Unified Fair Allocation for Indivisible Goods and Chores via Copies

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Abstract

We study a unified model of goods and chores with copies, as a framework to study (1) The existence of EFX and other fair allocation solution concepts for chores, and (2) The existence of EFX and other fair allocation solution concepts for goods with copies. Quite surprisingly, we show that these two issues are tightly related via a duality theorem that we develop. We demonstrate the usefulness of this theorem by using it to prove the existence of EFX allocations for chores and "leveled preferences". This unified framework relies on a new solution concept that we term EFX_{WC} and that we believe may be of independent interest. It is a strict relaxation of EFX that is appropriate for goods with copies, and its existence implies the existence of EFX for chores. Any allocation that is EFX_{WC} is at least $\frac{4}{11}$-MMS for goods with copies. In contrast, it is known that there exist EF1 allocations that are no better than $\frac{1}{n}$-MMS for goods, where n is the number of agents.

1 Introduction

This work stems from the question of existence of various solution concepts for fair allocation of indivisible items, for two different models that on the face of it do not seem tightly related. The first model has indivisible goods with multiple *copies*. While the seminal work of Budish [2011] includes copies as a key ingredient (motivated by applications like the allocation of university courses to students or the assignment of tasks to workers), most subsequent work focuses on the case of no copies. We consider a model with copies where each agent may receive at most one copy of each good; we term such allocations "exclusive" (they are termed "valid" allocations in [Kulkarni *et al.*, 2020], or "at most one" allocations in [Kroer and Peysakhovich, 2019]). For example, continuing with the main motivation of [Budish, 2011], when allocating courses to students, one student cannot be allocated multiple seats in the same course. Another example is digital goods with a license quota (imagine a university that has X software licenses and allocates at most one license per student). A third example is the task of paper reviewing (it is meaningless to assign the same paper twice to the same reviewer). As we shall see, many existence results significantly change when we consider exclusive allocations of goods with copies.

The second model that we consider has indivisible *chores* (or "bads"), which are items with negative values. Previous literature usually treats goods and chores differently (see, e.g., [Barman and Krishna Murthy, 2017; Chaudhury *et al.*, 2020b; Huang and Lu, 2019; Garg and Taki, 2020]). As it stands, our knowledge on chores is lacking compared to our knowledge on goods. For example, while it is known that EFX allocations always exists for three agents and any number of goods (without copies) [Chaudhury *et al.*, 2020b], almost nothing is known regarding EFX allocations of chores. While this specific gap is a knowledge gap, other gaps provably hold. For example, for goods, every optimal Nash-welfare allocation satisfies the so called "envy bounded by a single item" (EF1) notion as well as Pareto-optimality [Caragiannis *et al.*, 2016]. In contrast, for chores, there is no single valued welfare function that implies both fairness and efficiency [Bogomolnaia *et al.*,

$$
\begin{tikzcd}
 & & \text{EFL} \arrow[rrd] & & \\
\text{EFX} \arrow[r] \arrow[rru] & \text{EFX}_{\text{WC}} \arrow[r, "\text{Prop. 4}"] \arrow[d, "\text{Thm. 4}"] & \text{EFL}_{\text{WC}} \arrow[r] \arrow[d, "\text{Thm. 5}"] & \text{EF1}_{\text{WC}} \arrow[r] \arrow[d, "\text{Ex. 6}"] & \text{EF1} \\
 & \frac{4}{11}\text{-MMS} & \frac{1}{3}\text{-MMS} & \frac{1}{n}\text{-MMS} &
\end{tikzcd}
$$

Figure 1: The hierarchy established in Section 5.

2017a]. In the divisible case, there is a polynomial time algorithm to find a competitive division for goods, but this problem is PPAD-hard for chores [Chaudhury *et al.*, 2020a].

Given the above, one may conclude that studying existence of solution concepts for fair allocation of goods with copies and for chores is to be achieved by two disparate lines of research. However, a main contribution of this paper shows that these two models are tightly related, via a formal duality theorem that we develop for a large class of fairness notions that we define. This duality has been informally observed in the past. For example, [Bogomolnaia *et al.*, 2017a] nicely explain the intuitive connection: “Say that we must allocate 5 hours of a painful job [...] Working 2 hours [on the job] is the same as being exempt [from the job] for 3 hours”. Very recently, [**?**] employed this connection in their proofs. We believe that such an important duality should be formally highlighted and explored much beyond a technical tool inside a proof or a short intuitive remark. Furthermore, as we demonstrate, our formal treatment of this duality reveals several conclusions that were, to the best of our knowledge, previously unknown (e.g., the possible inexistence of EFX, the hierarchy of fairness notions that evolve as a result).

Our results. Consider (as a main example) the notions of EFX and EF1, where each agent i prefers her own bundle over the bundle of any other agent j if one arbitrary item (for EFX) or the most valued item (for EF1) is removed from it. As is standard in the literature, we assume additive values/costs, and study exclusive allocations. We first observe that EFX does not always exist with copies (while EF1 does [Biswas and Barman, 2018]). Our counterexample for EFX uses a simple setting with any number of $n \geq 3$ agents and identical values (Example 1).

In light of this inexistence problem, we define “EFX Without Commons” (EFX$_{\text{WC}}$), which is weaker than EFX and stronger than EF1 (i.e., EFX implies EFX$_{\text{WC}}$ but not vice versa). EFX$_{\text{WC}}$ requires each agent i to prefer her own bundle over the bundle of any other agent j, if an arbitrary item *which is not also in i's bundle* is removed from j's bundle. Note that a setting without copies is a special case of a setting with copies and exclusive allocations, and in a setting without copies EFX$_{\text{WC}}$ is identical to EFX by definition.

We define similar “without commons” notions for other solution concepts, in particular EF1$_{\text{WC}}$ and EFL$_{\text{WC}}$, and present multiple evidence for their usefulness (also see Figure 1):

- These concepts expand the hierarchy of solution concepts for fair allocations: we show EFX $\Longrightarrow$ EFX$_{\text{WC}}$ $\Longrightarrow$ EF1$_{\text{WC}}$ $\Longrightarrow$ EF1, but not vice versa.
- We prove that EFX$_{\text{WC}}$ approximates the maximin share (MMS) much better than EF1. MMS is an important share-based (rather than envy-based) fairness notion advocated by Budish [2011].

Moreover, the most important technical evidence for the usefulness of EFX$_{\text{WC}}$ comes from a surprising connection to the existence of a (standard) EFX allocation for chores. Specifically, for any setting with n agents and any tuple of n valuations/costs, we show that an EFX$_{\text{WC}}$ allocation for goods with $n-1$ copies for each good exists if and only if an EFX allocation for chores without copies exists. By this characterization result, existence of EFX$_{\text{WC}}$ allocations can be used as a technical tool to determine existence of (standard) EFX allocations for chores.

This connection is obtained using a “duality theorem” we develop of goods on the one hand, and chores with copies on the other, showing for example that any allocation for goods with copies is EFX$_{\text{WC}}$ if and only if its “dual” allocation for chores with copies is EFX$_{\text{WC}}$ (the theorem is formulated in a general way that fits other solution concepts as well). Our new characterization of existence of EFX allocations for chores is a main corollary of this theorem since the dual allocation for goods with $n-1$ copies is an allocation with one copy of each chore. The duality theorem is in fact flexible enough to apply to settings with a mixture of goods and chores (see Remark 1).

Do EFX_{WC} exclusive allocations for goods with copies always exist (while EFX exclusive allocations need not necessarily exist, even for three agents)? We prove an affirmative answer for the special case of "leveled" preferences [Babaioff *et al.*, 2017; Manjunath and Westkamp, 2021], where larger bundles are always preferred to smaller ones. This result implies the existence of EFX allocations for chores for this case via our duality theorem, thus demonstrating its usefulness.

1.1 Additional Related Literature

Solution concepts: An EF1 allocation is guaranteed to exist for both goods [Lipton *et al.*, 2004; Caragiannis *et al.*, 2019] and chores [Aziz *et al.*, 2019; Bhaskar *et al.*, 2020]. Envy-freeness up to any item (EFX) existence is still a wide open problem for $n \geq 4$ agents and additive goods. An MMS allocation does not always exist, as was observed by Kurokawa *et al.* [2016], but fractional approximations of it may be guaranteed, prompting the notion of α-MMS. Their work guarantees existence of $\frac{2}{3}$-MMS, and further works guarantee $\frac{3}{4}$-MMS or more for goods, depending on the number of agents [Ghodsi *et al.*, 2018; Garg and Taki, 2020]. For chores, Aziz *et al.* [2017] and Huang and Lu [2019] establish $\frac{11}{9}$-MMS. The more technical envy-free concept of *EFL*, which we find useful as an intermediary to argue about EFX, was introduced by Barman *et al.* [2018], who show its existence for additive goods. For goods, envy-freeness notions can be sorted into a hierarchy (namely, EFX $\implies$ EFL $\implies$ EF1), with the notions separated by their α-MMS guarantees, as shown by Amanatidis *et al.* [2018]. We are able to establish a similar hierarchy for our more general goods with copies model.

Allocation restrictions: Kroer and Peysakhovich [2019] consider optimization of the MNW (Maximal Nash Welfare) with the constraint of "at most one" allocations. They show this achieves an approximate CEEI in large enough instances. Barman and Krishnamurthy [2017] study identical ordinal preferences for both goods and chores. Mixed instances of goods and chores are considered by Aziz *et al.* [2019]. As we show, a mixed model of goods and chores is a special case of our goods with copies model. Biswas and Barman [2018] consider a model where each agent has cardinality constraints over the amount of a same-typed good she can be allocated. Our model is a special case and thus their general results hold, namely existence of EF1 allocations and $\frac{1}{3}$-MMS allocations. EFX and EFL allocations may not exist for certain instances in their model as well as in ours. We refine these notions by the introduction of "Without Commons" which enables our duality results, the hierarchy of concepts we describe, and our existence results. Other matroid constraints models over allocations are presented in [Biswas and Barman, 2019; Dror *et al.*, 2020]. Some works assume bundles must be connected in a given underlying graph [Bouveret *et al.*, 2017; Bilò *et al.*, 2018], but our model does not seem to immediately translate to these settings.

Duality framework: The possible duality between goods and chores was noted in [Bogomolnaia *et al.*, 2017b]. They give an example of how to transform a chores instance into a goods instance that seems to follow the transformation outlined in our Definition 8 of the dual allocation. Kulkarni *et al.* [2020] note that a chore can be reinterpreted as $n-1$ goods, and use it as a technical tool in their proofs, converting an instance of goods and chores to an optimization problem with goods and "valid allocations".

2 Preliminaries

Our model: Let $\mathcal{T}$ denote the set of all possible item *types* (we slightly abuse notation and also use $\mathcal{T}$ as an item set with exactly one copy of each item). Let $\mathcal{M}$ be the set of items to be allocated, which is a multiset of $\mathcal{T}$ that includes $k_t \leq n$ copies of each item type $t \in \mathcal{M}$. Let $\mathcal{N}$ be the set of agents and $n = |\mathcal{N}|$. Each agent $i \in \mathcal{N}$ has an additive *valuation* function $v_i : 2^{\mathcal{M}} \to \mathbb{R}$. An item $t \in \mathcal{M}$ is termed a "good" if $v_i(t) \geq 0$ for each agent i and > 0 for at least one agent, and a "chore" if $v_i(t) \leq 0$ for each agent i. We assume that each item is either a good or a chore, i.e., we do not allow an item to be a good for one agent and a chore for another agent. We use t when referring to a general item, g when referring to a good, and c when referring to a chore. An allocation $\mathbf{A} = (A_1, \ldots, A_n)$ is a partition of all items among the agents (no item can be left unallocated). We focus on allocations that satisfy the following constraint:

Definition 1. *An allocation* $\mathbf{A}$ *is* **exclusive** *if no two copies of the same item are allocated to the same agent.*[1]

Throughout, we use "allocation" to denote an exclusive allocation. We denote the set of all exclusive allocations of item set $\mathcal{M}$ by $\mathcal{EA}(\mathcal{M})$.

Definition 2. *An exclusive allocation* A' Pareto dominates *the exclusive allocation* A *if for all agents* i, $v_i(A'_i) \geq v_i(A_i)$ *and* $v_j\left(A'_j\right) > v_j(A_j)$ *for at least one agent* j.

An exclusive allocation $\mathbf{A} \in \mathcal{EA}(\mathcal{M})$ *is* **Pareto optimal** *if there is no exclusive allocation that Pareto dominates it.*

Fairness notions for goods and chores: Previous literature studies the following notions.

Definition 3. *An exclusive allocation* $\mathbf{A}$ of goods *is*

- EFX*:* $\forall\, i, j \in \mathcal{N}, \forall g \in A_j, v_i(A_i) \geq v_i(A_j \setminus \{g\})$.
- EF1*:* $\forall\, i, j \in \mathcal{N}, \exists g \in A_j, v_i(A_i) \geq v_i(A_j \setminus \{g\})$.
- EFL*:* $\forall\, i, j \in \mathcal{N}$*, either (1)* $|A_j| \leq 1$*, or (2)* $\exists g \in A_j, v_i(A_i) \geq \max\{v_i(A_j \setminus \{g\}), v_i(g)\}$.
- PROP*:* $\forall\, i \in \mathcal{N}$, $v_i(A_i) \geq \frac{1}{n} v_i(\mathcal{M})$.
- α-MMS *(for some* $0 < \alpha \leq 1$*):* $\forall i \in \mathcal{N}$,
$$v_i(A_i) \geq \alpha \cdot \max_{\mathbf{A} \in \mathcal{EA}(\mathcal{M})} \min_{j \in \mathcal{N}} v_i(A_j).$$
  1-MMS *is termed* MMS.

The last two notions, PROP and MMS, are "share-based". Their definition above holds for chores too, and in fact for any mixture of goods and chores. More generally, in share-based fairness, an agent's allocation is measured against what is deemed her fair share:

Definition 4. *Given a valuation* v*, the number of agents* n *and an item set* $\mathcal{M}$*, a* **share** *function* s *outputs a real value, i.e.,* $s(v, n, \mathcal{M}) \in \mathbb{R}$*. An exclusive allocation* $\mathbf{A}$ *is called* s**-share-fair** *if we have* $v_i(A_i) \geq s(v_i, n, \mathcal{M}), \forall i \in \mathcal{N}$.

For example, for PROP, $s(v_i, n, \mathcal{M}) = \frac{1}{n} v_i(\mathcal{M})$.

The first three notions above are "envy-based", and their definition above is for goods. The definition of envy-based notions for chores is different in that, when an agent i compares her bundle to the bundle of another agent j, she removes a single chore from her *own* bundle (while a good is removed from the other agent's bundle). For example, an exclusive allocation $\mathbf{A}$ for chores is EFX if for every $i, j \in \mathcal{N}$ and $c \in A_i$,

$$v_i(A_i \setminus \{c\}) \geq v_i(A_j). \tag{1}$$

The connection between envy-based fairness for goods and for chores is phrased more generally as follows.

Definition 5 (Comparison criteria for goods and chores)**.**

- *Given a valuation function* v*, and two bundles* B_I *and* B_U*, a* **comparison criterion** f *outputs* F *for "fair" and* U *for "unfair", i.e.,* $f(v, B_I, B_U) \in \{F, U\}$.
- *Given a comparison criterion* f*, the comparison criterion* f^c *satisfies* $f^c(v, B_I, B_U) = f(-v, B_U, B_I)$.

Note that, denoting by f the comparison criterion for EFX for goods, it holds that f^c is the comparison criterion for EFX for chores in Eq. (1). One can similarly define EF1 and EFL for chores. More generally:

Definition 6 (Envy-based fairness for goods and chores)**.**

- *Given a comparison criterion* f*, we say that an allocation* A *is* f**-fair** *iff* $\forall i, j \in \mathcal{N} : f(v_i, A_i, A_j) = F$.
- *A solution concept E is* **envy-based** *if there exists a comparison criterion* f *such that an allocation A for goods satisfies E iff A is* f*-fair, and an allocation A for chores satisfies E iff A is* f^c*-fair.*

[1] **??** termed this "valid allocations".

EFX, EF1, and EFL are all examples within this general class.

Remark 1. *We define these fair allocation solution concepts for the case where all items are goods and for the case where all items are chores. Our results can be extended to a setting where some items are goods while others are chores, using the envy-based fair allocation solution concepts of [Aziz* et al.*, 2019]. For brevity we defer this to the full version.*

3 Warm-up: EFX with Copies

We begin by showing that EFX may not exist with copies.

Example 1. *Consider $n \geq 3$ agents, $n+1$ goods, and $\frac{n}{2} < k < n$ copies of each good. All agents share the same valuation function v, and for every item type $g_w \in \mathcal{M}$:*

$$v(g_w) = \begin{cases} w & w < n+1; \\ n^2 & w = n+1. \end{cases}$$

Proposition 1. *For goods with copies, an* EFX *allocation may not exist even for 3 agents. The same holds for* EFL.

Proof. Consider Example 1. We show non-existence of EFL allocations. Since every EFX allocation is also EFL by [Barman *et al.*, 2018], this shows no EFX allocations exist for these settings. Fix an agent l who does not receive a copy of g_{n+1}. Since there are k copies of g_1 and k copies of g_{n+1} where $2k > n$, there exists an agent h which receives both g_1 and g_{n+1}. Agent l EFL-envies agent h since the first condition of EFL is not satisfied ($|A_h| \geq 2$), and for any $g \in A_h$ with $v_l(g) \leq v_l(A_l)$,

$$v_l(A_l) \leq \sum_{w \leq n} v_l(g_w) < v_l(g_{n+1}) \leq v_l(A_h \setminus \{g\}),$$

violating the second condition of EFL. □

To handle the inexistence problem we introduce a new fairness notion called EFX_{WC}, which is a generalization of EFX for a model with copies. Intuitively, when comparing two bundles, EFX_{WC} puts aside all items common to both bundles, and then compares the remaining items in an EFX way:

Definition 7 (EFX_{WC} for goods)**.** *An exclusive allocation* $\mathbf{A}$ *of goods is* EFX_{WC} *if for every agents $i, j \in \mathcal{N}$ and item $g \in A_j \setminus A_i$ it holds that $v_i(A_i \setminus A_j) \geq v_i((A_j \setminus A_i) \setminus \{g\})$.*

Remark 2. *We remark that:*

- EFX_{WC} *is an envy-based notion and therefore its definition extends to chores using Definition 6.*
- *A model with no copies is a special case of our model, and when there is one copy of each good in our model then* EFX_{WC} *is equivalent to* EFX.
- EFX *implies* EFX_{WC}*: By additivity of the values, the inequality in Def. 7 is equivalent to the EFX inequality $v_i(A_i) \geq v_i(A_j \setminus \{g\})$, but* EFX *allows g to be any item while* EFX_{WC} *restricts its choice.*

A main technical justification of EFX_{WC} is its ability to give a "dual" view of goods and chores, which also yields a new characterization of existence of standard EFX allocations for chores. This is summarized by the following result (it is a corollary of Theorem 2 stated in the next section).

Corollary 1 (Characterization of EFX existence for chores)**.** *An* EFX *allocation for chores exists in the standard setting without copies iff an* EFX_{WC} *allocation for goods exists in our setting with $k = n-1$ copies of each good.*

The following example illustrates the above.

Example 2 (Special case of Example 1)**.** *There are $n = 3$ agents and 4 goods with $n - 1 = 2$ copies each. All agents have valuation v as defined in Example 1 (the values of items t_1, t_2, t_3, t_4 are $1, 2, 3, 9$ respectively).*

In Example 2, the allocation $A_1 = \{t_1, t_2, t_4\}$, $A_2 = \{t_3, t_4\}$ and $A_3 = \{t_1, t_2, t_3\}$ is EFX$_{\text{WC}}$. Indeed, agent 3 EFX-envies the others (and moreover EFL-envies them), but after putting aside common items, only item t_4 is left for the other agents, implying EFX$_{\text{WC}}$. We now show that the "dual" allocation is EFX for chores. The dual allocation is given by $A_i^\circ = \mathcal{T} \setminus A_i$ for every i (see formal definition in Section 4). I.e., $A_1^\circ = \{t_3\}$, $A_2^\circ = \{t_1, t_2\}$ and $A_3^\circ = \{t_4\}$. When the goods are treated as chores by simply taking the negation $-v$ of the valuation (such that the values of t_1, t_2, t_3, t_4 are $-1, -2, -3, -9$ respectively), it is not hard to check that the dual allocation is EFX for these chores. Indeed, agent 3 envies the others, but removing any chore from agent 3's single-chore bundle alleviates the envy.

4 The Duality of Goods and Chores

In this section we show that allocations of goods and allocations of chores are "dual" in a formal sense, and that fairness notions translate between the dual allocations (Sec. 4.1 shows this for envy-based and Sec. 4.2 for share-based).

Definition 8 (Duality)**.** *The* **dual** *of a tuple* $(\mathbf{A}, v, \mathcal{M})$ *is:*

- *The dual allocation* $\mathbf{A}^\circ$ *is* $A_i^\circ = \mathcal{T} \setminus A_i$.
- *The dual valuation is* $v^\circ = -v$. *Thus goods become chores and vice versa.*
- *The dual item set* $\mathcal{M}^\circ$ *contains* $n - k_t$ *copies of every* $t \in \mathcal{T}$, *where* k_t *is the number of copies of* t *in* $\mathcal{M}$.

Proposition 2 (Properties of Dual Transformations)**.**

1. *If* $\mathbf{A}$ *is an exclusive allocation of item set* $\mathcal{M}$ *then* $\mathbf{A}^\circ$ *is an exclusive allocation of item set* $\mathcal{M}^\circ$.
2. *The dual of the dual is the original:* $\mathbf{A}^{\circ\circ} = \mathbf{A}$, $v^{\circ\circ} = v$ *and* $\mathcal{M}^{\circ\circ} = \mathcal{M}$.
3. *The dual operation is a one-to-one mapping.*

Proof.

1. $\mathbf{A}^\circ$ is an allocation since each item t is allocated k_t times in $\mathbf{A}$, so it does not appear in the bundles of $n - k_t$ agents. Every such agent is allocated a copy of t in $\mathbf{A}^\circ$, and other agents are not. $\mathbf{A}^\circ$ is exclusive since by construction each bundle A_i° is contained in $\mathcal{T}$ and has at most one copy of each item.
2. $A_i^{\circ\circ} = \mathcal{T} \setminus A_i^\circ = \mathcal{T} \setminus (\mathcal{T} \setminus A_i) = A_i,\ v^{\circ\circ} = -v^\circ = v$, and $\mathcal{M}^{\circ\circ}$ contains $n - (n - k_t) = k_t$ items $\forall t \in \mathcal{T}$.
3. Assume that $\mathbf{A}_1, \mathbf{A}_2, v_1, v_2, \mathcal{M}_1, \mathcal{M}_2$ satisfy $\mathbf{A}_1^\circ = \mathbf{A}_2^\circ$, $v_1^\circ = v_2^\circ$, $\mathcal{M}_1^\circ = \mathcal{M}_2^\circ$, then we have $\mathbf{A}_1 = \mathbf{A}_1^{\circ\circ} = \mathbf{A}_2^{\circ\circ} = \mathbf{A}_2$. The rest follows similarly. □

Theorem 1. *An exclusive allocation* $\mathbf{A}$ *is Pareto optimal iff the dual allocation* $\mathbf{A}^\circ$ *is Pareto optimal.*

Proof. First we prove a property of dual: If $v_i(B_i) > v_i(A_i)$ then $v_i^\circ(B_i^\circ) > v_i^\circ(A_i^\circ)$. From $v_i(B_i) > v_i(A_i)$ we have $v_i(\mathcal{M} \setminus B_i) < v_i(\mathcal{M} \setminus A_i)$, i.e. $v_i(B_i^\circ) < v_i(A_i^\circ)$. Because $v_i^\circ = -v_i$, we have $v_i^\circ(B_i^\circ) > v_i^\circ(A_i^\circ)$.

Suppose that allocation $\mathcal{B}$ Pareto dominates allocation $\mathbf{A}$. We prove that allocation $\mathcal{B}^\circ$ Pareto dominates allocation $\mathbf{A}^\circ$. By the property we just proved, we have $v_i^\circ(B_i) \geq v_i^\circ(A_i)$ for all $i \in \mathcal{N}$ and there is a j such that $v_j^\circ(B_j^\circ) > v_j^\circ(A_j^\circ)$

This directly implies the statement, as there is a Pareto dominating allocation over $\mathbf{A}$ iff there is such an allocation for the dual. □

4.1 A Meta-Theorem for Envy-based Notions

The idea of EFX$_{\text{WC}}$ generalizes to other envy-based notions:

Definition 9 (Comparison without commons)**.** *Given a comparison criterion* f, *the comparison criterion* f_{WC} *satisfies* $f_{\text{WC}}(v, B_I, B_U) = f(v, B_I \setminus B_U, B_U \setminus B_I)$, *for all* v, B_I, B_U.

As additional examples to EFX$_{\text{WC}}$, consider EF1$_{\text{WC}}$ and EFL$_{\text{WC}}$. Note that by Definition. 5, for any comparison criterion f we have $(f_{\text{WC}})^c = (f^c)_{\text{WC}} \equiv f^c_{\text{WC}}$. We now show our first main theorem – that envy-based fairness holds under a duality transformation from goods to chores.

Theorem 2. *Given a comparison criterion f, an exclusive allocation $\mathbf{A}$ is f_{WC}-fair with respect to v iff its dual $\mathbf{A}^\circ$ is f^c_{WC}-fair with respect to v°.*

Proof. We prove $f_{\text{WC}}(v_i, A_i, A_j) = f^c_{\text{WC}}(v_i^\circ, A_i^\circ, A_j^\circ)$ for all $i, j \in \mathcal{N}$. Let $O_i = A_i \setminus A_j$ and $O_j = A_j \setminus A_i$. Note that $O_i = A_j^\circ \setminus A_i^\circ$ and $O_j = A_i^\circ \setminus A_j^\circ$. Thus,

$$\begin{aligned} f_{\text{WC}}(v_i, A_i, A_j) &= f(v_i, O_i, O_j) = f^c(-v_i, O_j, O_i) \\ &= f^c_{\text{WC}}(v_i^\circ, A_i^\circ, A_j^\circ). \end{aligned}$$

□

It follows from Theorem 2 that an exclusive allocation is EFX_{WC} for goods if and only if its dual allocation is EFX_{WC} for chores. An important application is Corollary 1 in Section 3, since the dual of a setting of chores with no copies is a setting of goods with $n-1$ copies of each good, and in a setting of chores with no copies EFX_{WC} is identical to EFX.

4.2 A Meta-Theorem for Share-based Notions

Does a similar duality result hold for share-based fairness notions? We give an affirmative answer for a class of share-based notions that satisfy a property we term *linear shares*. MMS and PROP are both examples of notions in this class. The idea of a linear share-based notion is as follows. Focus on one agent and consider all values from her point of view. For any allocation, our dual transformation shifts the value of every bundle in her eyes by a constant d (in particular $d = v(\mathcal{T})$). It is natural that her fair share for this allocation also shifts by the same constant d. A linear share-based notion is one for which this natural property holds. The following definition formalizes this intuition, while generalizing the dual transformation to appropriate one-to-one mappings.

Definition 10 (Linear shares)**.** *Two item sets $\mathcal{M}$ and $\mathcal{M}'$ are* **linearly related** *w.r.t. valuations v, v' and a real constant d if there is a one-to-one mapping from exclusive allocations $\mathbf{A} \in \mathcal{EA}(\mathcal{M})$ to exclusive allocations $\mathbf{A}' \in \mathcal{EA}(\mathcal{M}')$ such that $v(A_i) = v'(A_i') + d$ for all $i \in \mathcal{N}$. A share is called* **linear** *if for any item sets $\mathcal{M}$ and $\mathcal{M}'$ that are linearly related w.r.t. v, v' and d we have $s(v, n, \mathcal{M}) = s(v', n, \mathcal{M}') + d$.*

An example of linearly related item sets and valuations is the following: consider $\mathcal{M}$ and $\mathcal{M}'$ where $\mathcal{M}'$ is equal to $\mathcal{M}$ with an additional n copies of a new item t, and v' is equal to v with an additional value for t. An example of a linear share s is one for which $s(v, n, \mathcal{M}) = s(v', n, \mathcal{M}') - v'(t)$.

Theorem 3. *For any linear share s, an exclusive allocation $\mathbf{A}$ is s-share-fair for $\mathcal{M}, v$ iff $\mathbf{A}^\circ$ is s-share-fair for $\mathcal{M}^\circ, v^\circ$.*

Proof. We prove that for every agent $i \in \mathcal{N}$, item sets $\mathcal{M}$ and $\mathcal{M}^\circ$ are linearly related w.r.t. valuations v_i, v_i° and $d = v_i(\mathcal{T})$. Given an allocation $\mathbf{A} \in \mathcal{EA}(\mathcal{M})$, let us consider the dual transformation, which is a one-to-one mapping that satisfies the condition in Definition. 10:

$$\begin{aligned} v_i(A_j) - v_i^\circ(A_j^\circ) &= v_i(A_j) - \left(-v_i\left(A_j^\circ\right)\right) \\ &= v_i(A_j) + v_i(\mathcal{T} \setminus A_j) = v_i(\mathcal{T}). \end{aligned}$$

Suppose that $\mathbf{A}$ is s-share-fair for $\mathcal{M}, v_i$. By definition, we have $v_i(A_i) \geq s(v_i, n, \mathcal{M})$ for all $i \in \mathcal{N}$. Since the share s is linear, $s(v_i, n, \mathcal{M}) = s(v_i^\circ, n, \mathcal{M}^\circ) + v_i(\mathcal{T})$. Therefore, $v_i^\circ(A_i^\circ) \geq s(v_i^\circ, n, \mathcal{M}^\circ)\ \forall i \in \mathcal{N}$, and $\mathbf{A}^\circ$ is s-share-fair for $\mathcal{M}^\circ, v_i^\circ$. The converse is proved similarly. □

Proposition 3. MMS *and* PROP *are linear.*

Proof. Suppose that item sets $\mathcal{M}$ and $\mathcal{M}'$ are linearly related with v, v' and d. We have a one to one mapping from exclusive allocations $\mathbf{A} \in \mathcal{EA}(\mathcal{M})$ to exclusive allocations $\mathbf{A}' \in \mathcal{EA}(\mathcal{M}')$ such that $v(A_i) = v'(A_i') + d$ for all $i \in \mathcal{N}$ and all allocations. For MMS, we have

$$\begin{aligned} \max_{\mathbf{A} \in \mathcal{EA}(\mathcal{M})} \min_{j \in \mathcal{N}} v_i(A_j) &= \max_{\mathbf{A} \in \mathcal{EA}(\mathcal{M}')} \min_{j \in \mathcal{N}} v_i'(A_j) + d \\ &= d + \max_{\mathbf{A} \in \mathcal{EA}(\mathcal{M}')} \min_{j \in \mathcal{N}} v_i'(A_j) \end{aligned}$$

For PROP, it holds that $v_i(\mathcal{M}) = \sum_{j \in \mathcal{N}} v_i(A_j) = \sum_{j \in \mathcal{N}} (v_i'(A_j') + d) = v_i'(\mathcal{M}') + n \cdot d$. So the value of PROP increases by d. □

Goods	H	a	b	c	d
Copies	1	2	1	1	1
v	1000	100	1	2	2

Table 1: Instance with an EFL allocation that is not EF1_{WC}

In contrast to MMS and PROP, α-MMS is not linear, since a shift d in the valuation translates to a shift αd in the share.

5 A Hierarchy of Envy-Based Concepts

We analyze some connections between the envy-based fairness notions introduced above, as summarized in Figure 1.

Proposition 4. *$EFX_{WC} \implies EFL_{WC} \implies EF1_{WC}$.*

Proof. The second implication is straight-forward since the only difference in the two definitions is an extra constraint over the choice of good g in the second EFL_{WC} condition. Other than that, the definitions identify.

For the first implication, fix an EFX_{WC} allocation $\mathbf{A}$ and $i, j \in \mathcal{N}$. We wish to show i does not EFL_{WC}-envy j. If $|A_i \setminus A_j| = 1$, the first EFL_{WC} condition holds. Otherwise, if there is any good $g \in A_i \setminus A_j$ s.t. $v_i(g) \geq v_i(A_j \setminus A_i)$, the EFX_{WC} condition fails when removing a good different than g. Therefore all goods satisfy $v_i(g) \leq v_i(A_j \setminus A_i)$, and again by the EFX_{WC} condition, $v_i(A_i \setminus (A_j \cup \{g\})) \leq v_i(A_j \setminus A_i)$. We can therefore choose an arbitrary good implying that the second EFL_{WC} condition holds. ☐

Example 3. *($EFL_{WC} \not\Rightarrow EFX_{WC}$) Consider two agents and five goods with a single copy each, and identical valuations $a = b = 1, c = d = 1 + \epsilon, e = \epsilon = 0.01$. The allocation $\{\{c, d, e\}, \{a, b\}\}$ is EFL_{WC} but not EFX_{WC}.*

Proposition 5. *$EFX_{WC} \not\Rightarrow EFL, EFL \not\Rightarrow EF1_{WC}$.*

The first negation is due to Example 2. The second negation is due to Example 4:

Example 4. *($EFL \not\Rightarrow EF1_{WC}$) Consider three agents, and goods as given in Table 1, where v is the identical valuation of all agents. Then $\mathbf{A} = \{\{a, b\}, \{a, c, d\}, \{H\}\}$ is EFL, but not $EF1_{WC}$, as agent 1 $EF1_{WC}$-envies agent 2.*

Example 5. *($EF1 \not\Rightarrow EF1_{WC}$) Consider three agents, fivegoods a, b, c, d, e with two copies each, and identical values $v(a) = 1$, $v(b) = v(c) = \frac{1}{2}, v(d) = v(e) = \epsilon = \frac{1}{100}$. Consider the allocation $\mathbf{A} = \{\{a, d, e\}, \{a, b, c\}, \{b, c, d, e\}$. It is EF1, and players' values are $(1.02, 2, 1.02)$. But agent 1 $EF1_{WC}$ envies agent 2 (the special good for the EF1 condition is a). For comparison, $\mathbf{A}' = \{\{a, b, e\}, \{a, c, d\}, \{b, c, d, e\}$ is both EF1 and $EF1_{WC}$.*

5.1 MMS Approximations

We next show that this hierarchical structure implies (strictly) different approximation guarantees to MMS.[2] We show upper bounds on the approximation guarantees of the three "Without Commons" notions that we introduced, and a lower bound for the approximation guarantee of EFL_{WC}, thus separating EFX_{WC} and EFL_{WC} from EF1_{WC} by a factor that grows to infinity with n. It is interesting to note that the bounds we show for goods with copies are strictly lower than known bounds for goods without copies: for EFX_{WC} we give an upper bound of 0.4 while without copies a lower bound of $\frac{4}{7}$ is known [Amanatidis *et al.*, 2018]; for EFL_{WC} we give an upper bound of $\frac{1}{3}$ while without copies a lower bound of $\frac{1}{2}$ is known [Barman *et al.*, 2018].

EF1_{WC}: [Amanatidis *et al.*, 2018] show that EF1_{WC} allocations do not guarantee an approximation strictly larger than $\frac{1}{n}$ to MMS for goods without copies (note that upper bounds on α for goods without copies immediately apply to goods with copies since the former is a special case of the latter). For completeness, we give an explicit example:

[2] Since our duality theorems do not hold for α-MMS, the results in this section cannot be directly transformed to chores.

Goods	H	x	x'	x"	y	y'	y_a	y_b	y_c
Copies	6	7	3	3	3	3	1	1	1
v	2.5	1	1	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

Table 2: EFX_{WC} 0.4-MMS upper bound

Example 6 (EF1_{WC} cannot guarantee strictly more than $\frac{1}{n}$-MMS for goods with copies.)**.** *Consider an example with n agents, all goods have one copy each, and identical valuations. Let $L_1, ..., L_n$ be goods, all with value 1, and let $H_1, ..., H_{n-1}$ be additional goods, all with value n. Then*

$$\mathbf{A} = \{\underbrace{\{H_\theta, L_\theta\}}_{1\leq\theta\leq n-1}, \{L_n\}\}.$$

is an $EF1_{WC}$ allocation with $v(A_n) = v(L_n) = 1$, while an MMS of n is guaranteed for agent n by

$$\mathbf{A}' = \{\underbrace{\{H_\theta\}}_{1\leq\theta\leq n-1}, \{L_1, ..., L_n\}\}.$$

$\textbf{EFX}_{\textbf{WC}}$**:** We give an upper bound of 0.4 and a lower bound of $\frac{4}{11}$.

Example 7 (There is an EFX_{WC} allocation which is at most 0.4-MMS for goods with copies)**.** *Consider 13 agents and 9 goods as given in Table 2, where v is the valuation of agent 13, and all other agents value all goods as 1. The following allocation is EFX_{WC}:*

$$\mathbf{A} = \{\underbrace{\{H, x\}}_{\times 6}, \underbrace{\{x', x''\}}_{\times 3}, \underbrace{\{y, y', y_\theta\}}_{\theta\in(a,b,c)}, \{x\}\}$$

In the following allocation all agents have a value of exactly 2.5 in terms of v:

$$\mathbf{A}' = \{\underbrace{\{H\}}_{\times 6}, \underbrace{\{x, x', y\}}_{\times 3}, \underbrace{\{x, x'', y'\}}_{\times 3}, \{x, y_a, y_b, y_c\}\}$$

We establish the following $\frac{4}{11}$-MMS lower bound by appropriately generalizing a proof of [Amanatidis *et al.*, 2018] to the case of goods with copies. We intentionally keep similar notations to allow easy comparison.

Theorem 4. *An EFX_{WC} allocation is at least $\frac{4}{11}$-MMS for goods with copies.*

Proof. Suppose that allocation $\mathbf{A}$ is an EFX_{WC} allocation. Let us takes the perspective from agent α. We divide the agents into three disjoint sets: $L_1 = \{i \in \mathcal{N} \mid |A_i \setminus A_\alpha| \leq 1\}$, $L_2 = \{i \in \mathcal{N} \mid |A_i \setminus A_\alpha| = 2\}$, $L_3 = \{i \in \mathcal{N} \mid |A_i \setminus A_\alpha| \geq 3\}$. Define the set of goods $\forall\theta \in \{1, 2, 3\}, S_\theta = \cup_{i\in L_\theta}(A_i \setminus A_\alpha)$.

Claim 1. *EFX_{WC} implies:*

1. *For any good $g \in S_2$, $v_\alpha(g) \leq v_\alpha(A_\alpha)$.*
2. *For any agent $i \in L_3$, $v_\alpha(A_i \setminus A_\alpha) \leq \frac{3}{2} \cdot v_\alpha(A_\alpha)$.*

Proof. For the first inequality, let i be an agent in the set L_2. By the condition of EFX_{WC},

$$\forall g \in A_i \setminus A_\alpha, v_\alpha((A_i \setminus A_\alpha) \setminus g) \leq v_\alpha(A_\alpha \setminus A_i) \leq v_\alpha(A_\alpha).$$

As $|A_i \setminus A_\alpha| = 2$, $v_\alpha(g) = v_\alpha((A_i \setminus A_\alpha) \setminus g') \leq v_\alpha(A_\alpha)$, where g' is another good in the set $A_i \setminus A_\alpha$.

For the second inequality, $\min_{g\in A_i\setminus A_\alpha} v_\alpha(g) \leq \frac{1}{2}v_\alpha(A_\alpha)$, since otherwise we have for any good $g' \in A_i \setminus A_\alpha$,

$$\begin{aligned}
v_\alpha((A_i \setminus A_\alpha) \setminus \{g'\}) &= \sum_{g\in(A_i\setminus A_\alpha)\setminus\{g'\}} v_\alpha(g) \\
&\geq \sum_{g\in(A_i\setminus A_\alpha)\setminus\{g'\}} \min_{g''\in A_i\setminus A_\alpha} v_\alpha(g'') \\
&\geq 2 \min_{g''\in A_i\setminus A_\alpha} v_\alpha(g'') > 2 \cdot \frac{1}{2}v_\alpha(A_\alpha) = v_\alpha(A_\alpha),
\end{aligned}$$

contradicting EFX_{WC}. For $g = \arg\min_{g' \in A_i \setminus A_\alpha} v_\alpha(g')$:

$$v_\alpha(A_i \setminus A_\alpha) = v_\alpha((A_i \setminus A_\alpha) \setminus \{g\}) + v_\alpha(g) \leq$$
$$v_\alpha(A_\alpha) + \min_{g \in A_i \setminus A_\alpha} v_\alpha(g) \leq \frac{3}{2} \cdot v_\alpha(A_\alpha)$$

□

Suppose that allocation $\mathbf{A}^*$ is a maximin share allocation for agent α. Let the allocation $\mathbf{A}' = \{A_i \in \mathbf{A}^* \mid |A_i \cap S_1| = 0 \text{ and } |A_i \cap S_2| \leq 1\}$. We remove any bundle contains at least one good in S_1 or at least two goods in S_2 from the allocation $\mathbf{A}^*$. Notice that allocation $\mathbf{A}'$ cannot be empty. As $\alpha \in L_1$, the number of bundles is at least $1 + |S_1| + \frac{|S_2|}{2}$. And we remove at most $|S_1| + \frac{|S_2|}{2}$ bundles. Let $n' = |\mathbf{A}'|$.

Next we prove that there is a bundle A'_j in the allocation $\mathbf{A}'$ such that $v_\alpha\left(A'_j\right) \leq \frac{11}{4} \cdot v_\alpha(A_\alpha)$. Let n' be the size of $|\mathbf{A}'|$, y be the size of $|L_3|$ and x be the number of goods in set S_2 appearing in the allocation $\mathbf{A}'$.

Claim 2. *We have the following quantity relationships: (1) $n' \geq x$, (2) $n' \geq \frac{x}{2} + y$.*

Proof. The first inequality holds as each bundle in $\mathbf{A}'$ has at most one good from S_2. For the second inequality, we have

$$n = |L_1| + |L_2| + |L_3| \geq |S_1| + \frac{|S_2|}{2} + y.$$

The number of removed bundles is at most $|S_1| + \frac{|S_2|-x}{2}$. Therefore, $n' \geq n - |S_1| - \frac{|S_2|-x}{2} \geq \frac{x}{2} + y$. □

By Claim 1, the total sum of goods from sets S_2 and S_3 in allocation $\mathbf{A}'$ for agent α is upper bounded by

$$x \cdot \max_{g \in S_2} v_\alpha(g) + y \cdot \max_{i \in L_3} v_\alpha(A_i \setminus A_\alpha)$$
$$\leq x \cdot v_\alpha(A_\alpha) + y \cdot \frac{3}{2} v_\alpha(A_\alpha)$$

By Claim 2, the average valuation is bounded by

$$\frac{x + \frac{3}{2}y}{n'} \cdot v_\alpha(A_\alpha) \leq \frac{x + \frac{3}{2} \cdot (n' - \frac{x}{2})}{n'} \cdot v_\alpha(A_\alpha)$$
$$= (\frac{1}{4} \cdot \frac{x}{n'} + \frac{3}{2}) \cdot v_\alpha(A_\alpha) \leq \frac{7}{4} \cdot v_\alpha(A_\alpha)$$

Therefore, there is an agent j such that $v_\alpha\left(A'_j \setminus A_\alpha\right) \leq \frac{7}{4} \cdot v_\alpha(A_\alpha)$. We have

$$v_\alpha\left(A'_j\right) = v_\alpha\left(A'_j \setminus A_\alpha\right) + v_\alpha\left(A'_j \cap A_\alpha\right)$$
$$\leq \frac{7}{4} \cdot v_\alpha(A_\alpha) + v_\alpha(A_\alpha) = \frac{11}{4} \cdot v_\alpha(A_\alpha).$$

MMS is $\min_{i \in \mathcal{N}} v_\alpha(A_i^*) \leq v_\alpha\left(A'_j\right) \leq \frac{11}{4} \cdot v_\alpha(A_\alpha)$. □

EFL_{WC}: We give an upper and a lower bound of one-third.

Example 8 (There is an EFL_{WC} allocation with at most $\frac{1}{3}$-MMS for goods with copies)**.** *Consider $2\ell + 1$ agents and goods as given in Table 3, where v is the valuation of agent $2\ell + 1$ and all other agents value all goods as* 1*. Then,*

$$\mathbf{A} = \{\underbrace{\{H, x\}}_{\times \ell}, \underbrace{\{x', y_\theta, z_\theta\}}_{1 \leq \theta \leq \ell}, \{x\}\}$$

is EFL_{WC}, but $v(A_{2\ell+1}) = 1$, while a MMS of at least $3 - \frac{2}{\ell}$ is guaranteed by

$$\mathbf{A}' = \{\underbrace{\{H\}}_{\times \ell}, \underbrace{\{x, x', y_\theta\}}_{1 \leq \theta \leq \ell}, \{x, z_1, ..., z_\ell\}\}$$

Goods	H	x	x'	$\forall_{1 \le i \le \ell}, \mathbf{y_i}$	$\forall_{1 \le i \le \ell}, \mathbf{z_i}$
Copies	ℓ	$\ell + 1$	ℓ	1	1
v	3	1	1	$1 - \frac{2}{\ell}$	$\frac{2}{\ell}$

Table 3: EFL_{WC} $\frac{1}{3}$-MMS upper bound

Theorem 5. *An EFL_{WC} allocation is at least $\frac{1}{3}$-MMS for goods with copies.*

Proof. Suppose that $\mathbf{A}$ is an EFL_{WC} allocation, and consider the perspective of an agent i^*. Among the remaining $n-1$ agents, let L_1 be the set of agents satisfying the first EFL_{WC} condition, and let L_2 be the remaining agents (that must thus satisfy the second condition). Denote $\ell = |L_1|, n - \ell - 1 = |L_2|$. Define $\forall \theta \in \{1, 2\}, S_\theta = \cup_{j \in L_\theta}(A_j \setminus A_{i^*})$ (we allow multiple copies in the same set). Notice that for any $j \in S_2$ we have by the second EFL_{WC} condition that there is such good g with

$$\max\{v_{i^*}(A_j \setminus (A_{i^*} \cup \{g\})), v_{i^*}(g)\} \le v_i(A_{i^*} \setminus A_j),$$

and so

$$\begin{aligned} v_{i^*}(A_j \setminus A_{i^*}) &= v_{i^*}(A_j \setminus (A_{i^*} \cup \{g\})) + v_{i^*}(g) \\ &\le 2v_{i^*}(A_{i^*} \setminus A_j) \\ &\le 2v_{i^*}(A_{i^*}), \end{aligned}$$

which implies

$$v_{i^*}(S_2) = \sum_{j \in L_2} v_{i^*}(A_j \setminus A_{i^*}) \le 2(n - \ell - 1)v_{i^*}(A_{i^*}).$$

All goods that are not in S_1, S_2 must have a copy in A_{i^*}. Denote all the remaining goods' copies R, then we overall have $\mathcal{M} = S_1 \cup S_2 \cup R$.

In any allocation $\mathbf{A}'$, there are at most ℓ agents with goods from S_1. That is since the first EFL_{WC} condition for an agent j requires $|A_j \setminus A_{i^*}| = 1$, and so we have $|S_1| = \sum_{j \in L_1} |A_j \setminus A_{i^*}| = \ell$. Let L' then be the set of at least $n - \ell$ agents with no good from S_1, and let $\{S_2^j\}_{j \in L'}, \{R^j\}_{j \in L'}$ be the allocations of S_2, R goods to these agents under $\mathbf{A}'$. By additivity, there must be some agent j with

$$\begin{aligned} v_{i^*}(S_2^j) &\le \frac{2(n - \ell - 1)}{|L'|} v_{i^*}(A_{i^*}) \\ &\le \frac{2(n - \ell - 1)}{n - \ell} v_{i^*}(A_{i^*}) < 2v_{i^*}(A_{i^*}). \end{aligned}$$

Since $\mathbf{A}'$ is an exclusive allocation, and since R includes only goods with a copy in A_{i^*}, agent j satisfies $v_{i^*}(R^j) \le v_{i^*}(A_{i^*})$, and overall $v_{i^*}(A_j) = v_{i^*}(S_1^j) + v_{i^*}(S_2^j) + v_{i^*}(R^j) \le 3v_{i^*}(A_{i^*})$. Since such an agent exists for any exclusive allocation, it exists for the MMS allocation, and so the minimal bundle in terms of v_{i^*} in that allocation is bounded by $3v_{i^*}(A_{i^*})$. This shows the $\frac{1}{3}$-MMS guarantee. □

6 Existence of EFX_{WC} for Leveled Preferences

Leveled preferences are defined as follows:

Definition 11. *A valuation v is a* leveled preference *for goods if for any two bundles, $|B_1| > |B_2| \implies v(B_1) > v(B_2)$.*

We prove our existence result for goods. By our duality framework (Theorem 2 in Section 4), these existence results hold for chores and in fact for mixed goods and chores.

Theorem 6. *There is an algorithm that always finds an EFX_{WC} exclusive allocation for goods with copies in the case of leveled preferences. Its runtime is $O(n|\mathcal{T}|^2)$.*

Proof. We can choose an initial allocation $\mathbf{A}$ such that $|A_i| - |A_j| \le 1$ for any $i, j \in \mathcal{N}$, e.g., by setting some arbitrary order $1, ..., n$ over the agents, an arbitrary order $1, ..., t$ over the good types,

Goods	H	a	b	c	d	e	f
Copies	2	1	1	2	2	2	1
v_1	2.5	1	1	1	1	0.1	0.1
v_2	2	1.5	1	0.7	0.7	0.7	0.7
v_3	2	1	0.5	0.5	0.1	0.1	0.1

Table 4: Envy-cycle cancelation failure for goods with copies

and allocating the k_i copies of the next good to the next k_i agents in a cyclic fashion. If there is exactly one level, then $\mathbf{A}$ is EFX$_{\text{WC}}$ (and moreover, EFX), and we are done.

Let the number of goods in the upper level be H, and thus the number of goods in the lower level is $H-1$. If the allocation $\mathbf{A}$ is not an EFX$_{\text{WC}}$ allocation, we perform the following operation. Suppose that agent i EFX$_{\text{WC}}$-envies agent j (that is, EFX-envies it after removing the goods they have in common). Agent i must be at a lower level than agent j, otherwise after removing a good from j, agent i is at a higher level and by the leveled preferences prefers its own bundle.

It must hold that $\max_{g \in A_j \setminus A_i} v_i(g) > \min_{g \in A_i \setminus A_j} v_i(g)$, otherwise we have for any good $g' \in A_j \setminus A_i$:

$$v_i((A_j \setminus A_i) \setminus \{g'\}) \leq |(A_j \setminus A_i) \setminus \{g'\}| \max_{g \in A_j \setminus A_i} v_i(g) \leq$$
$$|A_i \setminus A_j| \min_{g \in A_i \setminus A_j} v_i(g) \leq v_i(A_i \setminus A_j),$$

in contradiction to our assumption of EFX$_{\text{WC}}$-envy.

Let $g_{\max} = arg \max_{g \in A_j} v_i(g), g_{\in} = arg \min_{g \in A_i} v_i(g)$. Let agent i get the bundle $(A_i \setminus \{g_{\min}\}) \cup \{g_{\max}\}$, and let agent j get the bundle $(A_j \setminus \{g_{\max}\}) \cup \{g_{\min}\}$. After this operation, the allocation remains exclusive. Agent i gets a strictly improved bundle by its valuation, and both agents get a bundle with the same cardinality as before, thus maintaining the sets of lower-level and upper-level bundle agents unchanged.

We construct the potential function to show the number of steps is bounded and polynomial. For any good $g \in T$, let $\omega_i(g)$ be its ordinal position according to agent i's preference over the goods, e.g., for the minimal good $g \in T$ by i valuation we have $\omega_i(g) = 1$, and for the maximal good g' we have $\omega_i(g') = |\mathcal{T}|$. We consider the potential function

$$\psi(\mathbf{A}) = \sum_{\substack{i \in \mathcal{N} \\ |A_i| = H-1}} \sum_{g \in A_i} \omega_i(g).$$

Notice that at each step this potential function strictly increases as we replace some good with a strictly preferred good for some lower level agent. Also note that $0 \leq \min_{\mathbf{A} \in \mathcal{EA}(\mathcal{M})} \psi(\mathbf{A}) \leq \max_{\mathbf{A} \in \mathcal{EA}(\mathcal{M})} \psi(\mathbf{A}) \leq n|\mathcal{T}|^2$, and the function always returns an integer value. Thus the maximal number of substitution steps is in $O(n|\mathcal{T}|^2)$. □

Remark 3. *Many existence results of envy-based fairness notions for goods (without copies) rely on the primitive of envy-cycle canceling, first shown by [Lipton* et al.*, 2004]. We thus note an important technical difference in proving existence for f_{wc} notions. For goods with copies, it is sometimes impossible to cancel an envy cycle without breaking the fairness notion. This was first pointed out in [Bhaskar* et al.*, 2020]. We give below another such example that has two additional properties: First, the allocation is EFX$_{WC}$ before cancelling the envy-cycle, but not even EF1$_{WC}$ after the cancellation. Second, in our example the choice of which envy cycle to cancel is immaterial to the difficulty arising, as there is only one envy cycle.*

Example 9. *There are 3 agents and 7 good types. The valuation and the number of copies are listed in Table 4. Note that these valuations have identical ordinal preferences. Consider the allocation $A_1 = \{b, c, d, e, f\}$, $A_2 = \{H, a\}$, $A_3 = \{H, c, d, e\}$. It is an EFX$_{WC}$ allocation. We have agent 1 envies agent 2 and vice versa. Let us reallocate the bundles to resolve this envy cycle. Even though everyone gets a better bundle, it is not EF1$_{WC}$, as agent 1 then gets the bundle $\{H, a\}$ but EF1$_{WC}$-envies the bundle $\{H, c, d, e\}$.*

Goods	a	b	c	d
Copies	2	1	1	1
v_1	1	1	1	ϵ
v_2	1	ϵ	ϵ	ϵ
v_3	ϵ	ϵ	ϵ	1

Table 5: MNW allocation that is not EF1_{WC}

Remark 4. *Another intriguing property of goods with copies is that the Maximal Nash Welfare allocation is not necessarily EF1$_{WC}$, unlike the case with goods, as the following example shows.*

Definition 12. Nash Welfare *of an allocation is the product* $NW(\mathbf{A}) = \prod_{i \in \mathcal{N}} v_i(A_i)$
The Max Nash Welfare *allocation is* $MNW = \arg\max_{\mathbf{A}} NW(\mathbf{A})$.

Example 10. *Consider Table 5 with* $\epsilon = 10^{-6}$*. The MNW allocation is the exclusive allocation* $\{\{a, b, c\}, \{a\}, \{d\}\}$*. In this allocation agent 2 EF1$_{WC}$-envies agent 1.*

7 Discussion

To the best of our knowledge, we provide the first duality relationship between goods and chores, which establishes the equivalence of both settings for a broad class of fairness notions. It is interesting to further investigate this duality phenomenon for other fair division settings. For the new fairness notions such as $\text{EFX}_{\text{WC}}, \text{EFL}_{\text{WC}}$, the main open challenge is to settle their general existence for goods with copies. As a special case, the existence of EFX, EFL for chores is open. It is also worth investigating a more flexible model where the number of copies of each good is only loosely set, e.g., constrained between a minimal and maximal value.

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