

Weighted Envy-Freeness in Indivisible Item Allocation¹

Mithun Chakraborty, Ayumi Igarashi, Warut Suksompong, and Yair Zick

Abstract

We introduce and analyze new envy-based fairness concepts for agents with *weights* that quantify their entitlements in the allocation of indivisible items. We propose two variants of weighted envy-freeness up to one item (WEF1): *strong*, where envy can be eliminated by removing an item from the envied agent’s bundle, and *weak*, where envy can be eliminated either by removing an item (as in the strong version) or by replicating an item from the envied agent’s bundle in the envying agent’s bundle. We show that for additive valuations, an allocation that is both Pareto optimal and strongly WEF1 always exists and can be computed in pseudo-polynomial time; moreover, an allocation that maximizes the weighted Nash social welfare may not be strongly WEF1, but always satisfies the weak version of the property. Moreover, we establish that a generalization of the round-robin picking sequence algorithm produces in polynomial time a strongly WEF1 allocation for an arbitrary number of agents; for two agents, we can efficiently achieve both strong WEF1 and Pareto optimality by adapting the adjusted winner procedure. Our work highlights several aspects in which weighted fair division is richer and more challenging than its unweighted counterpart.

1 Introduction

The fair allocation of resources to interested parties is a central issue in economics and has increasingly received attention in computer science in the past few decades [Brams and Taylor, 1996, Moulin, 2003, Thomson, 2016, Markakis, 2017, Moulin, 2019]. The problem has a wide range of applications, from reaching divorce settlements [Brams and Taylor, 1996] and dividing land [Segal-Halevi et al., 2017] to sharing apartment rent [Gal et al., 2017]. *Envy-freeness* is one of the most commonly studied fairness criterion in the literature; it stipulates that all agents find their assigned bundle to be the best among all bundles in the allocation [Foley, 1967, Varian, 1974].

Envy-freeness is a compelling notion when all agents have equal entitlements—indeed, in a standard envy-free allocation, no agent would rather take the place of another agent with respect to the assigned bundles. However, in many division problems, agents may have varying claims on the resource. For instance, consider a facility that has been jointly funded by three investors—Alice, Bob, and Charlie—where Alice contributed $3/5$ of the construction expenses while Bob and Charlie contributed $1/5$ each. One could then expect Alice to envy either Bob or Charlie if she does not value her share at least three times as much as each of the latter two investors’ share when they divide the usage of the facility. Besides this interpretation as the *cost* of participating in the resource allocation exercise, the weights may also represent other publicly known and accepted measures of entitlement such as *eligibility* or *merit*. A prevalent example is inheritance division, wherein closer relatives are typically more entitled to the bequest than distant ones. Likewise, different countries

¹A preliminary version of the paper appeared in Proceedings of the 19th International Conference on Autonomous Agents and Multiagent Systems (AAMAS). All omitted proofs can be found in the full version: <http://arxiv.org/pdf/1909.10502.pdf>.

have differing entitlements when it comes to apportioning humanitarian aid. Envy-freeness can be naturally extended to the general setting in which agents have *weights* designating their entitlements. When the resource to be allocated is infinitely divisible (e.g., time to use a facility, or land in a real estate), it is known that a weighted envy-free allocation exists for any set of agents’ weights and valuations [Robertson and Webb, 1998, Zeng, 2000].

In this paper, we initiate the study of weighted envy-freeness for the ubiquitous setting where the resource consists of *indivisible* items. Indeed, inheritance division usually involves discrete items such as real estate, cars, and jewelry; similarly, facility usage is often allocated in fixed time slots (e.g., hourly). Since envy-freeness cannot always be fulfilled even in the canonical setting without weights, for example when all agents agree that one particular item is more valuable than the remaining items combined, recent works have focused on identifying relaxations of envy-freeness that can be satisfied in the case of equal entitlements. The most salient of these approximations is perhaps *envy-freeness up to one item (EF1)*: for any two agents i and j , if agent i envies agent j , then we can eliminate this envy by removing a single item from j ’s bundle [Budish, 2011]. Lipton et al. [2004] showed that an EF1 allocation exists and can be computed efficiently for any number of agents with monotone valuations.² Our goal in this work is to extend EF1 to the general case with arbitrary entitlements, and explore the relationship of these extensions to other important justice criteria such as proportionality and Pareto optimality. The richness of the weighted setting will be evident throughout our work; in particular, we demonstrate that while some protocols from the unweighted setting can be generalized to yield strong guarantees, others are less robust and cease to offer desirable properties upon the introduction of weights.

1.1 Our Contributions

We assume that agents have positive (not necessarily rational) weights representing their entitlements and, with the exception of Propositions 3.2, 6.1, and 6.2, that they are endowed with additive valuation functions. We begin in Section 2 by proposing two generalizations of EF1 to the weighted setting: (strong) weighted envy-freeness up to one item (**WEF1**) and weak weighted envy-freeness up to one item (**WWEF1**). While **WEF1** may appear as the more natural extension, we argue that it can impose a highly demanding constraint when the weights vastly differ, so that **WWEF1** is a useful alternative. In Section 3, we focus on two classical EF1 protocols. On the one hand, we show that the envy cycle elimination algorithm of Lipton et al. [2004] does not extend to the weighted setting except in the special case of identical valuations. On the other hand, we construct a weight-based picking sequence which allows us to compute a **WEF1** allocation efficiently—this generalizes a folklore result from the unweighted setting. The analysis of this algorithm is significantly more involved than for the unweighted version and requires making intricate algebraic manipulations. Nevertheless, the algorithm itself is simple both to explain and to implement, so we believe that it is suitable for maintaining fairness in practice.

In Sections 4 and 5, we examine the interplay between fairness and Pareto optimality. For two agents, we exhibit that a weighted variant of the adjusted winner procedure allows us to compute an allocation that is both **WEF1** and Pareto optimal in polynomial time—our algorithm provides a natural discretization of the classical procedure, which was designed for the divisible item setting. We then show by adapting an algorithm of Barman et al. [2018] that a Pareto optimal and **WEF1** allocation is guaranteed to exist and can be found in pseudo-polynomial time for any number of agents. Furthermore, we prove that while an allocation with maximum weighted Nash welfare may fail to satisfy **WEF1**, such an allocation

²EF1 is also remarkable for its robustness: it can be satisfied under cardinality constraints [Biswas and Barman, 2018] and connectivity constraints [Bilò et al., 2019], and is computable using few queries [Oh et al., 2019].

is both Pareto optimal and **WWEF1**, thereby generalizing an important result of Caragiannis et al. [2019]. Our proof for the **WWEF1** result follows a similar outline as that of Caragiannis et al., but we need to make a case distinction based on the comparison between weights. Finally, we conclude in Section 6 by discussing some obstacles that we faced when trying to extend our ideas and results beyond additive valuation functions; specifically, we show that a **WWEF1** allocation may not exist when agents have non-additive (submodular) valuations.

1.2 Related Work

There is a long line of work on the fair division of indivisible items; see, e.g., the surveys by Bouveret et al. [2016] and Markakis [2017] for an overview. Prior work on the fair allocation of indivisible items to asymmetric agents has tackled fairness concepts that are not based on envy. Farhadi et al. [2019] introduced *weighted maximin share (WMMS)* fairness, a generalization of an earlier fairness notion of Budish [2011]. Aziz et al. [2019] explored WMMS fairness in the allocation of indivisible *chores*—items that, in contrast to goods, are valued negatively by the agents—where agents’ weights can be interpreted as their shares of the workload. Babaioff et al. [2019] studied competitive equilibrium for agents with different budgets. Recently, Aziz et al. [2020] proposed a polynomial-time algorithm for computing an allocation of a pool of goods and chores that satisfies both Pareto optimality and *weighted proportionality up to one item (WPROP1)* for agents with asymmetric weights. Unequal entitlements have also been considered in the context of divisible items with respect to *proportionality* [Barbanel, 1995, Brams and Taylor, 1996, Cseh and Fleiner, 2020, Robertson and Webb, 1998, Segal-Halevi, 2019]. We remark here that (weighted) proportionality is a strictly weaker notion than (weighted) envy-freeness under additive valuations. However, while PROP1 is also implied by EF1 in the unweighted setting, the relationship between the corresponding weighted notions is much less straightforward, as we demonstrate in the full version of our paper.

In addition to expressing the entitlement of individual agents, weights can also be applied to settings where each agent represents a group of individuals [Benabbou et al., 2019, 2020a]—here, the *size* of a group can be used as its weight.³ Specifically, in the model of Benabbou et al. [2020a], groups correspond to ethnic groups (namely, the major ethnic groups in Singapore, i.e., Chinese, Malay, and Indian). Maintaining provable fairness guarantees amongst the ethnic groups is highly desirable; in fact, it is one of the principal tenets of Singaporean society.

2 Preliminaries

Throughout the paper, given a positive integer r , we denote by $[r]$ the set $\{1, 2, \dots, r\}$. We are given a set $N = [n]$ of *agents*, and a set $O = \{o_1, \dots, o_m\}$ of *items* or *goods*. Subsets of O are referred to as *bundles*, and each agent $i \in N$ has a *valuation function* $v_i : 2^O \rightarrow \mathbb{R}_{\geq 0}$ over bundles; the valuation function for every $i \in N$ is *normalized* (i.e., $v_i(\emptyset) = 0$) and *monotone* (i.e., $v_i(S) \leq v_i(T)$ whenever $S \subseteq T$). We denote $v_i(\{o\})$ simply by $v_i(o)$ for any $i \in N$ and $o \in O$.

An *allocation* A of the items to the agents is a collection of n disjoint bundles A_1, \dots, A_n such that $\bigcup_{i \in N} A_i \subseteq O$; the bundle A_i is allocated to agent i and $v_i(A_i)$ is agent i ’s *realized valuation* under A . Given an allocation A , we denote by A_0 the set $O \setminus (\bigcup_{i \in N} A_i)$, and its

³Note that in this model, each group has a valuation function that represents the overall preference of its members. Other group fairness notions do not assume the existence of such aggregate functions and instead take directly into account the preferences of the individual agents in each group [Conitzer et al., 2019, Segal-Halevi and Suksompong, 2019, Kyropoulou et al., 2020].

elements are referred to as *withheld items*. An allocation A is said to be *complete* if $A_0 = \emptyset$, and *incomplete* otherwise.

In our setting with different entitlements, each agent $i \in N$ has a fixed weight $w_i \in \mathbb{R}_{>0}$; these weights regulate how agents value their own allocated bundles relative to those of other agents, and hence bear on the overall (subjective) fairness of an allocation. More precisely, we define the *weighted envy* of agent i towards agent j under an allocation A as $\max\left\{0, \frac{v_i(A_j)}{w_j} - \frac{v_i(A_i)}{w_i}\right\}$. An allocation is *weighted envy-free* (WEF) if no agent has positive weighted envy towards another agent. Intuitively, agent i being weighted envy-free towards agent j means that i 's valuation for her share A_i , given that i 's entitlement is w_i , is at least as high as i 's valuation for A_j if i 's entitlement were w_j . Weighted envy-freeness reduces to traditional envy-freeness when $w_i = w, \forall i \in N$ for some positive real constant w . Since a complete envy-free allocation does not always exist, it follows trivially that a complete WEF allocation may not exist in general. We briefly remark here that with indivisible items, it is possible to define variations of weighted envy-freeness—for example, if $w_i = 1$ and $w_j = 2$, one could require that agent j 's bundle can be divided into two parts neither of which agent i finds more valuable than her own bundle. Nevertheless, the definition that we use is mathematically natural and can be directly applied to arbitrary (not necessarily rational) weights.

We now state the main definitions of our paper, which naturally extend *envy-freeness up to one item* (EF1) [Lipton et al., 2004, Budish, 2011] to the weighted setting.

Definition 2.1. An allocation A is said to be (*strongly*) *weighted envy-free up to one item* (WEF1) if for any pair of agents i, j with $A_j \neq \emptyset$, there exists an item $o \in A_j$ such that

$$\frac{v_i(A_i)}{w_i} \geq \frac{v_i(A_j \setminus \{o\})}{w_j}.$$

More generally, A is said to be *weighted envy-free up to c items* (WEF c) for an integer $c \geq 1$ if for any pair of agents i, j , there exists a subset $S_c \subseteq A_j$ of size at most c such that

$$\frac{v_i(A_i)}{w_i} \geq \frac{v_i(A_j \setminus S_c)}{w_j}.$$

Definition 2.2. An allocation A is said to be *weakly weighted envy-free up to one item* (WWEF1) if for any pair of agents i, j with $A_j \neq \emptyset$, there exists an item $o \in A_j$ such that

$$\text{either } \frac{v_i(A_i)}{w_i} \geq \frac{v_i(A_j \setminus \{o\})}{w_j} \text{ or } \frac{v_i(A_i \cup \{o\})}{w_i} \geq \frac{v_i(A_j)}{w_j}.$$

More generally, A is said to be *weakly weighted envy-free up to c items* (WWEF c) for an integer $c \geq 1$ if for any pair of agents i, j , there exists a subset $S_c \subseteq A_j$ of size at most c such that

$$\text{either } \frac{v_i(A_i)}{w_i} \geq \frac{v_i(A_j \setminus S_c)}{w_j} \text{ or } \frac{v_i(A_i \cup S_c)}{w_i} \geq \frac{v_i(A_j)}{w_j}.$$

In other words, an allocation is WEF1 if any (weighted) envy from an agent i towards another agent j can be eliminated by removing a single item from j 's bundle. Similarly, WWEF1 requires that any such envy can be eliminated by either removing an item from j 's bundle or adding a copy of an item from j 's bundle to i 's bundle.

A valuation function $v : 2^O \rightarrow \mathbb{R}_{\geq 0}$ is said to be *additive* if $v(S) = \sum_{o \in S} v(o)$ for every $S \subseteq O$. We will assume additive valuations for most of the paper; this is a very common assumption in the fair division literature and offers a tradeoff between expressiveness and simplicity [Bouveret and Lemaître, 2016, Caragiannis et al., 2019, Kurokawa et al., 2018].

Under this assumption, both WEF1 and WWEF1 reduce to EF1 in the unweighted setting. Moreover, one can check that under additive valuations, an allocation satisfies WWEF1 if and only if for any pair of agents i, j with $A_j \neq \emptyset$, there exists an item $o \in A_j$ such that

$$\frac{v_i(A_i)}{w_i} \geq \frac{v_i(A_j)}{w_j} - \frac{v_i(o)}{\min\{w_i, w_j\}}.$$

The criterion WEF1 can be criticized as being too demanding in certain circumstances, when the weight of the envied agent is much larger than that of the envying agent. To illustrate this, consider a problem instance where agent 1 has an additive valuation function and is indifferent among all items taken individually, e.g., $v_1(o) = 1$ for every $o \in O$. Now, if $w_1 = 1$ and $w_2 = 100$, then eliminating one item from agent 2’s bundle reduces agent 1’s weighted valuation of this bundle by merely 0.01. As such, we may trigger a substantial adverse effect on the overall welfare of the allocation by aiming to eliminate agent 1’s weighted envy towards agent 2. This line of thinking was our motivation for introducing the weak weighted envy-freeness concept. We also note that WWEF1 can be viewed as a stronger version of what one could refer to as “transfer weighted envy-freeness up to one item”: agent i is *transfer weighted envy-free up to one item* towards agent j under the allocation A if there is an item $o \in A_j$ that would eliminate the weighted envy of i towards j upon being transferred from A_j to A_i , i.e., $v_i(A_i \cup \{o\}) \geq \frac{w_i}{w_j} \cdot v_i(A_j \setminus \{o\})$.

In addition to fairness, we often want our allocation to satisfy an efficiency criterion. One important such criterion is Pareto optimality. An allocation A' is said to *Pareto dominate* an allocation A if $v_i(A'_i) \geq v_i(A_i)$ for all agents $i \in N$ and $v_j(A'_j) > v_j(A_j)$ for some agent $j \in N$. An allocation is *Pareto optimal* (or PO for short) if it is not Pareto dominated by any other allocation.

Allocations maximizing the *Nash welfare*—defined as $\text{NW}(A) := \prod_{i \in N} v_i(A_i)$ —are known to offer strong guarantees with respect to both fairness and efficiency in the unweighted setting [Caragiannis et al., 2019]. For our weighted setting, we define a natural extension called *weighted Nash welfare*— $\text{WNW}(A) := \prod_{i \in N} v_i(A_i)^{w_i}$. Since it is possible that the maximum attainable $\text{WNW}(A)$ is 0, we define a *maximum weighted Nash welfare* or *MWNW* allocation along the lines of Caragiannis et al. [2019] as follows: given a problem instance, we find a maximum subset of agents, say $N_{\max} \subseteq N$, to which we can allocate bundles of positive value, and compute an allocation to the agents in N_{\max} that maximizes⁴ $\prod_{i \in N_{\max}} v_i(A_i)^{w_i}$. To see why the notion of MWNW makes intuitive sense, consider a setting where agents have a value of 1 for each item; furthermore, assume that the number of items is exactly $\sum_{i=1}^n w_i$. In this case, one can verify (using standard calculus) that an allocation maximizing MWNW assigns to agent i exactly w_i items. Indeed, following the interpretation of w_i as the number of members of group i (see Section 1.2), the expression $v_i(A_i)^{w_i}$ can be thought of as each member of group i deriving the same value from the set A_i ; the group’s overall Nash welfare is thus $v_i(A_i)^{w_i}$.

3 WEF1 allocations

We commence our exploration of weighted envy-freeness by considering extensions of two standard methods for producing EF1 allocations in the unweighted setting: the *envy cycle elimination algorithm* and the *round-robin algorithm*. As we will see, these two procedures experience contrasting fortunes in the presence of weights: while the idea of eliminating envy cycle fundamentally fails, the round-robin algorithm admits an elegant generalization that can take into account arbitrary entitlements of the agents.

⁴There can be multiple maximum subsets N_{\max} having the same cardinality but different maximum weighted Nash welfare. Our main positive result for MWNW (Theorem 5.1) holds for all such subsets N_{\max} .

3.1 Envy Cycle Elimination Algorithm

Before we discuss the envy cycle elimination algorithm of Lipton et al. [2004], let us briefly recap how it works in the unweighted setting. The algorithm allocates one item at a time in an arbitrary order. It also maintains an “envy graph”, which captures the envy relation between the agents with respect to the (incomplete) allocation at each stage. The next item is allocated to an unenvied agent, and any envy cycle that forms as a result is eliminated by letting each agent take the bundle of the agent that she envies. This cycle elimination step allows the algorithm to ensure that there is an unenvied agent to whom it can allocate the next item.

As far as envy in the traditional sense is concerned, what an agent actually “envies” is an allocated bundle regardless of who owns that bundle. However, both the subjective valuations of allocated bundles and the relative weights interact in non-trivial ways to determine weighted envy. It is easy to see that weighted envy of i towards j does not imply traditional envy of i towards j , and vice versa. A crucial implication is that even if agent i ’s bundle is replaced with the bundle of an agent j towards whom i has weighted envy, i ’s realized valuation, and hence the ratio of her realized valuation to her weight, may decrease as a result. Indeed, consider a problem instance with $n = 2$ and $O = \{o_1, o_2, o_3\}$, weights $w_1 = 3$ and $w_2 = 1$, and identical, additive valuation functions such that $v_i(o) = 1$ for all $i \in N$ and $o \in O$. Under the complete allocation with $A_1 = \{o_1, o_2\}$, agent 1 has weighted envy towards agent 2 since $v_1(A_2)/w_2 = 1/1 = 1 > 2/3 = v_1(A_1)/w_1$, but agent 1 would not prefer to replace A_1 with A_2 since that reduces her realized valuation from 2 to 1. On the other hand, agent 2 could benefit from replacing A_2 with A_1 even though she does not have weighted envy towards agent 1. As such, the natural extension of the envy cycle elimination algorithm, where an edge exists from agent i to agent j if and only if i has weighted envy towards j , does not guarantee a complete WEF1 or even WWEF1 allocation.

Proposition 3.1. *The weighted version of the envy cycle elimination algorithm may not produce a complete WWEF1 allocation, even in a problem instance with two agents and additive valuations.*

Proof. Consider a problem instance with $n = 2$ and $m = 12$, weights $w_1 = 1$ and $w_2 = 2$, and valuation functions defined by

$$v_1(o_r) = \begin{cases} 1 & \text{for } r = 1; \\ 0.1 & \text{for } r = 12; \\ 0.21 & \text{otherwise;} \end{cases} \quad \text{and} \quad v_2(o_r) = \begin{cases} 1.1 & \text{for } r = 1; \\ 0.1 & \text{for } r = 12; \\ 0.2 & \text{otherwise.} \end{cases}$$

Suppose that the weighted envy cycle elimination algorithm iterates over o_1, o_2, \dots, o_{12} , and starts by allocating o_1 to agent 1 due to, say, lexicographic tie-breaking. At this point, agent 2 has weighted envy towards agent 1 and not vice versa; moreover, this condition persists until items o_2, \dots, o_{10} have all been allocated to agent 2. At this point, item o_{11} also goes to agent 2, resulting in valuations $v_1(A_1) = v_1(o_1) = 1$ and $v_2(A_2) = v_2(\{o_2, \dots, o_{11}\}) = 2$. Agent 2 still has weighted envy towards agent 1 since $v_2(A_2)/w_2 = 1 < 1.1/1 = v_2(A_1)/w_1$; on the other hand, agent 1 also develops weighted envy towards agent 2 since $v_1(A_2)/w_2 = 1.05 > 1 = v_1(A_1)/w_1$. Thus, there is a cycle in the induced weighted envy graph. For an unweighted envy graph, we would “de-cycle” the graph at this point by swapping bundles over the cycle and that would still maintain the invariant that the allocation is EF1. However, if we swap the bundles in this example so that the new allocated bundles are $A'_1 = A_2 = \{o_2, \dots, o_{11}\}$ and $A'_2 = A_1 = \{o_1\}$, agent 2 will end up having (weak) weighted envy up to more than one item towards agent 1 since

$v_2(A'_2)/w_2 = 1.1/2 = 0.55$ and $v_2(A'_1 \setminus \{o\})/w_1 = (0.2 \times 9)/1 = 1.8$ for every $o \in A'_1$, and this weighted envy persists no matter how we allocate o_{12} .⁵ \square

By replacing each of the items o_2, \dots, o_{11} with c smaller items of equal value, one can check that the envy cycle elimination algorithm cannot even guarantee $\text{WWEF}c$ for any fixed c . In spite of this negative result, the algorithm does work in the special case where the agents all have the same valuations.

Proposition 3.2. *The weighted version of the envy cycle elimination algorithm produces a complete $\text{WEF}1$ allocation whenever agents have identical (not necessarily additive) valuations, i.e., $v_i(S) = v(S)$ for some $v : 2^O \rightarrow \mathbb{R}_{\geq 0}$, $\forall i \in N$, $\forall S \subseteq O$.*

3.2 Picking Sequence Protocols

We now turn our attention to protocols that let agents pick their favorite item according to some predetermined sequence. When all agents have equal weight and additive valuations, it is well-known that a round-robin algorithm, wherein the agents take turns picking an item in the order $1, 2, \dots, n, 1, 2, \dots, n, \dots$, produces an $\text{EF}1$ allocation. This is in fact easy to see: If agent i is ahead of agent j in the ordering, then in every “round”, i picks an item that she likes at least as much as j ’s pick; by additivity, i does not envy j . On the other hand, if agent i picks after agent j , then by considering the first round to begin at i ’s first pick, it follows from the same argument that i does not envy j up to the first item that j picks.

We show next that in the general setting with weights, we can construct a weight-dependent picking sequence which guarantees $\text{WEF}1$ for any number of agents and arbitrary weights. The resulting algorithm is efficient, intuitive and can be easily explained to a layperson, so we believe that it has a strong practical appeal. Unlike in the unweighted case, however, the proof that the algorithm produces a fair allocation is much less straightforward and requires making several intricate arguments.

Theorem 3.3. *For any number of agents with additive valuations and arbitrary positive real weights, there exists a picking sequence protocol that computes a complete $\text{WEF}1$ allocation in polynomial time.*

ALGORITHM 1: Weighted Picking Sequence Protocol

- 1: Remaining items $\widehat{O} \leftarrow O$.
 - 2: Bundles $A_i \leftarrow \emptyset$, $\forall i \in N$.
 - 3: $t_i \leftarrow 0$, $\forall i \in N$. /*number of times each agent has picked so far*/
 - 4: **while** $\widehat{O} \neq \emptyset$ **do**
 - 5: $i^* \leftarrow \arg \min_{i \in N} \frac{t_i}{w_i}$, breaking ties lexicographically.
 - 6: $o^* \leftarrow \arg \max_{o \in \widehat{O}} v_{i^*}(o)$, breaking ties arbitrarily.
 - 7: $A_{i^*} \leftarrow A_{i^*} \cup \{o^*\}$.
 - 8: $\widehat{O} \leftarrow \widehat{O} \setminus \{o^*\}$.
 - 9: $t_{i^*} \leftarrow t_{i^*} + 1$.
 - 10: **end while**
-

⁵Another interesting feature of this example is that the two agents have *commensurable* valuations, i.e., both agents have the same valuation for the entire collection of items O : $\sum_{o \in O} v_1(o) = \sum_{o \in O} v_2(o) = 3.2$. This shows that the negative result of Proposition 3.1 holds even if we impose the additional restriction of commensurability on the valuation functions.

To prove Theorem 3.3, we construct a picking sequence such that in each turn, an agent with the lowest weight-adjusted picking frequency picks the next item (Algorithm 1). We claim that after the allocation of each item, for any agent i , every other agent is weighted envy-free towards i up to the item that i picked first.

To this end, first note that choosing an agent who has had the minimum (weight-adjusted) number of picks thus far ensures that the first n picks are a round-robin over all of the agents; in this phase, the allocation is obviously WEF1 since each agent has at most one item at any point. We will show that, after this phase, the algorithm generates a picking sequence over the agents with the following property:

Lemma 3.4. *Consider an agent i chosen by Algorithm 1 to pick an item at some iteration t , and suppose that this is not her first pick. Let t_i and t_j be the numbers of times agent i and some other agent j appear in the prefix of iteration t in the sequence respectively, not including iteration t itself. Then $\frac{t_j}{t_i} \geq \frac{w_j}{w_i}$.*

Proof. Since agent i is picked at iteration t , it must be that $i \in \arg \min_{k \in N} \frac{t_k}{w_k}$. This means that $\frac{t_i}{w_i} \leq \frac{t_j}{w_j}$, i.e., $\frac{t_j}{t_i} \geq \frac{w_j}{w_i}$ since $t_i > 0$. \square

The property guaranteed by Lemma 3.4 is sufficient to ensure that the latest picker does not attract weighted envy up to more than one item towards herself after her latest pick:

Lemma 3.5. *Suppose that, for every iteration t in which agent i picks an item after her first pick, the numbers of times that agent i and some other agent j appear in the prefix of the iteration in the sequence, not including iteration t itself—call them t_i and t_j respectively—satisfy the relation $\frac{t_j}{t_i} \geq \frac{w_j}{w_i}$. Then, in the partial allocation up to and including i 's latest pick, agent j is weighted envy-free towards i up to the first item i picked.*

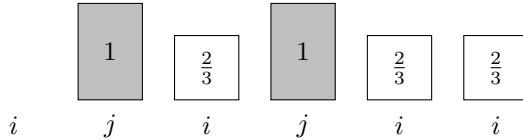


Figure 1: Illustration of the intuition behind the proof of Lemma 3.5. Here, $i < j$, $w_i = 3$, and $w_j = 2$. The rectangles represent the agents' buckets, and the numbers therein correspond to their capacities. Note that agent i does not receive a bucket in her first pick. Agent j 's buckets are filled, while those of agent i are empty.

We provide a high-level intuition of the proof of Lemma 3.5. Recall the argument for the unweighted case at the beginning of Section 3.2. One way to visualize this argument is that when we consider envy from agent j towards agent i , every time agent j picks an item, we give her a bucket with 1 unit of water, while every time agent i picks an item from the second time onwards, we give her an empty bucket of capacity 1. Agent j is allowed to pour water from any of her buckets into any of i 's buckets that comes later in the sequence. Since j values an item that she picks at least as much as any item that i picks in a later turn, in order to establish EF1, it suffices to show that j can fill up all of i 's buckets using such operations. A similar idea can be used in the weighted setting, except that in order to account for the weights, every time agent i picks after the first time, we give her an empty bucket of capacity w_j/w_i units (see Figure 1 for an example when $w_i > w_j$). Note in particular that this bucket setup is entirely independent of the agents' valuations for the items. However, unlike in the unweighted setting, where agent j can accomplish the task by simply pouring all the water from each of her buckets into i 's following bucket, in the

weighted case, j may need to pour water from a bucket into several of i 's buckets, even those coming after j 's subsequent bucket.

With Lemmas 3.4 and 3.5 in hand, we are now ready to prove Theorem 3.3.

Proof of Theorem 3.3. It is easy to see that directly after an agent picks an item, her envy towards other agents cannot get any worse than before. Since the partial allocation after the initial round-robin phase is WEF1 and every agent is weighted envy-free up to one item towards every subsequent picker due to Lemmas 3.4 and 3.5, the allocation is WEF1 at every iteration, and in particular at the end of the algorithm. This establishes the correctness of the algorithm.

For the time complexity, note that there are $O(m)$ iterations of the **while** loop. In each iteration, determining the next picker takes $O(n)$ time, while letting the picker pick her favorite item takes $O(m)$ time. Since we may assume that $m > n$ (otherwise it suffices to allocate at most one item to every agent), the algorithm runs in time $O(m^2)$. \square

If w_i equals a positive constant w for every $i \in N$, then Algorithm 1 degenerates into the traditional round-robin procedure which is guaranteed to return an EF1 allocation for additive valuations, but may not be Pareto optimal; as such, Algorithm 1 may not produce a PO allocation either. This is easily seen in the following example: $n = m = 2$, $w_1 = w_2 = 1$, $v_1(o_1) = v_1(o_2) = 0.5$, $v_2(o_1) = 0.8$, and $v_2(o_2) = 0.2$. With lexicographic tie-breaking for both agents and items, our algorithm will give us $A_1 = \{o_1\}$ and $A_2 = \{o_2\}$, which is Pareto dominated by $A'_1 = \{o_2\}$ and $A'_2 = \{o_1\}$. On the other hand, if each agent has the same value for all items, the algorithm is equivalent to an apportionment method called *Adams' method* [Balinski and Young, 2001].⁶ In the apportionment setting, agents correspond to states of a country, and items to seats in a parliament. Since all seats are considered identical, the states can simply “pick” any seat from the remaining seats in apportionment, whereas for item allocation it is important that each agent picks her favorite item in her turn.

4 WEF1 and PO allocations

As the picking sequence that we construct in Section 3.2 yields an allocation that is WEF1 but may fail Pareto optimality, our next question is whether WEF1 can be achieved in conjunction with the economic efficiency notion. We show that this is indeed possible, by generalizing the classic adjusted winner procedure for two agents and an algorithm of Barman et al. [2018] for higher numbers of agents.

4.1 Two Agents

When agents have equal entitlements, it is known that fairness and efficiency are compatible: Caragiannis et al. [2019] showed that an allocation maximizing the Nash social welfare satisfies both PO and EF1. Unfortunately, this approach is not applicable to our setting—we show that the MWNW allocation may fail to be WEF1. In fact, we prove a much stronger negative result: for any fixed c , the allocation may fail to be WEF c even for two agents with identical valuations.

Proposition 4.1. *Let c be an arbitrary positive integer. There exists a problem instance with two agents having identical additive valuations for which any MWNW allocation is not WEF c .*

⁶We are grateful to Ulrike Schmidt-Kraepelin for pointing out this connection.

Proof. Suppose that $n = 2$, and the weights are $w_1 = 1$ and $w_2 = k$ for some positive integer k such that $\left(1 + \frac{1}{k+c}\right)^k > 2$; such an integer k exists because $\lim_{k \rightarrow \infty} \left(1 + \frac{1}{k+c}\right)^k = e$. Let $m = k + c + 2$, so $O = \{o_1, o_2, \dots, o_{k+c+2}\}$. The agents have identical, additive valuations defined by $v_i(o) = 1$ for all $i \in N$ and $o \in O$. Since $\left(1 + \frac{1}{k+c}\right)^k > 2$, we have $1 \cdot (k + c + 1)^k > 2 \cdot (k + c)^k$. Moreover, for $2 \leq i \leq k + c$, we have

$$\left(1 + \frac{1}{k + c + 1 - i}\right)^k > \left(1 + \frac{1}{k + c}\right)^k > 2 > \frac{i + 1}{i},$$

and so $i(k + c + 2 - i)^k > (i + 1)(k + c + 1 - i)^k$. This means that any MWNW allocation A must give one item to agent 1, say $A_1 = \{o_1\}$, and the remaining items to agent 2, i.e., $A_2 = \{o_2, \dots, o_{k+c+2}\}$. However, even if we remove a set S_c of at most c items from A_2 , we would still have $v_1(A_2 \setminus S_c)/w_2 \geq 1 + 1/k > 1 = v_1(A_1)/w_1$, so the allocation is not WEF c . \square

Given that a MWNW allocation may not be WEF1 in our setting, a natural question is whether there is an alternative approach for guaranteeing the existence of a PO and WEF1 allocation. We first show that this is indeed the case for two agents: we establish that such an allocation exists and can be computed in polynomial time for two agents, by adapting the classic adjusted winner procedure [Brams and Taylor, 1996] to the weighted setting.

Theorem 4.2. *For two agents with additive valuations and arbitrary positive real weights, a complete WEF1 and PO allocation always exists and can be computed in polynomial time.*

4.2 Any Number of Agents

Having resolved the existence question of PO and WEF1 for two agents, we now investigate whether such an allocation always exists for any number of agents, answering the question in the affirmative. To this end, we employ a weighted modification of the algorithm by Barman et al. [2018], which finds a PO and EF1 allocation in pseudo-polynomial time for agents with additive valuations in the unweighted setting. Like Barman et al., we consider an artificial market where each item has a price and agents purchase a bundle of items with the highest ratio of value to price, called “bang per buck ratio”. This allows us to measure the degree of fairness of a given allocation in terms of the prices.

Formally, a *price vector* is an m -dimensional non-negative real vector $\mathbf{p} = (p_1, p_2, \dots, p_m) \in \mathbb{R}_{\geq 0}^Q$; we call p_o the *price* of item $o \in O$, and write $p(X) = \sum_{o \in X} p_o$ for a set of items X . Let A be an allocation and \mathbf{p} be a price vector. For each $i \in N$, we call $p(A_i)$ the *spending* and $\frac{1}{w_i} p(A_i)$ the *weighted spending* of agent i . We now define a weighted version of the price envy-freeness up to one item (pEF1) notion introduced by Barman et al. [2018].

Definition 4.3. Given an allocation A and a price vector \mathbf{p} , we say that A is *weighted price envy-free up to one item* (WpEF1) with respect to \mathbf{p} if for any pair of agents i, j , either $A_j = \emptyset$ or $\frac{1}{w_i} p(A_i) \geq \frac{1}{w_j} \min_{o \in A_j} p(A_j \setminus \{o\})$.

The *bang per buck ratio* of item o for agent i is $\frac{v_i(o)}{p_o}$; we write the maximum bang per buck ratio for agent i as $\alpha_i(\mathbf{p})$. We refer to the items with maximum bang per buck ratio for i as i 's *MBB items* and denote the set of such items by $\text{MBB}_i(\mathbf{p})$ for each $i \in N$. The following lemma is a straightforward adaptation of Lemma 4.1 in [Barman et al., 2018] to our setting; it ensures that one can obtain the property of WEF1 by balancing among the spending of agents under the MBB condition.

Lemma 4.4. *Given a complete allocation A and a price vector \mathbf{p} , suppose that allocation A satisfies WpEF1 with respect to \mathbf{p} and agents are assigned to MBB items only, i.e., $A_i \subseteq \text{MBB}_i(\mathbf{p})$ for each $i \in N$. Then A is WEF1 .*

It is also known that if each agent i only purchases MBB items, so that i maximizes her valuation under the budget $p(A_i)$, then the corresponding allocation is Pareto optimal.

Lemma 4.5 (First Welfare Theorem; Mas-Colell et al. [1995], Chapter 16). *Given a complete allocation A and a price vector \mathbf{p} , suppose that agents are assigned to MBB items only, i.e., $A_i \subseteq \text{MBB}_i(\mathbf{p})$ for each $i \in N$. Then A is PO .*

With Lemmas 4.4 and 4.5, the problem of finding a PO and WEF1 allocation reduces to that of finding an allocation and price vector pair satisfying the MBB condition and WpEF1 . We show that there is an algorithm that finds such an outcome in pseudo-polynomial time. Our algorithm follows a similar approach as that of Barman et al. [2018]; thus the proof of Theorem 4.6 is deferred to the full version of our paper.

Theorem 4.6. *For any number of agents with additive valuations and arbitrary positive real weights, there exists a WEF1 and PO allocation. Furthermore, such an allocation can be computed in time $\text{poly}(m, n, v_{\max}, w_{\max})$ for any integer-valued inputs, where $v_{\max} := \max_{i \in N, o \in O} v_i(o)$ and $w_{\max} := \max_{i \in N} w_i$.*

The outline of the algorithm is as follows. Our algorithm alternates between two phases: the first phase involves reallocating items from large to small *spenders* (where the “spending” of an agent is defined as the ratio between the price for her bundle of items and her weight; see the formal definition in our full version), and the second phase involves increasing the prices of the items owned by small spenders. We show that by increasing prices gradually, the algorithm converges to an allocation and price vector pair satisfying the desired criteria when both input weights and valuations are expressed as integral powers of $(1 + \epsilon)$ for some $\epsilon > 0$. Similarly to Barman et al. [2018], we apply our algorithm to the ϵ -approximate instance of the original input and show that for small enough ϵ , the output of the algorithm satisfies the original MBB condition and WpEF1 . We note that compared to Barman et al. [2018], the analysis becomes more involved due to the presence of weights. In particular, each price-rise phase takes into account not only the valuations but also the weights; as a result, ϵ needs to be much smaller in order to ensure the equivalence.

5 WEF1 and PO allocations: Maximum Weighted Nash Welfare

In the previous section, we saw that MWNW allocations may fail to satisfy WEF1 , showing that the result of Caragiannis et al. [2019] from the unweighted setting does not extend to the weighted setting via WEF1 (or even $\text{WEF}c$ for any fixed c). Given that these allocations maximize a natural objective, it is still tempting to ask whether they provide any fairness guarantee. The answer is indeed positive: we show that a MWNW allocation satisfies WWEF1 , a weaker fairness notion that also generalizes EF1 .

Theorem 5.1. *For any number of agents with additive valuations and arbitrary positive real weights, a MWNW allocation is always WWEF1 and PO .*

The proof of Theorem 5.1 follows a similar outline as the corresponding proof of Caragiannis et al. [2019]. PO follows easily from the definition of MWNW . For WWEF1 , we assume for contradiction that an agent i weakly envies another agent j up to more than one item in a MWNW allocation. If every agent has a positive value for every item, we pick an item

in agent j 's bundle for which the ratio between i 's value and j 's value is maximized. By distinguishing between the cases $w_i \geq w_j$ and $w_i \leq w_j$, we show that we can achieve a higher weighted Nash welfare upon transferring this item to agent i 's bundle, which yields the desired contradiction. The case where agents may have zero value for items is then handled separately.

6 Discussion and Future Work

In this paper, we have introduced and studied envy-based notions for the allocation of indivisible items in a general setting where agents can have different entitlements. As most of our results hold for additive valuation functions, the reader may wonder whether they can be extended to more general classes—after all, in the absence of weights, an EF1 allocation is known to exist for arbitrary monotone valuations [Lipton et al., 2004]. We therefore point out some hurdles that we faced while trying to generalize our weighted envy concepts beyond additive valuations. First, we show that even for simple non-additive valuations, the existence of a WEF1 or WWEF1 allocation can no longer be guaranteed. Since WWEF1 is weaker than WEF1, it suffices to prove the claim for WWEF1.

Proposition 6.1. *There exists an instance with $n = 2$ agents such that one of the agents has a (normalized and monotone) submodular valuation,⁷ the other agent has an additive valuation, and a complete WWEF1 allocation does not exist.*

By increasing the lower bound on the number of items in the instance of the proof of Proposition 6.1 to $5c$, one can show that a complete WWEF c allocation is also not guaranteed to exist for any constant c .

One of the key ideas in our analysis of the maximum weighted Nash welfare allocation (Theorem 5.1) is what we call the *transferability* property: If agent i has weighted envy towards agent j under additive valuations, then there is at least one item o in j 's bundle for which agent i has positive (marginal) valuation—in other words, the item o could be transferred from j to i to augment i 's realized valuation.⁸ Unfortunately, this property no longer holds for non-additive valuations.

Proposition 6.2. *There exists an instance such that an agent i with a non-additive valuation function has weighted envy towards an agent j under some allocation A , but there is no item in j 's bundle for which i has positive marginal valuation—i.e., $\nexists o \in A_j$ such that $v_i(A_i \cup \{o\}) > v_i(A_i)$.*

In light of these negative results, an important direction for future research is to identify appropriate weighted envy notions for non-additive valuations. Other interesting directions include establishing conditions under which WEF allocations are likely to exist, investigating weighted envy in the allocation of *chores* (items with negative valuations), and considering weighted versions of other envy-freeness approximations such as *envy-freeness up to any item (EFX)* [Caragiannis et al., 2019, Plaut and Roughgarden, 2020]. From a broader point of view, our work demonstrates that fair division with different entitlements is richer and more challenging than its traditional counterpart in several ways, and much interesting work remains to be done.

⁷A valuation function $v : 2^O \rightarrow \mathbb{R}_{\geq 0}$ is said to be *submodular* if for any $O_1 \subseteq O_2 \subseteq O$ and any item $o \in O \setminus O_2$, we have $v(O_1 \cup \{o\}) - v(O_1) \geq v(O_2 \cup \{o\}) - v(O_2)$.

⁸Transferability and related properties have been studied by Babaioff et al. [2020] and Benabbou et al. [2020b] in the context of EF1 and P0 allocations for a subclass of submodular valuations.

References

- Haris Aziz, Hau Chan, and Bo Li. Weighted maxmin fair share allocation of indivisible chores. In *Proceedings of the 28th International Joint Conference on Artificial Intelligence (IJCAI)*, pages 46–52, 2019.
- Haris Aziz, Hervé Moulin, and Fedor Sandomirskiy. A polynomial-time algorithm for computing a Pareto optimal and almost proportional allocation. *Operations Research Letters*, 48(5):573–578, 2020.
- Moshe Babaioff, Noam Nisan, and Inbal Talgam-Cohen. Fair allocation through competitive equilibrium from generic incomes. In *Proceedings of the 2nd ACM Conference on Fairness, Accountability, and Transparency (ACM FAT*)*, page 180, 2019.
- Moshe Babaioff, Tomer Ezra, and Uriel Feige. Fair and truthful mechanisms for dichotomous valuations. *arXiv preprint arXiv:2002.10704*, 2020.
- Michel L. Balinski and H. Peyton Young. *Fair Representation: Meeting the Ideal of One Man, One Vote*. Brookings Institution Press, 2001.
- Julius B. Barbanel. Game-theoretic algorithms for fair and strongly fair cake division with entitlements. *Colloquium Mathematicae*, 69(1):59–73, 1995.
- Siddharth Barman, Sanath Kumar Krishnamurthy, and Rohit Vaish. Finding fair and efficient allocations. In *Proceedings of the 19th ACM Conference on Economics and Computation (EC)*, pages 557–574, 2018. Extended version available as arXiv:1707.04731.
- Nawal Benabbou, Mithun Chakraborty, Edith Elkind, and Yair Zick. Fairness towards groups of agents in the allocation of indivisible items. In *Proceedings of the 28th International Joint Conference on Artificial Intelligence (IJCAI)*, pages 95–101, 2019.
- Nawal Benabbou, Mithun Chakraborty, Xuan-Vinh Ho, Jakub Sliwinski, and Yair Zick. The price of quota-based diversity in assignment problems. *ACM Transactions on Economics and Computation*, 8(3):14:1–14:32, 2020a.
- Nawal Benabbou, Mithun Chakraborty, Ayumi Igarashi, and Yair Zick. Finding fair and efficient allocations when valuations don’t add up. In *Proceedings of the 13th International Symposium on Algorithmic Game Theory (SAGT)*, pages 32–46, 2020b.
- Vittorio Bilò, Ioannis Caragiannis, Michele Flammini, Ayumi Igarashi, Gianpiero Monaco, Dominik Peters, Cosimo Vinci, and William S. Zwicker. Almost envy-free allocations with connected bundles. In *Proceedings of the 10th Innovations in Theoretical Computer Science Conference (ITCS)*, pages 14:1–14:21, 2019.
- Arpita Biswas and Siddharth Barman. Fair division under cardinality constraints. In *Proceedings of the 27th International Joint Conference on Artificial Intelligence (IJCAI)*, pages 91–97, 2018.
- Sylvain Bouveret and Michel Lemaître. Characterizing conflicts in fair division of indivisible goods using a scale of criteria. *Autonomous Agents and Multi-Agent Systems*, 30(2):259–290, 2016.
- Sylvain Bouveret, Yann Chevaleyre, and Nicolas Maudet. Fair allocation of indivisible goods. In Felix Brandt, Vincent Conitzer, Ulle Endriss, Jérôme Lang, and Ariel D. Procaccia, editors, *Handbook of Computational Social Choice*, chapter 12, pages 284–310. Cambridge University Press, 2016.

- Steven J. Brams and Alan D. Taylor. *Fair Division: From Cake-Cutting to Dispute Resolution*. Cambridge University Press, 1996.
- Eric Budish. The combinatorial assignment problem: Approximate competitive equilibrium from equal incomes. *Journal of Political Economy*, 119(6):1061–1103, 2011.
- Ioannis Caragiannis, David Kurokawa, Hervé Moulin, Ariel D. Procaccia, Nisarg Shah, and Junxing Wang. The unreasonable fairness of maximum Nash welfare. *ACM Transactions on Economics and Computation*, 7(3):12:1–12:32, 2019.
- Vincent Conitzer, Rupert Freeman, Nisarg Shah, and Jennifer Wortman Vaughan. Group fairness for the allocation of indivisible goods. In *Proceedings of the 33rd AAAI Conference on Artificial Intelligence (AAAI)*, pages 1853–1860, 2019.
- Ágnes Cseh and Tamás Fleiner. The complexity of cake cutting with unequal shares. *ACM Transactions on Algorithms*, 16(3):29:1–29:21, 2020.
- Alireza Farhadi, Mohammad Ghodsi, MohammadTaghi Hajiaghayi, Sebastien Lahaie, David Pennock, Masoud Seddighin, Saeed Seddighin, and Hadi Yami. Fair allocation of indivisible goods to asymmetric agents. *Journal of Artificial Intelligence Research*, 64:1–20, 2019.
- Duncan K. Foley. Resource allocation and the public sector. *Yale Economics Essays*, 7(1):45–98, 1967.
- Ya’akov (Kobi) Gal, Moshe Mash, Ariel D. Procaccia, and Yair Zick. Which is the fairest (rent division) of them all? *Journal of the ACM*, 64(6):39:1–39:22, 2017.
- David Kurokawa, Ariel D. Procaccia, and Junxing Wang. Fair enough: Guaranteeing approximate maximin shares. *Journal of the ACM*, 64(2):8:1–8:27, 2018.
- Maria Kyropoulou, Warut Suksompong, and Alexandros A. Voudouris. Almost envy-freeness in group resource allocation. *Theoretical Computer Science*, 841:110–123, 2020.
- Richard J. Lipton, Evangelos Markakis, Elchanan Mossel, and Amin Saberi. On approximately fair allocations of indivisible goods. In *Proceedings of the 5th ACM Conference on Electronic Commerce (EC)*, pages 125–131, 2004.
- Evangelos Markakis. Approximation algorithms and hardness results for fair division with indivisible goods. In Ulle Endriss, editor, *Trends in Computational Social Choice*, chapter 12, pages 231–247. AI Access, 2017.
- Andreu Mas-Colell, Michael D. Whinston, and Jerry R. Green. *Microeconomic Theory*. Oxford University Press, 1995.
- Hervé Moulin. *Fair Division and Collective Welfare*. MIT Press, 2003.
- Hervé Moulin. Fair division in the internet age. *Annual Review of Economics*, 11:407–441, 2019.
- Hoon Oh, Ariel D. Procaccia, and Warut Suksompong. Fairly allocating many goods with few queries. In *Proceedings of the 33rd AAAI Conference on Artificial Intelligence (AAAI)*, pages 2141–2148, 2019.
- Benjamin Plaut and Tim Roughgarden. Almost envy-freeness with general valuations. *SIAM Journal on Discrete Mathematics*, 34(2):1039–1068, 2020.

- Jack Robertson and William Webb. *Cake-Cutting Algorithms: Be Fair if You Can*. Peters/CRC Press, 1998.
- Erel Segal-Halevi. Cake-cutting with different entitlements: How many cuts are needed? *Journal of Mathematical Analysis and Applications*, 480(1):123382, 2019.
- Erel Segal-Halevi and Warut Suksompong. Democratic fair allocation of indivisible goods. *Artificial Intelligence*, 277:103167, 2019.
- Erel Segal-Halevi, Shmuel Nitzan, Avinatan Hassidim, and Yonatan Aumann. Fair and square: Cake-cutting in two dimensions. *Journal of Mathematical Economics*, 70(8):1–28, 2017.
- William Thomson. Introduction to the theory of fair allocation. In Felix Brandt, Vincent Conitzer, Ulle Endriss, Jérôme Lang, and Ariel D. Procaccia, editors, *Handbook of Computational Social Choice*, chapter 11, pages 261–283. Cambridge University Press, 2016.
- Hal R. Varian. Equity, envy, and efficiency. *Journal of Economic Theory*, 9:63–91, 1974.
- Dao-Zhi Zeng. Approximate envy-free procedures. In Fioravante Patrone, Ignacio García-Jurado, and Stef Tijs, editors, *Game Practice: Contributions from Applied Game Theory*, chapter 17, pages 259–271. Springer, 2000.

Mithun Chakraborty
University of Michigan
Ann Arbor, MI, USA
Email: dcsmc@umich.edu

Ayumi Igarashi
National Institute of Informatics
Tokyo, Japan
Email: ayumi_igarashi@nii.ac.jp

Warut Suksompong
National University of Singapore
Singapore
Email: warut@comp.nus.edu.sg

Yair Zick
University of Massachusetts, Amherst
Amherst, MA, USA
Email: yzick@umass.edu