_{i \in N}$ we construct the price system $\{p_i\}_{i \in N}$ witnessing strict SP as follows. For each $i \in N$ and $c \in W$ we set $p_i(c) = \gamma_i(c)$; for $c \notin W$ we set $p_i(c) = 0$. We now verify that $\{p_i\}_{i \in N}$ satisfies the conditions of stable-priceability. (C1) follows from (S5*), and we will prove it later on. (C2) follows from (Lin-UM(a)) (feasibility in the utility maximization condition). (C3) follows from profit maximization (Lin-PM) and cost-efficiency, and (C4) follows directly from the construction of the payment functions.

Let us now consider (S5*). Let us fix $W' \subseteq C \setminus W$, set of voters $S \subseteq N$, collection $\{R_i\}_{i \in S}$ and collection $\{p'_i\}_{i \in S}$ satisfying for each $i \in S$: $\sum_{c \in W'} p'_i(c) + \sum_{c \in W \setminus R_i} p_i(c) \leqslant 1$. Observe that if $u_i((W \setminus R_i) \cup W') > u_i(W)$, then by (Lin-UM) $\sum_{c \in (W \setminus R_i) \cup W'} \gamma_i(c) > 1$ and so:

$$\begin{aligned}\sum_{c \in W'} p'_i(c) &\leqslant 1 - \sum_{c \in W \setminus R_i} p_i(c) = 1 - \sum_{c \in W \setminus R_i} \gamma_i(c) \\ &= 1 - \sum_{c \in (W \setminus R_i) \cup W'} \gamma_i(c) + \sum_{c \in W'} \gamma_i(c) < \sum_{c \in W'} \gamma_i(c).\end{aligned}$$

On the other hand, if $u_i((W \setminus R_i) \cup W') = u_i(W)$ and $\sum_{c \in W'} p'_i(c) < \sum_{c \in R_i} p_i(c)$, then either $\sum_{c \in (W \setminus R_i) \cup W'} \gamma_i(c) > 1$ (and we obtain the estimation as above), or $\sum_{c \in W} \gamma_i(c) \leqslant \sum_{c \in (W \setminus R_i) \cup W'} \gamma_i(c)$. In the latter case we get that:

$$\begin{aligned}\sum_{c \in W'} p'_i(c) &< \sum_{c \in R_i} p_i(c) = \sum_{c \in R_i} \gamma_i(c) = \sum_{c \in W} \gamma_i(c) - \sum_{c \in W \setminus R_i} \gamma_i(c) \\ &\leqslant \sum_{c \in (W \setminus R_i) \cup W'} \gamma_i(c) - \sum_{c \in W \setminus R_i} \gamma_i(c) = \sum_{c \in W'} \gamma_i(c).\end{aligned}$$

In any case, we get that if $\left(u_i((W \setminus R_i) \cup W'), \sum_{c \in W \setminus R_i} p_i(c) + \sum_{c \in W'} p'_i(c)\right) \succ \left(u_i(W), \sum_{c \in W} p_i(c)\right)$ then $\sum_{c \in W'} p'_i(c) < \sum_{c \in W'} \gamma_i(c)$. By (Lin-PM) we get that for each $c \in W'$ we have $\sum_{i \in N} \gamma_i(c) \leqslant p$. Thus, we can continue as:

$$\sum_{i \in S} \sum_{c \in W'} p'_i(c) < \sum_{i \in S} \sum_{c \in W'} \gamma_i(c) = \sum_{c \in W'} \sum_{i \in S} \gamma_i(c) \leqslant |W'| \cdot p.$$

Hence, there needs to exist a candidate $c \in W'$ such that $\sum_{i \in S} p'_i(c) < p$, which proves that (S5*) is indeed satisfied.

Second, we show that a committee W that is strictly SP is in a cost-efficient Lindahl equilibrium. Consider a committee $W \subseteq C$ that is strictly SP and let $(p, \{p_i\}_{i \in N})$ be the corresponding price system. Let us fix ε to a small positive value (we will specify it later on). We set the price of a candidate to p and construct a price system $\{\gamma_i\}_{i \in N}$ that will witness that W is in the cost-efficient Lindahl equilibrium. For each $i \in N$ and each $c \in W$ we set $\gamma_i(c) = p_i(c)$. For $c \notin W$ and $i \in N(c)$ we set:

$$\gamma_i(c) = \max\left(\max_{c' \in W} (p_i(c')) - \varepsilon, r_i\right) + \varepsilon.$$

Otherwise (for $c \notin W$ and $i \notin N(c)$) we set $\gamma_i(c) = 0$. By (C2) we know that for each voter i the choice W is feasible (Lin-UM (a)). Observe that for each voter i and each $c \notin W$ buying c costs at least the same as buying any candidate from W; by (S5*) she does not want to stop paying for any candidate from W in order to obtain W' (even if we decreased the individual prices of each $c \in W'$ so that they sum up to p). Thus, (Lin-UM (b)) is satisfied.

From (C3) we get cost-efficency and profit maximization for $c \in W$. The profit maximization for $c \notin W$ follows from (S5*) applied for $W' = \{c\}$ and $S = N(c)$ (for each $i \in S$, R_i is set either to the i's most expensive candidate or to $\emptyset$):

$$\forall \varepsilon' > 0 \sum_{i \in N(c)} \max\left(\max_{c' \in W} (p_i(c')) - \varepsilon', r_i\right) < p,$$

Thus, it is possible to set ε to a value such that:

$$\sum_{i \in N(c)} \left(\max\left(\max_{c' \in W} (p_i(c')) - \varepsilon, r_i\right) + \varepsilon\right) < p,$$

from which it follows that $\sum_{i \in N} \gamma_i(c) < p$. □

A.8 Proof of Theorem 2

For sake of clarity of the presentation, let us first introduce some auxiliary notation. For each candidate $c \in C$, by $N(-c)$ we denote $N \setminus N(c)$, and for a set of candidates $S \subseteq C$ we will write $N(S)$ as a shorthand for $\bigcap_{c \in S} N(c)$ and $N(-S)$ for $\bigcap_{c \in S} N(-c)$. Further, for $q_1, \ldots, q_t \in C \cup 2^C$ we will write $N(\pm q_1, \ldots, \pm q_t)$ to denote $N(\pm q_1) \cap \ldots \cap N(\pm q_k)$.

Before we prove Theorem 2 we prove a few auxiliary lemmas.

Lemma 1. *For each election instance E, each committee W that is supported by a stable price system $ps = \{p, \{p_i\}_{i \in N}\}$, each candidate $a \notin W$ and each set of candidates $S \subseteq W$ such that every voter from $N(a)$ approves at most one candidate from S, the following inequality holds:*

$$\sum_{b \in S} \sum_{i \in N(a,b)} p_i(b) + \sum_{i \in N(a,-S)} \frac{1}{|W \cap A_i| + 1} \leqslant p$$

Proof. Note that i pays for at most $|W \cap A_i|$ members of W. Hence, either she pays at least $\frac{1}{|W \cap A_i|+1}$ for her most expensive representative, or she has at least $\frac{1}{|W \cap A_i|+1}$ money left. □

Corollary 6. *For each election instance E, each committee W that is supported by a stable price system $ps = \{p, \{p_i\}_{i \in N}\}$, and each candidate $a \notin W$, the following inequality holds:*

$$\sum_{i \in N(a)} \frac{1}{|W \cap A_i| + 1} \leqslant p$$

Proof. This follows directly from Lemma 1, applied to $S = \emptyset$. □

For positive integers s, t, z, by $E(s, t, z)$ we denote an election instance with $N = [sz + 2tz]$, $C = \{x_1, \ldots, x_{2z}, y\}$, and where the approval sets are as follows:

- s voters approve $\{x_{2i-1}, x_{2i}, y\}$, for each $i \in [z]$,
- t voters approve $\{x_i\}$, for each $i \in [2z]$.

We will now prove a few useful properties of these instances.

Lemma 2. *Consider an election instance $E(s, t, z)$ such that $z > 1$ and $\frac{s}{3} < \frac{t}{z-1}$, and a committee W supported by a stable price system $ps = \{p, \{p_i\}_{i \in N}\}$. Then:*

(1) if $|W| = 2z + 1$, then $p \leqslant \frac{2z}{2z+1}(t + \frac{s}{2}) < t + \frac{s}{2}$,

(2) if $|W| = 2z - 1$, then $p \geqslant t + \frac{s}{2}$.

(3) if $|W| = 2z$, then $p \in [\frac{sz}{3}; \frac{zt}{z-1}]$,

Proof of (1). All the voters have in total $sz + 2tz$ dollars. Since they need to pay p dollars for $2z + 1$ candidates, it holds that $p \leqslant \frac{sz+2tz}{2z+1} = \frac{2z}{2z+1}(t + \frac{s}{2})$. □

Proof of (2). Let us consider three cases:

Case 1. $y \notin W$. Then, there exist $j \in [2z]$ such that $x_j \notin W$. For simplicity, assume without loss of generality that $j = 1$. From Corollary 6 we have that:

$$\sum_{i \in N(x_1)} \frac{1}{|W \cap A_i| + 1} = t + \frac{s}{2} \leqslant p$$

as we have t supporters of x_1 who have no representatives in W and s supporters of x_1 who have only one representative, x_2.

Case 2. there exists $j \in [z]$ such that both $x_{2j-1}, x_{2j} \notin W$. For simplicity, assume without loss of generality that $j = 1$. Then we can repreat the reasoning from the previous case—from Corollary 6:

$$\sum_{i \in N(x_1)} \frac{1}{|W \cap A_i| + 1} = t + \frac{s}{2} \leqslant p$$

as we have t supporters of x_1 who have no representatives in W and s supporters of x_1 who have only one representative, y.

Case 3. there exist $j_1, j_2 \in [2z]$ such that both $x_{j_1}, x_{j_2} \notin W$ and no voter approves both x_{j_1}, x_{j_2}. For simplicity, assume without loss of generality that $j_1 = 1, j_2 = 3$.

From Corollary 6 we have that:

$$\sum_{i \in N(x_1)} \frac{1}{|W \cap A_i| + 1} = t + \frac{s}{3} \leqslant p \tag{7}$$

We will now try to find an upper bound for p. From Lemma 1 for $a = x_1$, $S = \{y\}$ we obtain:

$$\sum_{i \in N(y, x_1)} p_i(y) \leqslant p - \sum_{i \in N(x_1, -y)} \frac{1}{|W \cap A_i| + 1} = p - t \tag{8}$$

and analogously for $a = x_3$, $S = \{y\}$:

$$\sum_{i \in N(y, x_3)} p_i(y) \leqslant p - t \tag{9}$$

Consider now any candidate x_j for $j \notin \{1, 3\}$. Then it holds that:

$$p - t = \sum_{i \in N(x_j, y)} p_i(x_j) + \sum_{i \in N(x_j, -y)} p_i(x_j) - t \leqslant \sum_{i \in N(x_j, y)} p_i(x_j) \tag{10}$$

Now we can write the main inequality:

$$
\begin{aligned}
p &= \sum_{i \in N(y)} p_i(y) = \sum_{i \in N(y,x_1)} p_i(y) + \sum_{i \in N(y,x_3)} p_i(y) + \sum_{j \in [z] \setminus [2]} \sum_{i \in N(y,x_{2j-1})} p_i(y) \\
&\overset{(8),(9)}{\leqslant} 2(p-t) + (z-2)\left(s - \sum_{j \in [z] \setminus [2]} \sum_{i \in N(y,x_{2j-1})} p_i(x_{2j-1}) + p_i(x_{2j}) \right) \\
&\overset{(10)}{\leqslant} 2(p-t) + (z-2)(s - 2(p-t)) = 2p - 2t + (z-2)s - 2p(z-2) + 2t(z-2) \\
&= p(6-2z) + sz - 2s + 2tz - 6t
\end{aligned}
$$

which can be simplified to:

$$p(2z-5) \leqslant sz - 2s + 2tz - 6t \tag{11}$$

Combining inequality (11) with inequality (7), we obtain:

$$
\begin{gathered}
\left(t + \frac{s}{3}\right)(2z-5) \leqslant sz - 2s + 2tz - 6t \\
2tz - 5t + \frac{2sz}{3} - \frac{5s}{3} \leqslant sz - 2s + 2tz - 6t \\
t \leqslant \frac{sz}{3} - \frac{s}{3} = \frac{s}{3}(z-1) \\
\frac{t}{z-1} \leqslant \frac{s}{3}
\end{gathered}
$$

However, from our assumptions it holds that $\frac{t}{z-1} > \frac{s}{3}$. Hence, this case leads to a contradiction, which completes the proof.

□

Proof of (3). Let us consider two cases:

Case 1. $y \in W$. Then, there exists $j \in [2z]$ such that $x_j \notin W$. For simplicity, assume without loss of generality that $j = 1$. From Corollary 6 we have that:

$$\sum_{i \in N(x_1)} \frac{1}{|W \cap A_i| + 1} = t + \frac{s}{3} \leqslant p \tag{12}$$

as we have t supporters of x_1 who have no representatives in W and s supporters of x_1 who have two representatives, y and x_2.

Now we will try to find an upper bound for p. From Lemma 1 for $a = x_1$, $S = \{y\}$ we obtain:

$$\sum_{i \in N(y,x_1)} p_i(y) \leqslant p - \sum_{i \in N(x_1,-y)} \frac{1}{|W \cap A_i| + 1} = p - t. \tag{13}$$

Consider any candidate x_j for $j \neq 1$. Then it holds that:

$$p - t = \sum_{i \in N(x_j,y)} p_i(x_j) + \sum_{i \in N(x_j,-y)} p_i(x_j) - t \leqslant \sum_{i \in N(x_j,y)} p_i(x_j). \tag{14}$$

Now we can write the main inequality:

$$\begin{aligned}p &= \sum_{i\in N(y)} p_i(y) = \sum_{i\in N(y,x_1)} p_i(y) + \sum_{j\in[z]\setminus[1]} \sum_{i\in N(y,x_{2j-1})} p_i(y)\\ &\overset{(13)}{\leqslant} p - t + (z-1)\left(s - \sum_{j\in[z]\setminus[1]} \sum_{i\in N(y,x_{2j-1})} p_i(x_{2j-1}) + p_i(x_{2j})\right)\\ &\overset{(14)}{\leqslant} p - t + (z-1)(s - 2(p-t)) = p - t + (z-1)s - 2p(z-1) + 2t(z-1)\end{aligned}$$

which can be simplified to:

$$2p(z-1) \leqslant 2t(z-1) + (z-1)s - t$$

$$p \leqslant t + \frac{s}{2} - \frac{t}{2(z-1)} \tag{15}$$

Combining inequality (15) with inequality (12), we obtain:

$$t + \frac{s}{3} \leqslant t + \frac{s}{2} - \frac{t}{2(z-1)}$$

$$\frac{t}{2(z-1)} \leqslant \frac{s}{6}$$

$$\frac{t}{z-1} \leqslant \frac{s}{3}$$

However, from our assumptions it holds that $\frac{t}{z-1} > \frac{s}{3}$. Hence, this case leads to a contradiction.

Case 2. $W = \{x_1, \dots, x_{2z}\}$. From Corollary 6 we have that:

$$\sum_{i\in N(y)} \frac{1}{|W\cap A_i| + 1} = \frac{sz}{3} \leqslant p \tag{16}$$

as we have sz supporters of y, each of whom has two representatives in W.

On the other hand, for each $j \in [2z]$ we have that:

$$p - t = \sum_{i\in N(x_j,y)} p_i(x_j) + \sum_{i\in N(x_j,-y)} p_i(x_j) - t \leqslant \sum_{i\in N(x_j,y)} p_i(x_j). \tag{17}$$

and from Lemma 1 for $a = y$, $S = \{x_1, x_3, \dots, x_{2z-1}\}$ it holds that:

$$\begin{aligned}\sum_{j\in[z]} \sum_{i\in N(y,x_{2j-1})} p_i(x_{2j-1}) + \sum_{i\in N(y,-S)} \frac{1}{|W\cap A_i| + 1}&\\ = \sum_{j\in[z]} \sum_{i\in N(y,x_{2j-1})} p_i(x_{2j-1}) \leqslant p&\end{aligned} \tag{18}$$

Combining inequalities (17) and (18) we have that:

$$z(p-t) \leqslant p$$

$$p \leqslant \frac{tz}{z-1} \tag{19}$$

Indeed, from (16) and (19) we obtain that $p \in [\frac{sz}{3}; \frac{zt}{z-1}]$, which completes the proof.

□

Theorem 2. *There exists an election for which no feasible commitee is supported by a stable price system.*

Proof. Consider two election instances $E(72, 97, 5)$ and $E(88, 89, 4)$. Note that both instances satisfy $\frac{s}{3} < \frac{t}{z-1}$, hence we can apply Lemma 2 to them. For $E(72, 97, 5)$ and committee W_1 elected for this instance supported by a stable price system with price p, we obtain that:

- if $|W_1| = 11$, then $p < 133$,
- if $|W_1| = 9$, then $p \geqslant 133$,
- if $|W_1| = 10$, then $p \in [120; \frac{485}{4}] = [120; 121.25]$

Analogously, for $E(88, 89, 4)$ and committee W_2 elected for this instance supported by a stable price system with price p, we obtain that:

- if $|W_2| = 9$, then $p < 133$,
- if $|W_2| = 7$, then $p \geqslant 133$,
- if $|W_2| = 8$, then $p \in [\frac{352}{3}; \frac{356}{3}] \approx [117.33; 118.67]$.

Now consider an election instance obtained by merging $E(72, 97, 5)$ and $E(88, 89, 4)$ (treating candidates $x_1, \ldots, x_{2z}, y$ from $E(72, 97, 5)$ and $E(88, 89, 4)$ as different copies). We will prove that there exists no committee of size 18 supported by a stable price system for this instance. Indeed, assume for the sake of contradiction that such a committee W exists. Because our instance consists of two disjoint subinstances, we have that $W = W_1 \cup W_2$, where W_1, W_2 are disjoint committees for resp. $E(72, 97, 5)$ and $E(88, 89, 4)$, supported by stable price systems with a common price. Besides, $|W_1| + |W_2| = 18$ and $|W_1| \leqslant 11$, $W_2 \leqslant 9$, which means that we have three cases:

- $|W_1| = 11$ and $|W_2| = 7$. Then we have $p < 133$ and $p \geqslant 133$, a contradiction.
- $|W_1| = 10$ and $|W_2| = 8$. Then we have $p \in [120; 121.25]$ and $p \in [117.33; 118.67]$, a contradiction.
- $|W_1| = 9$ and $|W_2| = 9$. Then we have $p \geqslant 133$ and $p < 133$, a contradiction.

Hence, no stable price system for W exists.

□

A.9 Proof of Theorem 6

Theorem 6. *Inequality* (4) *together with condition (E1) is equivalent to condition (E5).*

Proof. We first prove that condition (E5) implies (4). Indeed, assume that for some $c \in C, S \subseteq N(c)$ equation (4) does not hold. As a result, it holds that $p/|S| \leqslant \min_{i \in S} \max\left(\max_{c' \in W} (p_i(c')), r_i\right)$. Set $W' = \{c\}$. For each $i \in S$ set $u'_i(c) = 1$. Besides, if $r_i \geqslant \max_{c' \in W} (p_i(c'))$ then we set $R_i = \emptyset$, otherwise we set R_i to the candidate for which i pays most. Moreover, for each $i \in S$ we set $p'_i(c) = p/|S|$. Then, we can clearly see that $(W', \{u'_i\}_{i \in S})$, $\{R_i\}$ and $\{p'_i\}_{i \in S}$ satisfy all the conditions of (E5), hence they witness the lack of stability.

To prove that (E1) and (4) imply (E5), assume that for some election instance with approval preferences there exist a set of voters S, a committee $(W', \{u_i\}_{i \in S})$, and collections $\{p'_i\}_{i \in N}$ and $\{R_i\}_{i \in N}$, witnessing the lack of stability. We will prove that in this case condition (4) is not satisfied. Set c to the candidate from W' with the minimal ρ_c. Set $S' = \{i \in S \colon u_i(c) = 1\}$. Note that by definition $S' \subseteq N(c)$. The second condition from (E5) can be equivalently written for c as:

$$|S'| \cdot p'_i(c) = |S'| \cdot \rho_c > p \tag{20}$$

Now consider a voter $i \in S'$. We know that $u_i(R_i) \leqslant u'_i(W')$. If we have that $|R_i| = u_i(R_i) = u'_i(W')$, then $\sum_{c' \in W'} p'_i(c') \leqslant \sum_{c' \in R_i} p_i(c')$ and:

$$\begin{aligned}
\rho_c = \frac{\rho_c \cdot u'_i(W')}{u'_i(W')} &= \frac{\sum_{c' \in W'} u'_i(c') \cdot \rho_c}{u_i(W')} \leqslant \frac{\sum_{c' \in W'} u'_i(c') \cdot \rho_{c'}}{u_i(W')} \\
&= \frac{\sum_{c' \in W'} p'_i(c')}{u_i(W')} \leqslant \frac{\sum_{c' \in R_i} (p_i(c'))}{u_i(W')} \\
&\leqslant \frac{\sum_{c' \in R_i} (\max_{c' \in W} (p_i(c')))}{u_i(W')} \leqslant \frac{|R_i| \cdot (\max_{c' \in W} (p_i(c')))}{u_i(W')} \\
&= \max_{c' \in W} (p_i(c')) \leqslant \max \left(r_i, \max_{c' \in W} (p_i(c')) \right)
\end{aligned}$$

Otherwise, we have that $|R_i| = u_i(R_i) < u'_i(W')$ and $\sum_{c \in W'} p'_i(c) \leqslant p - \sum_{c' \in W \setminus R_i} p_i(c') = r_i + \sum_{c' \in R_i} p_i(c')$. In this case:

$$\begin{aligned}
\rho_c = \frac{\rho_c \cdot u'_i(W')}{u'_i(W')} &= \frac{\sum_{c' \in W'} u'_i(c') \cdot \rho_c}{u_i(W')} \leqslant \frac{\sum_{c' \in W'} u'_i(c') \cdot \rho_{c'}}{u_i(W')} \\
&= \frac{\sum_{c' \in W'} p'_i(c')}{u_i(W')} \leqslant \frac{r_i + \sum_{c' \in R_i} p_i(c')}{u_i(W')} \\
&\leqslant \frac{r_i + \sum_{c' \in R_i} \max_{c' \in W} (p_i(c'))}{u_i(W')} \leqslant \frac{r_i + |R_i| \cdot \max_{c' \in W} (p_i(c'))}{u_i(W')} \\
&\leqslant \frac{(|R_i| + 1) \cdot \max (r_i, \max_{c' \in W} (p_i(c')))}{u_i(W')} \leqslant \max \left(r_i, \max_{c' \in W} (p_i(c')) \right)
\end{aligned}$$

Hence, in both cases we have that:

$$\rho_c \leqslant \min_{i \in S} \left(\max \left(r_i, \max_{c' \in R_i} (p_i(c')) \right) \right) \tag{21}$$

Combining (20) and (21) we obtain:

$$|S| \min_{i \in S} \left(\max \left(r_i, \max_{c' \in R_i} (p_i(c')) \right) \right) > p$$

which completes the second part of the proof. □

A.10 Proof of Theorem 7

We start by presenting a sequential voting rule that for each instance returns exactly the BSP committees. We call it *Rule X*, as it is very similar to the rule of the same name defined by [Peters and Skowron, 2020]

The definition of the variant of Rule X we consider here[4] is the following: we say that a group of voters Q_c is *demanding* if every member of Q_c has at least $p/|Q_c|$ dollars left and

[4]It is slighly different, yet very similar to the classical definition of this rule from [Peters and Skowron, 2020]. The only two differences is the possibility to avoid demanding groups which are not strongly demanding, and that in our paper the payments need to be equal, while the original definition allowed unequal payments in specific borderline situations.

```
W ← ∅
for i ∈ N:
    for c ∈ C:
        p_i(c) ← 0
        u_i(c) ← 0
repeat:
    Q_a ← the largest demanding group
    Q_b ← the largest strongly demanding group
    Q_c ← max(Q_a, Q_b), according to some tie-breaking strict linear order over demanding groups
    if Q_c = ∅:
        break
    W ← W ∪ {c}
    for i ∈ Q_c:
        p_i(c) ← p/|Q_c|
        u_i(c) ← 1
return (W, {u_i}_{i∈N})
```

Listing 1: Rule X algorithm

approves some candidate $c \notin W$. We say that Q_c is *strongly demanding* if it demanding and every voter from Q_c has more than $p/|Q_c|$ dollars left.

Intuitively, every demanding group can afford to additionally buy a new candidate to the committee. Strongly demanding groups can afford to buy a candidate even for a price higher than p. Rule X can be viewed as a rule greedily satisfying these demands, starting from the largest groups. If the group is not strongly demanding, it can be either considered or skipped. The algorithm stops when there are no nonempty, non-skipped demanding groups. Formally, we perform the algorithm presented in Listing 1.

Theorem 7. *It can be verified in polynomial time whether a given committee is BSP. Besides, for given election instance and price p, a BSP committee can be found in polynomial time.*

Proof. First we show that every committe elected by Rule X is BSP. It will prove that for given instance and price p, we can find a BSP committee in polynomial time.

The corresponding price system is constructed implicitly by the algorithm. It is clear that this price system satisfies (E1) and (C2)–(C4). Now, we show that it also satisfies (4). Indeed, for the sake of contradiction suppose that it does not hold and let $\varepsilon > 0$, $c \notin W$, and $V \subseteq N(c)$ be witnessing the violation of stable priceability. Hence, every voter $v \in V$ has more than $p/|V|$ money left or pays for some candidate from W more than $p/|V|$. If such a candidate does not exist, then at the end of execution of the algorithm V is a strongly demanding group, thus the algorithm would not stop. Otherwise, let c' be the candidate added to W at the earliest step such that each voter pays for c' more than $p/|V|$. As the individual price for c' is greater than for c, it needs to be the case that c' was added to W for some demanding group of size smaller than $|V|$. However, before that step, group V was a strongly demanding group. Hence, the demanding group supporting c' was not the largest one, a contradiction.

Now we show that every BSP committee is elected by Rule X with proper tie-breaking. For the sake of contradiction suppose W is a stable-priceable committee that cannot be elected by Rule X. Let $|W| = \ell$. Enumerate the candidates $c_1, \ldots, c_\ell$ in W by number of voters who pay for them in the descending order and denote these groups by $Q_{c_1}, \ldots, Q_{c_\ell}$. Consider a rule, which is similar to Rule X, but instead of taking the largest demanding group, considers only groups $Q_{c_1}, \ldots, Q_{c_\ell}$. The only difference between such a rule and Rule

X may appear if at some i^{th} iteration, there exists some strongly demanding group Q_c which is strictly larger than Q_{c_i}. Such a group, if ignored, either remains strongly demanding at the end of the execution of the algorithm (and then is a witness for violating stability of W) or stops being strongly demanding at some step (after some voters from Q_c have their initial budgets decreased). However, individual payments for the candidates added to W in further steps need to be strictly higher than $p/|Q_c|$. Hence, Q_c is still the witness for violating stability of W.

Finally, note that we can use the rule defined above to verify whether a given committee is BSP—identifying and sorting groups $Q_{c_1}, \dots, Q_{c_\ell}$ can be clearly done in polynomial time. If at some i^{th} iteration, group Q_{c_i} cannot be elected by Rule X, then we know that the committee is not BSP, otherwise it is BSP. □

A.11 Proof of Theorem 8

Lemma 3. *Consider an election instance, two voters i, j and a committee W supported by a BSP price system. If $A_i = A_j$, then for each $c \in C$ we have that $p_i(c) = p_j(c)$.*

Proof. Let p be the price in the BSP price system. Suppose for the sake of contradiction that there exists a candidate a for which i pays $\frac{p}{q_a}$ dollars and j pays nothing. Then either j has at least $\frac{p}{q_a}$ dollars left or there exists a candidate b for which j pays and i does not. In the first case, j prefers to join the other q_a voters paying for a (then the individual price for a decreases to $\frac{p}{q_a+1}$, which is strictly less than the amount of the savings of j). In the second case, compare the individual payment for b (equal to $\frac{p}{q_b}$) to $\frac{p}{q_a}$. Without loss of generality, assume that $\frac{p}{q_a} \leqslant \frac{p}{q_b}$. Then j has an incentive to stop paying for b and join other q_a voters paying for a—then again the individual price for a decreases to $\frac{p}{q_a+1}$, which is strictly less than $\frac{p}{q_b}$. We obtained a contradiction with the assumption that the price system is BSP, which proves that for each $c \in C$ it holds that $p_i(c) = p_j(c)$. □

Theorem 8. *There exists an election for which no feasible committee is supported by an BSP price system.*

Proof. Let us consider the following election instance for $n = 49, m = 46, k = 44$:

Group 1 (16 voters): $\{x_1, \dots, x_{18}\}$
Group 2 (8 voters): $\{y_1, \dots, y_{12}\}$
Group 3 (3 voters): $\{z_1, \dots, z_{12}\}$
Group 4 (4 voters): $\{a, x_1, \dots, x_{18}\}$
Group 5 (6 voters): $\{a, b, y_1, \dots, y_{12}\}$
Group 6 (4 voters): $\{c, z_1, \dots, z_{12}\}$
Group 7 (4 voters): $\{d, z_1, \dots, z_{12}\}$
Group 8 (2 voters): $\{b, c, z_1, \dots, z_{12}\}$
Group 9 (2 voters): $\{b, d, z_1, \dots, z_{12}\}$

In total, 20 voters approve candidates $\{x_1, \dots, x_{18}\}$, 14 voters approve candidates $\{y_1, \dots, y_{12}\}$, 15 voters approve candidates $\{z_1, \dots, z_{12}\}$, 10 voters approve a, 10 voters approve b, 6 voters approve c and 6 voters approve d. Further, the sets of voters who approve x-, y-, and z-candidates are disjoint. Hence, Rule X will elect candidates $x_1, \dots, x_{18}, z_1, \dots, z_{12}, y_1, \dots, y_{12}$ first. After that, we have at most 42 candidates elected. As our goal is to elect a committee of size 44, the value of price p should allow all those 42 candidates to be elected. After that:

1. Voters from groups 1-3 run out of approved candidates.
2. Each voter from group 4 has $\frac{10-9p}{10}$ dollars left.
3. Each voter from group 5 has $\frac{7-6p}{7}$ dollars left.
4. Each voter from groups 6-9 has $\frac{5-4p}{5}$ dollars left.

We need to elect exactly 2 more candidates. Assume for the sake of contradiction that it is possible and let us consider all possible pairs of candidates from $\{a, b, c, d\}$ as the ones that can be included in the final committee.

Case 1: $\{a, b\}$. As we have at least 4 voters (group 6) who approve c and have $\frac{5-4p}{5}$ dollars left, the following inequality holds:

$$\begin{aligned}\frac{p}{4} &\geqslant \frac{5-4p}{5}\\ 5p &\geqslant 20-16p\\ p &\geqslant \frac{20}{21} \approx 0.952\end{aligned} \tag{22}$$

Suppose that there exist a voter paying for both a and b (from group 5). As each voter from this group has $\frac{7-6p}{7}$ dollars left, at most 10 voters pay for a (groups 4 and 5) and at most 10 voters pay for b (groups 5, 8, 9) we would have the following inequality:

$$\begin{aligned}\frac{p}{10}+\frac{p}{10} &\leqslant \frac{7-6p}{7}\\ 14p &\leqslant 70-60p\\ p &\leqslant \frac{70}{74} \approx 0.946\end{aligned} \tag{23}$$

which contradicts (22). Hence, we need to assume that no voter pays for both a and b. By Lemma 3 we have only two cases: either no voters from group 5 pays for b or no voters from group 5 pays for a.

If no voters from group 5 pays for b, then only at most 4 voters do so. Suppose that at least one voter from group 8 pays for b (the case of group 9 is analogous) the individual price which is at least $\frac{p}{4}$ dollars. Then, $\frac{p}{4} \leqslant \frac{5-4p}{5}$. In this case, all 6 voters approving c would prefer to pay $\frac{p}{6}$ dollars for c instead of paying for b or (in case of voters not paying for b) from their savings—which is sufficient, as $\frac{p}{6} < \frac{p}{4} \leqslant \frac{5-4p}{5}$. We obtain a contradiction.

Now assume that no voters from group 5 pay for a. Then only at most 4 voters from group 4 do so. Then we have a following inequality:

$$\begin{aligned}\frac{p}{4} &\leqslant \frac{10-9p}{10}\\ 5p &\leqslant 20-18p\\ p &\leqslant \frac{20}{23} \approx 0.87\end{aligned} \tag{24}$$

which also contradicts (22).

Case 2: $\{a,c\}$. In this case, 6 voters from groups 7 and 9 shall not be able to pay for d. Hence we have the following inequality:

$$\frac{p}{6} \geqslant \frac{5-4p}{5}$$

$$5p \geqslant 30 - 24p$$
$$p \geqslant \frac{30}{29} > 1 \tag{25}$$

Suppose now that only voters from group 5 pay for a. Then the following inequality needs to hold:

$$\frac{p}{6} \leqslant \frac{7-6p}{7}$$
$$7p \leqslant 42 - 36p$$
$$p \leqslant \frac{42}{43} \tag{26}$$

which contradicts (25).

Now suppose that at least one voter from group 4 pays for a. As in total there are 10 voters approving a, the following inequality holds:

$$\frac{p}{10} \leqslant \frac{10-9p}{10} = 1 - \frac{9p}{10} = 1 - p + \frac{p}{10}$$
$$p \leqslant 1 \tag{27}$$

which also contradicts (25).

Case 3: $\{a,d\}$. The reasoning here is analogous as in Case 2 (inequality (25) still holds because of candidate c and voters from groups 6 and 8).

Case 4: $\{b,c\}$. The reasoning in this case is similar to the one in Case 1. First, note that 10 voters approving a shall not be able to pay for this candidate. Hence:

$$\frac{p}{10} \geqslant \frac{10-9p}{10}$$
$$p \geqslant 1 \tag{28}$$

(an opposite inequality to (27)).

Suppose that there exist a voter paying for both b and c (from group 8). As each voter from this group has $\frac{5-4p}{5}$ dollars left, at most 10 voters pay for b (groups 5, 8, 9) and at most 6 voters pay for c (groups 6, 8) we have:

$$\frac{p}{6} + \frac{p}{10} \leqslant \frac{5-4p}{5}$$
$$20p + 12p \leqslant 120 - 96p$$
$$p \leqslant \frac{120}{128} \tag{29}$$

which contradicts (28). Hence, we assume that no voter pays for both c and b. Note that by Lemma 3 we have only two cases: either voters from group 8 pay for b or for c. The second option is not possible—if the voters pay for c, then they pay at least $\frac{p}{6}$, while voters paying for b pay $\frac{p}{8}$. Hence, they have an incentive to stop paying for c and start paying for b.

Now assume that voters from group 8 pay for b. Then only at most 4 voters from group 6 pay for c. Then we have a following inequality:

$$\frac{p}{4} \leqslant \frac{5-4p}{5}$$
$$p \leqslant \frac{20}{21} \tag{30}$$

(the opposite inequality to (22)), which also contradicts (28).

Case 5: $\{b, d\}$. This case is analogous to Case 4 (swapping candidates c and d).

Case 6: $\{c, d\}$. In this case the voters from groups 6 and 8 pay for c and the voters from groups 7 and 9 pay for d.

Consider the group of all 10 voters approving b who may want to start paying for b (and in case of groups 8 and 9, stop paying for c and d—as $\frac{p}{10} < \frac{p}{6}$, it is always profitable for them). To prevent them, the price needs to be high enough so that voters from group 5 do not have enough money to spend. Hence we have that:

$$\frac{p}{10} > \frac{7-6p}{7}$$
$$7p \geqslant 70 - 60p$$
$$p \geqslant \frac{70}{67} \approx 1.045 \tag{31}$$

On the other hand, as at most 6 voters pay for c, we have that:

$$\frac{p}{6} \leqslant \frac{5-4p}{5}$$
$$p \leqslant \frac{30}{29} \approx 1.034 \tag{32}$$

which contradicts (31). □

B Stable Priceability for Other Types of Voters' Preferences

B.1 The Choice of the Preference Order for the Stability Condition

Condition (S5) uses a linear order $\succ$ that indicates when a voter is willing to change a committee. In the main text we have used the following preference relation:

$$(x, p) \succ (y, q) \iff x > y \text{ or } (x = y \text{ and } p < q). \tag{2}$$

Since in our model the only information we get from a voter is her preference relation over the candidates (the initial endowments of the voters do not correspond to true money that the voters could spend or save), another natural order seems to be the following:

$$(x, p) \succ (y, q) \iff x > y. \tag{33}$$

In the remainder of this section we explain how our results change if we use this preference relation in condition (S5). In the further part of this section we assume that the preference relation used in (S5) is given by (33).

Proposition Proposition 4 is similar to Theorem 3 except we need to addiitonally assume that $p \neq n/k$.

Proposition 4. *A feasible (strict) SP committee for the preference relation* (33) *with price* $p \neq n/k$ *is in the core.*

Proof. Consider a committee W that is stable priceable for the preference relation (33) with price $p \neq n/k$, and for the sake of contradiction assume W is not in the core. Since the committee is feasible, we get that $p < n/k$. Then, there exists $S \subseteq N$ and $T \subseteq C$ with $|T| \leqslant k \cdot |S|/n$ such that $u_i(W) < u_i(T)$ for all $i \in S$. We set $W' = T$, and $R_i = W$ for all $i \in S$. Then we can set $\{p'_i\}_{i \in N}$ such that for each $i \in S$ and $c \in T$ it holds that $p'_i(c) = 1/|T|$. It holds that:

1. For each $c \in T$: $\sum_{i \in S} p'_i(c) = {|S|}/{|T|} \geqslant {n}/{k} > p$
2. For each $i \in S$: $\sum_{c \in W'} p'_i(c) + \sum_{c \in W \setminus R_i} p_i(c) \leqslant 1$,
3. For each $i \in S$: $u_i(W') = u_i(T) > u_i(W) = u_i(R_i)$.

This gives a contradiction, and completes the proof. $\square$

SP with (33) implies a slightly worse (yet, still very high) proportionality degree.

Theorem 9. *SP for the preference relation* (33) *implies the proportionality degree of* $\ell - 2\sqrt{\ell}$.

Proof. Fix an election instance, and consider a size-k committee W that is SP. Let S be an ℓ-cohesive group of voters and let T be a set of ℓ candidates who are approved by all members of S. Let us fix x, $1 \leqslant x \leqslant \ell$, and let S' denote the set of voters from S who have at most $x - 1$ representatives in W. Let W' consist of some x candidates from T. For each voter $i \in S'$ we set $R_i = |W \cap A_i|$, that is to the set of all candidates the voter pays for; by stop paying for R_i the voter can have in total one unit of unspent money. For $i \in S \setminus S'$ we set R_i to the set of $x - 1$ candidates i pays most for. Clearly, by stop paying for the candidates from R_i the voter will end up with at least $\frac{x-1}{|A_i \cap W|}$ units of money. By (S5) we infer that the money of all voters must sum up to less than $x \cdot p$:

$$x \cdot p > \sum_{i \in S \setminus S'} \frac{x-1}{|A_i \cap W|} + \sum_{i \in S'} 1$$
$$= \sum_{i \in S \setminus S'} \frac{x-1}{|A_i \cap W|} + \sum_{i \in S'} \frac{x-1}{|A_i \cap W| + x - 1 - |A_i \cap W|}.$$

Further, by the inequality between the harmonic and arithmetic mean, we get that:

$$x \cdot p > (x-1) \cdot \frac{|S|^2}{\sum_{i \in S} |A_i \cap W| + \sum_{i \in S'} (x - 1 - |A_i \cap W|)}.$$

Since $|S| \geqslant {n\ell}/{k}$ and $p \leqslant {n}/{k}$, we get that:

$$x > (x-1) \cdot \frac{\frac{n}{k} \cdot \ell^2}{\sum_{i \in S} |A_i \cap W| + \sum_{i \in S'} (x - 1 - |A_i \cap W|)}.$$

After reformulating:

$$\sum_{i \in S} |A_i \cap W| + \sum_{i \in S'} (x - 1 - |A_i \cap W|) > \frac{n}{k} \cdot \ell^2 \cdot \frac{x-1}{x}.$$

For the sake of contradiction assume that $\sum_{i \in S} |A_i \cap W| < {n\ell}/{k} \cdot (\ell - 2\sqrt{\ell})$. Then:

$$\sum_{i \in S'} (x - 1 - |A_i \cap W|) > \frac{n}{k} \cdot \ell \cdot \left(2\sqrt{\ell} - \frac{\ell}{x}\right).$$

The above inequality must hold for each x, in particular, if we set $x = \lceil \sqrt{\ell} \rceil$. Then we get that:

$$\sum_{i \in S'} (\sqrt{\ell} - |A_i \cap W|) > \frac{n}{k} \cdot \ell \cdot \left(2\sqrt{\ell} - \frac{\ell}{\sqrt{\ell}}\right) = \frac{n}{k} \cdot \ell\sqrt{\ell}.$$

However, this is impossible, since $|S'| \leqslant {n\ell}/{k}$ (otherwise, the whole set of voters from S' could replace their representatives with T). This gives a contradiction, and completes the proof. $\square$

Finally, we note that for SP using (33) we have a much weaker relation with the concept of Lindahl equilibrium. In the proof of Proposition 5 below we show a simple example where an SP committee W is not in a cost-efficient Lindahl equilibrium. In this example the committee W is, intuitively, worse than other committees, and it should not be picked by any reasonable rule. This, once again, shows that the concept of SP with (2) is the more appealing one.

Proposition 5. *For every committee W that is in a cost-efficient Lindahl equilibrium for the preference relation* (33)*, there exists a price system $(p, \{p_i\}_{i \in N})$ that satisfies (C2)–(C4) and (S5). There exists an instance with an SP committee W for* (33)*, which is not in a cost-efficient Lindahl equilibrium.*

Proof. The proof of the first statement is similar to the first part of the proof of Theorem 5.

For the second statement, consider a profile with three candidates a, b, c and five voters:

2 voters: $\{a, b\}$ 3 voters: $\{a, b, c\}$.

Here, committee $\{c\}$ is SP for price $p = 3$. Indeed, only 2 voters would like to transfer money towards $\{a\}$ or $\{b\}$, which is not enough to cover the cost of each such a committee.

However, $\{c\}$ is not in the cost-efficient Lindahl equilibrium. For the sake of contradiction, assume it is. First, observe that the price of a candidate must equal at most $p \leqslant 3$. For each voter i among the first two, we have that $\gamma_i(a) > 1$ and $\gamma_i(b) > 1$. For each i among the last three voters, we have that $\gamma_i(a) + \gamma_i(b) > 1$. Thus, $\sum_{i \in [5]}(\gamma_i(a) + \gamma_i(b)) > 7$. Thus, one of these candidates gets the total payment of more than 3, which contradicts (Lin-PM). □

Let us conclude this section with a short discussion comparing the two preference relations, (2) and (33). Money in our model is fictional and therefore voters do not have any natural incentive to save it, when it does not improve their satisfaction. Given that, a reader may find order (2) counterintuitive. There are two main reasons why we find it well-justified.

1. Consider the context of participatory budgeting, where the candidates come with their actual monetary costs. Assume that the unspent money will stay in the municipal budget and can be spent on some other citizens' needs (but voter i will not be able to influence how the saved money is spent). Fix the initial endowements of the voters to 1 dollar, and consider a voter $i \in N$. Consider two committees—W and W'—which i likes equally ($u_i(W) = u_i(W')$). By picking a committee W rather than W', the voter would choose a more efficient outome—the one which she likes equally as W', but which allows a larger amount of money to be spent on the other voters' needs.

2. The condion (2) appears technically more interesting and easier to be used in practice. For example, it can be verified in polynomial time. It implies higher proportionality degree (Theorem 4 versus Theorem 9). Further, it excludes many committees which are intuitively not fair (see, e.g., the construction in the proof of Proposition 5) and is more closely related to the concept of Lindahl equilibrium (Theorem 5 versus Proposition 5).

C Algorithms for Finding Stable Priceable Committees

In this section we describe two algorithms for finding SP and BSP committees.

C.1 An Integer Linear Program for Finding SP Committees.

Below we formulate an ILP for finding SP committees. For each candidate $c \in C$ we have a binary variable x_c which indicates whether c is a part of the SP committee W. Inequality

(35) encodes the feaibility constraint for the committee W. For each $c \in C$ and $i \in N$ we have a variable $p_{i,c}$ which denotes the amount of money that voter i pays for c. Inequality (36) says that a voter can pay only for selected committee members and Inequality (37) ensures that a voter will not spend more than its initial budget. Finally, (38) ensures that the total payment for the selected candidates equals p (note that in any feasible solution p must be lower or equal than $|N|$). For each voter $i \in N$ we also have a variable m_i, which intuitively equals to $\max\left(\max_{a \in W}(p_{i,a}), r_i\right)$—this interpretation is encoded in (39) and (40). The last inequality (41) encodes the constraint of stability (3).

$$
\begin{aligned}
\textbf{constraints:}\quad & x_c \in \{0,1\} && \text{for } c \in C && (34)\\
& \sum_{c \in C} x_c = k && && (35)\\
& 0 \leqslant p_{i,c} \leqslant x_c && \text{for } i \in N,\, c \in C && (36)\\
& \sum_{c \in C} p_{i,c} \leqslant 1 && \text{for } i \in N && (37)\\
& p + (x_c - 1)|N| \leqslant \sum_{i \in N} p_{i,c} \leqslant p && \text{for } c \in C && (38)\\
& m_i \geqslant p_{i,c} && \text{for } i \in N,\, c \in C && (39)\\
& m_i \geqslant 1 - \sum_{c \in C} p_{i,c} && \text{for } i \in N && (40)\\
& \sum_{i \in N(c)} m_i \leqslant p + x_c|N| && \text{for } c \in C && (41)
\end{aligned}
$$

C.2 A Heuristic Algorithm for Finding BSP Committees

Our algorithm for finding weakly BSP committees is based on the characterization of such committees given in Theorem 7. We use binary search to find the price for which Rule X (as defined in Section 4.3) finds the committees of the closest size to k as possible. Besides, we do not skip demanding groups which are not strongly demanding, until $|W| = k$.

This algorithm is heuristic, for two reasons: (i) our adapted tie-breaking rule is not the only possible one, and (ii) the size of committees elected by Rule X is not monotonous with respect to the price, as it is show in the following example.

However, our experiments show that this algorithm very often successfully manages to find BSP committees.

Example 5. Consider the following election with $n = 14$ voters and $m = 15$ candidates.

$$
\begin{aligned}
&\text{4 voters: } \{a_1, ..., a_8, b_1\}\\
&\text{4 voter: } \{a_1, ..., a_8, b_2\}\\
&\text{1 voter: } \{c_1, c_2, c_3, b_1, b_2\}\\
&\text{2 voters: } \{c_1, c_2, c_3\}\\
&\text{1 voter: } \{d_1, d_2, d_3, b_1, b_2\}\\
&\text{2 voters: } \{d_1, d_2, d_3\}
\end{aligned}
$$

If we set the price to $p = 1$, then Rule X chooses $a_1, \ldots, a_8, c_1, c_2, c_3, d_1, d_2, d_3$, that is it selects 14 candidates. If we set the price to $p = 5/6$, then Rule X first chooses $a_1, \ldots, a_8$, and next it selects b_1 and b_2. For b each voter pays $1/6$. Thus, the voters who approve both b_1 and b_2 are left with $2/3$ dollars. If they were to pay for 3 candidates, then the price each voter would need to pay for such candidates would be equal to at most $2/9$. However, then each 3 voters, by paying this amount of money, would only collect $2/3$ dollars, less than p. Thus, only two c- and d-candidates can be selected. At most 13 candidates can be selected, which shows the lack of monotonicity.

D Existence of Stable Priceable Committees: Experiments

In this section we complement our theoretical analysis with experiments aimed at assessing how often SP and BSP committees exist for given k. Our results show that, except for few models, stable committees very often exist.

For instances where they do not exist, the next natural question is what is the greatest value $k' \leqslant k$, for which [balanced] stable priceable committees exist—note that in such case, the largest possible fraction of the committee is fair according to the concepts of stable priceability. Then we can either complete the remaining $k - k'$ seats in the committee in a different way or decrease its size to k'.

However, sometimes we have the opposite situation—the value k is the lowest possible number of seats in a committee and we could agree to increase it (as little as possible) if it would improve proportionality. The real-life example here could be German parliamentary elections. The election system (a mixed-member proportional representation system) combines seats elected by a district using first-past-the-post (plurality) together with a proportional party list election. As a result, there may be an *overhang*, when a party wins more seats in the district elections than its overall share of party list votes at the national level. Since 2013, in Germany this problem is solved by increasing the size of the parliament so that the final division of seats is roughly proportional. These additional compensational seats are called *leveling seats*. Due to this system, the German parliament elected in 2017 has 709 members, while the minimal legal number of seats is 598.

Hence, in our experiments we also calculated the lowest value $k'' \geqslant k$, for which [balanced] stable priceable committees exist. Finally, we considered also the minimum of these two values k', k'' in case of the size of the committee can be both increased and decreased.

For finding SP committees, we used the ILP defined in Appendix C.1. For finding BSP committees we used the heuristic algorithm described in Appendix C.2.

D.1 Datasets

We generated voters' preferences over candidates randomly from the following models:

1D-Euclidean model. In this model the voters and candidates are represented as points in the one-dimensional Euclidean space. The points were selected uniformly at random form the interval $[0, 1]$. The approval ballots were created in one of the following two ways:

1. For each candidate we chose uniformly at random the length of the radius of the approval ball—every candidate was approved only by the voters inside her ball.
2. The radiuses were chosen for each voter, and every voters approved the candidates inside her ball.

In our results, we refer to these two models as '1D Euclidean 1' and '1D Euclidean 2'.

2D-Euclidean model. Here, we represent the voters and the candidates as points in the Euclidean plane $[0, 1] \times [0, 1]$. The points were generated as follows. We first generated between 1 and 5 points of concentration of the voters and the candidates. Next, we randomly divided the voters and the candidates so that each of them was assigned to one point of concentration. Finally, for each voter and candidate we selected their position from the normal distribution with the ceter at the corresponding point of concentration and with the standard deviation set to 0.2.

We generated the approval-ballots from the positions of the voters and the candidates similarly as in the first case in the 1D-Euclidean model.

Impartial Culture model. Here, each candidate was approved by each voter with probability $1/2$.

Mallow's model. Here we first generate a ranking-based preference profile according to the mixture of three Mallow's models [Mallows, 1957]. The parameters ϕ for the three models were generated uniformly at random from $[0, 1]$; the reference rankings were also selected uniformly at random. Next, for each voter we selected uniformly at random a position $p \in [0; 0.25 \cdot m]$, and made this voter approve the first p candidates in her ranking.

Pólya-Eggenberger urn model. Here our model is parameterized by the size of the approval sets α and the replacement value β. At first, we consider an urn containing all the candidates; for each voter we draw α candidates from the urn uniformly at random, and each time we return to the urn β copies of the selected candidate—increasing the probability that next time the same candidate would be chosen again. In our tests parameter α was chosen uniformly at random from interval $[1; 0.1 \cdot m]$, and parameter β was chosen in two ways:

1. uniformly at random from interval $[0; 0.1 \cdot m]$,
2. uniformly at random from interval $[0; 0.25 \cdot m]$.

In our results, we refer to these two models as 'Urn 1' and 'Urn 2'.

In case of BSP, we used also some real instances (from 2020 participatory budgeting in Warsaw) and we naturally adapted BSP to the PB model with different costs of candidates (see Appendix E for details). For complexity reasons, we did not computed SP instances on them.

D.2 Results for BSP

Here we performed tests for $n = 100$ voters and $m = 30$ candidates. The value of the k was sequentially increased, covering all the values from $[m]$. For each k and each model with specific values of the parameters was run at least 1000 experiments. Thus, in total we checked $1000 \cdot 30 \cdot 7 = 210000$ election instances.

In Table 1 we present the results for existence for every model with fixed parameters and every value of k. Summarizing, in most cases BSP committees exist— 2D Euclidean model appeared to be the worst one, especially when $k > 20$. Therefore, in Table 2 we present the additional results for that model, showing that in most cases, even if BSP committees do not exist, they exist for slightly modified committee size k.

We also performed experiments for 19 real instances, coming from 2020 Warsaw participatory budgeting. The instances correspond to 19 Warsaw districts. The overall budget was 25000000 PLN. In our experiments we distributed it to the districts proportionally to the number of voters. The results are presented in Table 3. We can see that for every district, it was possible to elect an SP outcome which either spends nearly all the budget or only slightly exceeds it.

k	1D Euclidean 1	1D Euclidean 2	2D Euclidean	IC	Urn 1	Urn 2	Mallows
1	1000	1000	1000	1000	1000	1000	1000
2	1000	1000	1000	1000	1000	1000	1000
3	1000	999	999	1000	1000	1000	1000
4	996	998	999	1000	995	1000	1000
5	998	999	993	1000	999	1000	1000
6	995	999	996	1000	999	1000	999
7	991	998	989	1000	992	999	999
8	996	998	991	1000	991	998	998
9	991	999	993	1000	986	999	998
10	995	997	985	1000	981	998	998
11	997	999	992	1000	976	998	995
12	995	999	994	1000	972	999	999
13	994	999	988	1000	978	997	995
14	993	999	989	1000	981	999	998
15	994	998	983	1000	971	999	997
16	995	1000	980	1000	980	999	996
17	994	1000	970	1000	976	999	994
18	992	998	962	1000	979	998	996
19	994	997	960	1000	994	997	998
20	994	998	930	1000	983	1000	998
21	995	998	905	1000	998	1000	997
22	995	999	887	1000	997	999	996
23	989	999	872	1000	1000	1000	995
24	994	998	855	1000	1000	1000	996
25	997	997	865	1000	1000	1000	997
26	992	998	856	1000	1000	1000	1000
27	994	998	891	1000	1000	1000	999
28	998	1000	891	1000	1000	1000	1000
29	1000	994	934	1000	1000	1000	1000
30	1000	1000	1000	1000	1000	1000	1000

Table 1: Existence of BSP committees

k	**Max absolute deviation (lower)**	**Average absolute deviation (lower)**	**Max absolute deviation (upper)**	**Average absolute deviation (upper)**	**Max absolute deviation (total)**	**Average absolute deviation (total)**
1	-	-	-	-	-	-
2	-	-	-	-	-	-
3	1	1	1	1	1	1
4	1	1	1	1	1	1
5	1	1	1	1	1	1
6	1	1	1	1	1	1
7	1	1	1	1	1	1
8	1	1	1	1	1	1
9	1	1	1	1	1	1
10	1	1	2	1.2	1	1
11	2	1.125	1	1	1	1
12	1	1	1	1	1	1
13	1	1	1	1	1	1
14	2	1.182	1	1	1	1
15	3	1.294	1	1	1	1
16	2	1.1	4	1.3	2	1.05
17	3	1.267	2	1.3	2	1.03
18	3	1.263	3	1.132	2	1.026
19	3	1.183	3	1.3	1	1
20	3	1.129	3	1.171	3	1.057
21	3	1.116	3	1.221	2	1.011
22	4	1.15	4	1.283	2	1.053
23	3	1.227	3	1.227	2	1.031
24	3	1.131	5	1.221	2	1.034
25	2,	1.119	5,	1.296	2,	1.015
26	2,	1.125	4,	1.271	2,	1.028
27	2,	1.119	3,	1.211	2,	1.028
28	2	1.055	4	1.165	2	1.028
29	1	1	3	1.152	1	1
30	-	-	-	-	-	-

Table 2: Maximal and average absolute deviation of BSP committe size in instances generated from 2D Euclidean model for which no BSP committee of size k existed. We consider the case when we (i) require that k is the lower bound for committee size (lower), (ii) require that k is upper bound for committee size (upper), (iii) do not have any requirements (total).

D.3 Results for SP

In case of SP, we were able to perform tests for $n = 100$ voters and $m = 30$ candidates (analogous as in case of BSP) for all models except for Impartial Culture. In case of this model, because of complexity issues we decided to reduce the number of voters to 50 and the number of candidates to 15 (the number of experiments for each $k \in [m]$ remained 1000). Furthermore, the results (when k is small) were the worst for this model, even on such reduced instances (which is somehow surprising, as they were the best in case of BSP). The detailed results—in particular, checking how much we need to deviate k to obtain SP—are presented in Table 4.

For other models—also for Euclidean 2D, which appeared to be the worst for the existence of BSP—it was hard to even find any instance not satisfying SP, as we can see in Table 5.

E Stable Priceability for Participatory Budgeting with Cardinal Utilities

In this section we explain how our notions extend to the more general setting of participatory budgeting. We consider the following model.

An election is a tripple (C, N, b), where $C = \{c_1, \ldots, c_m\}$ is a set of *candidates*, $N = \{1, \ldots, n\}$ is a set of *voters*, and $b \in \mathbb{R}_+$ is the total voters' *budget*. For each voter $i \in N$ and for each candidate $c \in C$ by $u_i(c)$ we denote the cardinal *utility* that i assigns to c; intuitively, $u_i(c)$ quantifies the satisfaction the voter experiences if c is among the selected candidates. Further, each candidate $c \in C$ is associated with a *cost*, denoted as $\mathrm{cost}(c)$. We adapt this notation to sets, setting $u_S(T) = \sum_{i \in S} \sum_{c \in T} u_i(c)$ and $\mathrm{cost}(T) = \sum_{c \in T} \mathrm{cost}(c)$ for each $S \subseteq N$ and $T \subseteq C$.

We adapt the conditions of priceability accordingly. A price system is a pair $\mathrm{ps} = (p, \{p_i\}_{i \in [n]})$, where $p \geqslant {}^{b}\!/\!_{n}$ is the initial *endowment* (the initial budget) of each voter, and for each voter $i \in N$, $p_i \colon C \to [0, 1]$ is a *payment function* that specifies the amount of money a particular voter pays for the elected candidates. In our main text, and in the original paper of Peters and Skowron [2020] it is assumed that each voter is initially given one dollar, which corresponds to setting $p = 1$, but that there is an additional variable that specifies the total price that needs to be paid for an elected candidate. These two formulations are equivalent, but we chose to use the one with fixed prices and adjustable voters' initial budgets, since this formulation more naturally extends to the general PB model. The requirement that $p \geqslant {}^{b}\!/\!_{n}$ ensures that the voters have at least enough money to buy candidates with the total cost of b. Without this requirement, e.g., an empty outcome $W = \emptyset$ would be priceable (with $p = 0$). Formally, we say that an outcome W is supported by a price system $\mathrm{ps} = (p, \{p_i\}_{i \in [n]})$ if the following conditions hold:

(C1). A voter can pay only for the approved candidates: $p_i(c) = 0$ for each $i \in N$ and $c \notin A_i$.

In the general PB model the condition is: $u_i(c) = 0 \implies p_i(c) = 0$.[5]

(C2). Each voter has the same initial budget, equal to p dollars: $\sum_{c \in C} p_i(c) \leqslant p$ for each $i \in N$.

(The condition remains unchanged in the general model.)

[5]While condition (C1) is well-justified in the approval-based setting, in the general PB model it will have a significantly limited scope of impact. Indeed, the condition does not put any restrictions on the payments when the utilities $u_i(c)$ are very small, yet still positive. However, this condition will not play an instrumental role in the further part of our study, and will be implied by some conditions that we will consider later on.

District name	Unspent budget	Exceeded budget	Minimum
Bemowo	10.8%	23.0%	10.8%
Bialoleka	0.5%	0.4%	0.4%
Bielany	0.5%	1.1%	0.5%
Mokotow	0.0%	1.9%	0.0%
Ochota	0.4%	3.4%	0.4%
Ogolnomiejski	0.3%	0.8%	0.3%
Praga-Polnoc	7.3%	1.0%	1.0%
Praga-Poludnie	1.6%	2.1%	1.6%
Rembertow	0.3%	2.4%	0.3%
Srodmiescie	25.1%	2.0%	2.0%
Targowek	16.0%	7.8%	7.8%
Ursus	5.2%	3.5%	3.5%
Ursynow	0.8%	1.1%	0.8%
Wawer	0.5%	1.7%	0.3%
Wesola	31.2%	0.5%	0.5%
Wilanow	1.4%	0.3%	0.3%
Wlochy	4.8%	0.3%	0.3%
Wola	0.8%	0.1%	0.1%
Zoliborz	1.8%	2.2%	1.8%

Table 3: BSP outcomes for Warsaw PB instances. We aim to keep the outcome cost as close to the budget value as possible, and consider cases when we prefer to have some money unspent or to possibly exceed the budget.

k	Existence	Max absolute deviation (lower)	Average absolute deviation (lower)	Max absolute deviation (upper)	Average absolute deviation (upper)	Max absolute deviation (total)	Average absolute deviation (total)
1	1000	-	-	-	-	-	-
2	587	6	2.065	1	1	1	1
3	649	5	1.607	2	1.761	2	1.362
4	842	5	1.411	3	2.576	3	1.361
5	948	4	1.327	4	3.462	4	1.288
6	968	3	1.1875	5	4.40625	3	1.1875
7	999	2	2	6	6	2	2
8	1000	-	-	-	-	-	-
9	1000	-	-	-	-	-	-
10	1000	-	-	-	-	-	-
11	1000	-	-	-	-	-	-
12	1000	-	-	-	-	-	-
13	1000	-	-	-	-	-	-
14	1000	-	-	-	-	-	-
15	1000	-	-	-	-	-	-

Table 4: Maximal and average deviation of SP committe size in instances generated from Impartial Culture model for which no SP committee of size k existed. We consider the case when we (i) require that k is the lower bound for committee size (lower), (ii) require that k is upper bound for committee size (upper), (iii) do not have any requirements (total).

k	**1D Euclidean 1**	**1D Euclidean 2**	**2D Euclidean**	**Urn 1**	**Urn 2**	**Mallows**
1	1000	1000	1000	1000	1000	1000
2	1000	1000	996	1000	990	996
3	1000	1000	1000	1000	1000	997
4	1000	1000	1000	1000	1000	1000
5	1000	1000	1000	1000	1000	1000
6	1000	1000	1000	1000	1000	1000
7	1000	1000	1000	1000	1000	1000
8	1000	1000	1000	1000	1000	1000
9	1000	1000	1000	1000	1000	1000
10	1000	1000	1000	1000	1000	1000
11	1000	1000	1000	1000	1000	1000
12	1000	1000	1000	1000	1000	1000
13	1000	1000	1000	1000	1000	1000
14	1000	1000	1000	1000	1000	1000
15	1000	1000	1000	1000	1000	1000
16	1000	1000	1000	1000	1000	1000
17	1000	1000	1000	1000	1000	1000
18	1000	1000	1000	1000	1000	1000
19	1000	1000	1000	1000	1000	1000
20	1000	1000	1000	1000	1000	1000
21	1000	1000	1000	1000	1000	1000
22	1000	1000	1000	1000	1000	1000
23	1000	1000	1000	1000	1000	1000
24	1000	1000	1000	1000	1000	1000
25	1000	1000	1000	1000	1000	1000
26	1000	1000	1000	1000	1000	1000
27	1000	1000	1000	1000	1000	1000
28	1000	1000	1000	1000	1000	1000
29	1000	1000	1000	1000	1000	1000
30	1000	1000	1000	1000	1000	1000

Table 5: Existence of SP committees

(C3). Each elected candidate gets the total payment of 1: $\sum_{i\in[n]} p_i(c) = 1$ for each $c \in W$.

In the general model: $\sum_{i\in[n]} p_i(c) = \mathrm{cost}(c)$ for each $c \in W$.

(C4). The voters do not pay for non-elected candidates: $\sum_{i\in[n]} p_i(c) = 0$ for each $c \notin W$.

(The condition also remains unchanged in the general PB model.)

(C5). For each unelected candidate, her supporters have a remaining unspent budget of at most 1:
$\sum_{i\in N(c)} \left(p - \sum_{c'\in W} p_i(c')\right) \leqslant 1$ for each $c \notin W$.

For the general model: $\sum_{i\in N(c)} \left(p - \sum_{c'\in W} p_i(c')\right) \leqslant \mathrm{cost}(c)$ for each $c \notin W$.

The condition of stability (S5) also naturally extends.

(S5). Condition for Stability: There exists no coalition of voters $S \subseteq N$, no subset $W' \subseteq C \setminus W$ and no collections, $\{p'_i\}_{i\in S}$ ($p'_i \colon W' \to [0,1]$) and $\{R_i\}_{i\in S}$ (with $R_i \subseteq W$ for all $i \in S$) such that all the following conditions hold:

1. For each $c \in W'$: $p'_i(S) > \mathrm{cost}(c)$.
2. For each $i \in S$: $p_i(W \setminus R_i) + p'_i(W') \leqslant p$.
3. For each $i \in S$:

$$\big(u_i(W \setminus R_i \cup W'), p_i(W \setminus R_i) + p'_i(W')\big) \succeq \big(u_i(W), p_i(W)\big).$$

In this section, similarly as in the main text we focus on the preference relation (2).

Proposition 6. *An outcome that is either strictly SP or SP with $p \neq b/n$ is in the core.*

Proof. Consider an outcome W that is SP, and for the sake of contradiction assume W is not in the core. Then, there exists $S \subseteq N$ and $T \subseteq C$ with $\sum_{c\in T} \mathrm{cost}(c) \leqslant b \cdot |S|/n$ such that $u_i(W) < u_i(T)$ for all $i \in S$. We set $W' = T$, and $R_i = W$ for all $i \in S$; for $i \notin S$ we set $R_i = \emptyset$. Then we can set $\{p'_i\}_{i\in N}$ such that for each $i \in S$ and $c \in T$ it holds that $p'_i(c) = p\cdot\mathrm{cost}(c)/\mathrm{cost}(T)$. Clearly, for $i \in S$:

$$u_i((W \setminus R_i) \cup W') = u_i(T) > u_i(W),$$

and so:

$$\sum_{i\in S}\sum_{c\in T} p'_i(c) = p|S| \geqslant b/n \geqslant \sum_{c\in T} \mathrm{cost}(c).$$

For strict stable priceability we get the contradiction. For stable priceability with $p > b/n$, we note that the second inequality in the above sequence is strict, and we get a contradiction. This completes the proof. $\square$

The PB adaptation of [cost-efficient] Lindahl equilibrium is similar as in case of priceability—now parameter p does not equals a candidate's production cost (which instead equals $\mathrm{cost}(c)$) but each voter's endowment (which earlier was equal to 1 dollar).

Theorem 10. *Every outcome that is in a cost-efficient Lindahl equilibrium for price pis strictly stable priceable. The other implication does not hold.*

Proof. Consider an outcome $W \subseteq C$ that is in the cost-efficient Lindahl equilibrium, and let $\{\gamma_i\}_{i \in N}$ be the corresponding price system. From $\{\gamma_i\}_{i \in N}$ we construct the price system $\{p_i\}_{i \in N}$ witnessing stable priceability as follows. For each $i \in N$ and $c \in W$ we set $p_i(c) = \gamma_i(c)$; for $c \notin W$ we set $p_i(c) = 0$. We now verify that $\{p_i\}_{i \in N}$ satisfies the conditions of stable-priceability. (C1) follows from (S5), and we will prove it later on. (C2) follows from (Lin-UM(a)) (feasibility in the customer-stability condition). (C3) follows from profit maximization (Lin-PM) and cost-efficiency, and (C5) follows directly from the construction of the paymnet functions.

Let us now consider (S5). Let us fix $W' \subseteq C \setminus W$ and a collection $\{R_i\}_{i \in N}$. Observe that if $u_i((W \setminus R_i) \cup W') > u_i(W)$, then by (Lin-UM) $\sum_{c \in (W \setminus R_i) \cup W'} \gamma_i(c) > p$ and so:

$$\begin{aligned}\sum_{c \in W'} p'_i(c) &\leqslant p - \sum_{c \in W \setminus R_i} p_i(c) = p - \sum_{c \in W \setminus R_i} \gamma_i(c) \\ &= p - \sum_{c \in (W \setminus R_i) \cup W'} \gamma_i(c) + \sum_{c \in W'} \gamma_i(c) < \sum_{c \in W'} \gamma_i(c).\end{aligned}$$

On the other hand, if $u_i((W \setminus R_i) \cup W') = u_i(W)$ then either $\sum_{c \in (W \setminus R_i) \cup W'} \gamma_i(c) > p$ (and we obtain the estimation as above), or $p - \sum_{c \in W} \gamma_i(c) \geqslant p - \sum_{c \in (W \setminus R_i) \cup W'} \gamma_i(c)$. In the latter case we get that:

$$\begin{aligned}\sum_{c \in W'} p'_i(c) \leqslant \sum_{c \in R_i} p_i(c) &= \sum_{c \in R_i} \gamma_i(c) = \sum_{c \in W} \gamma_i(c) - \sum_{c \in W \setminus R_i} \gamma_i(c) \\ &\leqslant \sum_{c \in (W \setminus R_i) \cup W'} \gamma_i(c) - \sum_{c \in W \setminus R_i} \gamma_i(c) = \sum_{c \in W'} \gamma_i(c).\end{aligned}$$

In any case, we get that $\sum_{c \in W'} p'_i(c) \leqslant \sum_{c \in W'} \gamma_i(c)$. By (Lin-PM) we get that for each $c \in W'$ we have $\sum_{i \in N} \gamma_i(c) \leqslant \mathrm{cost}(c)$. Thus, we can continue as:

$$\sum_{i \in N} \sum_{c \in W'} p'_i(c) < \sum_{i \in N} \sum_{c \in W'} \gamma_i(c) \leqslant \sum_{c \in W'} \mathrm{cost}(c') = \mathrm{cost}(W').$$

Which proves that (S5) is indeed satisfied.

Second, we show an instance and an outcome that is strictly stable priceable, but which is not in a Lindahl equilibrium. We have four candidates, a_1, a_2, b_1, b_2, and two voters. The budget is $b = 2$. The voters' preferences and the costs of the candidates are summarized in the table, below.

candidate	cost	$u_1(\cdot)$	$u_2(\cdot)$	$p_1(\cdot)$	$p_2(\cdot)$
a_1	ε	6	0	ε	0
a_2	$2-\varepsilon$	1	3	$1-\varepsilon$	1
b_1	1	2	2	0	0
b_2	1	2	2	0	0

For this instance, outcome $A = \{a_1, a_2\}$ is stable priceable with the initial endowments $p = 1$. The corresponding price system is also given in the above table. The conditions (C1)–(C5) are clearly satisfied. To see that (S5) is also satisfied, consider all possible values of W'—namely $\{b_1\}$, $\{b_2\}$ and $\{b_1, b_2\}$.

If $W' = \{b_1\}$ or $W' = \{b_2\}$, then it needs to hold that $S = \{1\}$ and $R_1 \subseteq \{a_2\}$ (voter 2 does not pay for a_1 and will decrease her utility if she resigned from a_2; voter 1 will decrease her utility if she resigned from a_1). Then $p'_1(W') \leqslant p_1(a_2) \leqslant 1 - \varepsilon < \mathrm{cost}(W')$, a contradiction.

If $W' = \{b_1, b_2\}$ and $S = \{1, 2\}$, then $R_1 \subseteq \{a_2\}$ (voter 1 will decrease her utility if she resigned from a_1) and $R_2 \subseteq \{a_2\}$ (voter 2 does not pay for a_1). Then $\sum_{i \in S} p'_i(W') \leqslant \sum_{i \in S} p_i(a_2) \leqslant 2 - \varepsilon < \mathrm{cost}(W')$, a contradiction. Naturally, taking smaller set S will even decrease value $\sum_{i \in S} p'_i(W')$.

Yet, given the initial endowment $p = 1$, the outcome $\{a_1, a_2\}$ is not in the cost-efficient Lindahl equilibrium. For the sake of contradiction, assume that $\{a_1, a_2\}$ is in the cost-efficient Lindahl equilibrium. Then, $\gamma_1(a_1) = \varepsilon$ and $\gamma_2(a_1) = 0$. Further, it must also be the case that $\gamma_1(a_2) = 1 - \varepsilon$ and $\gamma_2(a_2) = 1$. Further, since voter 1 prefers b_1 to a_2 and b_2 to a_2, it must hold that $\gamma_1(b_1) > 1 - \varepsilon$ and that $\gamma_1(b_2) > 1 - \varepsilon$. Also, $\gamma_2(b_1) + \gamma_2(b_2) > 1$. This means that the sum of the prices for b_1 and b_2 is at least $3 - 2\varepsilon$, thus it exceeds the cost of producing b_1 and b_2, and so it violates (Lin-PM). This gives a contradiction, and completes the proof. $\square$