

Random Paths To Popularity In Two-Sided Matching

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Abstract

We study practically relevant aspects of popularity in two-sided matching where only one side has preferences. A matching is called *popular* if there does not exist another matching that is preferred by a simple majority. We show that for a matching to be popular it is necessary and sufficient that no coalition of size up to 3 decides by simple majority to exchange their houses. We then show that a market where such coalitions meet at random converges to a popular matching whenever it exists.

Keywords. two-sided matching, popular matching, random paths, house allocation, assignment problem

1 Introduction

Various real-life economic situations can be modeled as two-sided matching markets where agents have preferences over indivisible objects and such that each agent gets at most one object, also known as the house allocation problem. These situations include housing markets, assigning students to primary schools, donor organ exchange programs, job placement and graduates, and so forth.

Among different notions of efficiency for these matching markets, recent literature highlights the concept of popularity. A matching is called *popular* if majority of agents weakly prefers it over any other matching.¹ Popularity has mainly served as a normative property as it is a natural non-Paretian selection from a (possibly very large) set of efficient matchings.

The seminal paper by [1] that introduced popularity for the house allocation problem proposed a simple characterization of popular matchings. A matching is popular if and only if (1) each agent gets either his best house among all houses (called his *first house*), or the best house among all houses that are not someone's best (called his *second house*), and (2) all first houses are allocated among agents that deem them as the best.

The subsequent literature (see [6, 23] and the subsection below) focused mainly on issues relevant for centralized markets. In contrast to that, in the current paper we shift the focus from popularity in centralized markets to decentralized markets.

Our first result (Theorem 1) shows that a popular matching only needs to be popular locally: the matching is popular if and only if no group of up to three agents decides (by simple majority) to exchange their houses, keeping the matching of other agents untouched.² The original characterization in [1] directly follows from our result (Corollary 1).

¹One can also see popular matchings as weak Condorcet winners in a voting problem where candidates are all possible matchings.

²This result can also be interpreted from the cooperative standpoint. If for each coalition we take the majority rule as the solution concept, then for a matching to be in the core it is enough to check coalitions of size up to three. The analogous result for the marriage market states that the set of pairwise stable matchings coincides with the core [33].

Our paper also suggests a positive rationale behind popularity: we show that locally popular improvements lead to globally popular matchings and thus popularity is likely to be eventually observed in realistic situations. Specifically, we consider a decentralized market where agents at random meet in groups of size up to three and exchange their houses when this is supported by majority of them. Our second result (Theorem 2) shows that this market eventually converges to a popular matching whenever it exists.

Our finding is analogous to the result in [34] about convergence in a marriage market. There, one matching is modified locally by a blocking pair of a man and woman that prefer each other over their current matches. As this man and this woman match, their previous partners become unmatched, and these changes constitute a new matching. Then a new blocking pair is considered, a new matching is formed, and so forth. [34] show that the sequence of these matchings lead to a stable matching.

In our case blocking takes a slightly different form: each time the group of agents exchanges their houses within themselves. This strengthens the analogy with the real estate market where agents exchange their houses within small groups and no agent leaves without a house.³

Another closely related paper is [2] that considers the popularity-improvement paths from an arbitrary matching. The main finding is that, given a popular matching exists, it can be attained by at most two steps using an efficient algorithm. However, a realistic application of these exchanges even in a centralized market faces two difficulties. First, for exchanges that large one might need much data about agent’s preferences, and this data might be difficult to obtain.⁴ Second, large exchanges are very risky and unstable as everything depends on each person agreeing to be part of the deal.

1.1 Background

The assignment game where agents exchange indivisible objects (houses) without money was first introduced in [35], the assignment problem where all houses are social endowments was first studied in [15]. The concept of popularity was first introduced by [11] for the two-sided matching problem [10], where popularity coincides with stability, and was applied to two-sided matching problem only recently by [1]. In their paper [1] characterized the set of popular matchings as matchings where each agent gets either his first house or his second house and proposed an efficient algorithm to find a popular matching.⁵

Existence and multiplicity of popular matchings was studied from several sides. First, [24] shows that a popular matching is likely to exist whenever preferences are uniformly random and the number of houses is approximately 1.42 times larger than the number of agents. For settings where a popular matching does not exist, [22] studied how to minimally augment the preference profile so that the existence is guaranteed; this problem is, in general, NP-hard.

Another way to ensure popularity is to consider mixed matchings, i.e. lotteries over matchings, and a straightforward generalization of the popularity property; [18] show that a popular mixed matching always exists and propose an efficient algorithm to find one.

As an alternative approach, [27] proposes least-unpopularity criteria to find the “most” popular matching; finding his least-unpopular matchings is, in general, NP-hard.

³For instance, according to [32], in 2017 the secondary housing market in the largest European city, Moscow, the share of alternative deals, i.e. not house-for-money but house-for-house, was approximately 90%.

⁴According to [37], in 1990 subsidized housing markets in Beijing where residents exchange their houses had size of 80 000.

⁵The result in [1] also allows ties. This setting was further generalized to the case with ties and matroid constraints by [17] and to the case with two-sided preferences and one-sided ties by [8] (the latter problem turns out to be NP-hard). The many-to-one matching problem, where each house has a capacity was studied in [36], and the many-to-many problem was studied by [31].

The problem of counting the number of popular matchings has been addressed in [28] for the case of strict preferences and in [30] and [3] for the case of weak preferences.

2 The Model

Let A be a set of agents and H be a (larger) set of houses, $|H| \geq |A|$. Each agent $a \in A$ is endowed with a strict preference relation \succ_a over the set of houses $H \cup \{\emptyset\}$ (i.e. \succ_a is a linear order), and a prefers each house $h \in H$ over having no house, $h \succ_a \emptyset$. The collection of individual preferences of all agents $\succ = (\succ_a)_{a \in A}$ is referred to as **the preference profile**. The triple (A, H, \succ) constitutes the two-sided matching problem (aka house allocation problem), or simply **a problem**. In what follows we assume that the sets A and H are fixed and the problem is given by the preference profile \succ .

A solution to the problem is a **matching** μ – a mapping from $A \cup H \cup \emptyset$ on itself: by definition agent $a \in A$ is said to be matched to a house $h \in H$ in matching μ if $\mu(a) = h$ and also $\mu(h) = a$. If some agent or house remain unmatched, we say that they are matched to \emptyset . Let \mathcal{M} denote the set of all possible matchings.

For any two matchings $\mu, \mu' \in \mathcal{M}$ and a subset of agents $B \subset A$ define **pairwise comparison** $PC_B(\mu, \mu')$ as the number of agents in B that strictly prefer their house in μ over their house in μ' .

A matching $\mu \in \mathcal{M}$ is called **popular (among set A)** if there does not exist another matching $\mu' \in \mathcal{M}$ such that μ' is preferred over μ by simple majority within entire set of agents A : $PC_A(\mu', \mu) > PC_A(\mu, \mu')$.

For each agent a let us call his most preferred house in H as a 's **first house**: $FH(a) = h$ such that for each $h' \in H$ and $h' \neq h$ it holds that $h \succ_a h'$. **The set of all first houses** is denoted as $FH(A) = \{FH(a)\}_{a \in A}$. For each house h let us call agents for whom h is the first house as h 's **first agents**: $FA(h) = \{a \in A | h = FH(a)\}$.

For each agent a let us call his most preferred house among all non-first houses as a 's **second house**: $SH(a) = h$ such that for each $h' \in H \setminus FH(A)$ and $h' \neq h$ it holds that $h \succ_a h'$. **The set of all second houses** is denoted as $SH(A) = \{SH(a)\}_{a \in A}$. For each house h let us call agents for whom h is the second house as h 's **second agents**: $SA(h) = \{a \in A | h = SH(a)\}$.

Note that sets $FH(A)$ and $SH(A)$ are disjoint, i.e. no agent's second house can be a first house for any other agent.

3 Characterization of Popular Matching

Note that a matching cannot be popular if at least one agent is unmatched. Therefore throughout the paper we can focus only on full matchings, $\mu(A) \subset H$.

Our first main result characterizes the popular matching as a matching that is popular among each triple of agents.

For a profile \succ , we say that a matching μ is *popular among each three agents* if for each three agents $a, b, c \in A$ there does not exist a matching $\mu' \in \mathcal{M}$ same as μ for each other agent $a' \notin \{a, b, c\}$ $\mu'(a') = \mu(a')$ and such that it wins μ in pairwise comparison within this triple of agents $PC_{\{a, b, c\}}(\mu', \mu) > PC_{\{a, b, c\}}(\mu, \mu')$.

Theorem 1. A matching is popular if and only if it is popular among each three agents.

Proof. The "only if" part is straightforward: each popular matching μ is popular among each triple of agents. For a contradiction, assume that there is a triple of agents $a, b, c \in A$ and another matching μ' same as μ for all other agents and such that it is preferred over μ : $PC_{\{a, b, c\}}(\mu', \mu) > PC_{\{a, b, c\}}(\mu, \mu')$. Then μ cannot be popular among all agents since

all other agents are indifferent and thus: $PC_A(\mu', \mu) - PC_A(\mu, \mu') = PC_{\{a,b,c\}}(\mu', \mu) - PC_{\{a,b,c\}}(\mu, \mu') > 0$.

The "if" part we also prove by contradiction. For a contradiction, assume that there is a matching μ that is popular among each triple of agents, but it loses in pairwise comparison to some other matching μ' : $PC_A(\mu', \mu) > PC_A(\mu, \mu')$. Consider all agents that have different houses in these two matchings, denote the set of these agents as $A_1 = \{a \in A : \mu(a) \neq \mu'(a)\}$. (In what follows we will change the notation of these agents for convenience).

We partition all agents into those who participate in a trading cycle, i.e. exchange their matched houses among themselves, and trading chains, i.e. those that are matched in μ' to a previously empty house or whose house in μ becomes empty in μ' .

We first deal with chains. Consider an arbitrary agent $b_1 \in A_1$ that received a previously empty house $\mu'(b_1) \notin \mu(A)$, $\mu(\mu'(b_1)) = \emptyset$. If b_1 's house is empty in μ' , $\mu'(\mu(b_1)) = \emptyset$, then we get a chain of size 1. Otherwise there is some agent b_2 such that $\mu'(b_2) = \mu(b_1)$. If b_2 's house is empty in μ' , $\mu'(\mu(b_2)) = \emptyset$, then we get a chain of size 2. Otherwise, we continue in the same way until we find the last agent in the chain. Similarly, determine chains for each agent that receives a previously empty house. Denote the set of agents participating in a chain as B_1 .

We then deal with cycles. Consider an arbitrary agent not from any chain $a_1 \in A_1 \setminus B_1$, $\mu(a_1) \neq \mu'(a_1)$. Consider agent a_2 that owns house $\mu'(a_1)$, $a_2 = \mu(\mu'(a_1))$. Agent a_2 also does not belong to any chain, $a_1 \in A_1 \setminus B_1$ and as $\mu(a_2) = \mu'(a_1)$, then $a_2 \neq a_1$. If the two agents just exchanged their houses, $\mu'(a_2) = \mu(a_1)$, then we get a trading cycle $(\mu(a_1), a_1, \mu'(a_1), a_2)$ of length 2. Otherwise, if $\mu'(a_2) \neq \mu(a_1)$, then consider agent $a_3 = \mu(\mu'(a_2))$. Since $\mu(a_3) = \mu'(a_2) \neq \mu(a_1)$, then $a_2 \neq a_3$, $a_1 \neq a_3$ и $a_3 \in A_1$.

And so forth until we get a cycle of length at least 2 and at most $|A_1 \setminus B_1|$. In the same way we find all trading cycles among all other agents.

Thus, the set A_1 and the set of corresponding houses $\mu(A_1) \cup \mu'(A_1)$ is partitioned into trading chains of size at least 1 and cycles of size at least 2.

By assumption $PC_A(\mu', \mu) > PC_A(\mu, \mu')$, there is at least one trading chain or one trading cycle such that more than half of its agents prefer μ' over μ . Formally, if A_{TC} denotes the set of agents in this chain or cycle, $PC_{A_{TC}}(\mu', \mu) > PC_{A_{TC}}(\mu, \mu')$.

If A_{TC} form a cycle, then we can find two neighbouring agents $i, j \in A_{TC}$, $j = \mu(\mu'(i))$, that both prefer μ' over μ . If this trading cycle is of length 2, then consider a new matching μ'' that is identical to μ for each agent except $a = \{i, j\}$ and same as μ' for these pair $\mu''(a) = \mu'(a)$. Then by adding one other arbitrary agent we get a triple of agents that prefer μ'' over μ by majority – contrary to our premise. If this trading cycle is of length more than 2, then consider the next neighbouring agent $l = \mu(\mu'(j))$. Consider now a new matching μ'' that is identical to μ for each agent except $a = \{i, j, l\}$ and $\mu''(i) = \mu'(i)$, $\mu''(j) = \mu'(j)$, and $\mu''(l) = \mu(i)$. The triple of agents i, j, l prefers μ'' over μ by majority: $PC_{\{i,j,l\}}(\mu'', \mu) > PC_{\{i,j,l\}}(\mu, \mu'')$, contrary to our premise.

If A_{TC} forms a chain of length 1, $A_{TC} = \{a_1\}$, then consider a new matching μ'' constructed as before: μ'' is identical to μ for each agent except for a_1 , $\mu''(a_1) = \mu'(a_1)$. A triple of agents a_1 and two arbitrary agents a_2, a_3 prefers μ'' over the original matching μ : $PC_{\{a_1,a_2,a_3\}}(\mu'', \mu) > PC_{\{a_1,a_2,a_3\}}(\mu, \mu'')$, contrary to our premise.

If A_{TC} forms a chain of length 2, then both agents in A_{TC} are better off in μ' compared to μ . By adding one other arbitrary agent we get a triple of agents that prefers a similarly constructed μ'' over μ by majority, contrary to our premise.

If the length of the chain is above 2, then either (1) we can find two neighbouring agents $i, j \in A_{TC}$, $j = \mu(\mu'(i))$, that both prefer μ' over μ , or (2) the chain begins and ends with agents that are better off in μ' compared to μ (and agents in between interchange). In case (1) we take the triple of these agents i, j and the previous owner of j 's house $l = \mu(\mu'(j))$ (if j 's house was empty, then take an arbitrary l). This triple i, j, l prefers a similarly constructed μ'' over μ by majority, contrary to our premise.

In case (2) we take the triple of agents as the first agent in the chain a_1 , $\mu(\mu'(a_1)) = \emptyset$, the last agent a_k , $\mu'(\mu(a_k)) = \emptyset$, and the one before the last a_{k-1} . The triple a_1, a_{k-1}, a_k prefers a similarly constructed μ'' over μ by majority, contrary to our premise. ■

As an immediate corollary we get the characterization of popular matchings from [1].

Corollary 1 [1] A matching is popular if and only if (1) each agent gets either his first house or his second house, and (2) each first house is matched with one of its first agents.

Proof. The "if" part is straightforward since it is enough to check only triples of agents. In each such triple only an agent a with a second house can become better off, but each better house $f \succ_a SH(a)$ is already matched to one of its first agents $b = \mu(f) \in FA(f)$, making a better off requires making b worse off, which cannot be supported by majority.

We prove the "only if" part by contradiction. Let condition (2) be violated: some first house f is not allocated to one of its first agents. Then each f 's first agent $a \in FA(f)$, the owner of f $b = \mu(f)$ and the owner of b 's first house $c = \mu(FH(b))$ form a triple for which μ is not popular.

Hence, in any popular matching, each agent gets his first house, second house, or a bad house.

Let condition (1) be violated: some agent a_1 gets a bad house t in matching μ , there is a triple of agents a_1 , the owner of a_1 's second house $a_2 = \mu(SH(a_1))$, and the owner of a_2 's first house $a_3 = \mu(FH(a_2))$ for whom μ is not popular. ■

4 Random Paths to Popularity

A popular market is a finite Markov chain. The set space is the set of matchings \mathcal{M} . The transition probabilities between the states are not symmetric and depend on how many agents become better off in one state compared to the other. For each matching $\mu \in \mathcal{M}$ we consider all "neighbouring" matchings $\mu' \in \mathcal{M}$ that is matchings where at most three agents are matched to different house than in μ . If $k = 1, 2, 3$ agents are matched differently in μ and μ' , then we say that μ and μ' are connected by a k -way exchange. If the k -way exchange makes more than half of these k agents better off, then the transition probability is positive, otherwise the transition probability is zero. Then the set of absorbing states coincides with the set of popular matchings.

Next we present our second main result.

Theorem 2. A popular market with groups of size up to 3 converges to a popular matching whenever it exists.

Sketch of the proof. According to the theory of Markov chains it is sufficient to show that a popular matching – given it exists – can be reached in a finite number of allowed exchanges. We propose a simple finite algorithm that does it only by using one-, two- and three-way exchanges.

The algorithm has two stages. In the first stage it matches each first house to some of its first agents. This is done in a greedy serial dictatorship fashion. According to a fixed order each agent a takes his first house f unless this house is already matched to one of its other first agents (in this case no exchange takes place and we proceed to the next agent in the order). In the same time, the agent owning house f takes his own first house g and the owner of this house $\mu(g)$ takes the house of agent a . This three-way exchange is supported by at least two agents a and $\mu(f)$, and, possibly, also by agent $\mu(g)$.

In the second stage of the algorithm we use another simple greedy procedure where owners of bad houses are forcibly given their second houses. Each agent a owning some bad house t takes his second house s , while the owner of s takes his first house f ,⁶ and house t

⁶Note that s cannot be owned by his first agent, otherwise s does not qualify as a second house for agent a .

goes to the owner of f . This three-way exchange is supported by at least two agents a and $\mu(s)$, but the exchange also might be "bad" if t is a bad house for both agents $\mu(s)$ and $\mu(f)$. A bad exchange like that leads to the same situation: out of three agents one owns his first house, one owns his second house and one owns a bad house. It remains to show that the sequence of the bad exchanges is finite.

The finiteness follows from that the sequence of bad exchanges eventually arrives to a house that was in the sequence earlier (due to finiteness of A). At this step k of the sequence we have $k - 2$ agents that have the same $\lfloor \frac{k-2}{2} \rfloor$ as their first and second houses. By the Hall's theorem (applied to the characterization in Corollary 3) these houses can only be matched to these agents. Thus agent k that gets house t after a series of bad exchanges can only start a new sequence of bad exchanges but the other $k - 1$ agents remain untouched with their matched houses until the end of the algorithm. Thus the procedure converges to some matching and, by Theorem 1, this matching is popular. ■

The restriction to the groups of three agents is not compulsory as the same algorithm works when groups of larger size are also allowed.

Corollary 2. A popular market with groups of arbitrary size converges to a popular matching whenever it exists.

5 Conclusions

In the current paper we propose a novel characterization of "global" popularity via "local" popularity, and also show that locally popular exchanges lead to a globally popular matching.

One important open question is about the convergence speed of popular markets. To answer this question one may need to design a more efficient algorithm: our greedy algorithm does many unnecessary steps, for instance when it repeatedly runs the same chains. We cannot simply avoid these steps as then we cannot build a triple that blocks the current matching. However, it might be possible if we use alternative algorithms.

Another open question is about popular markets in situations when popular matchings do not exist. Perhaps, these markets converge to some stationary probabilistic distribution over the set of matchings, and it is reasonable to deem the more probable matchings as more popular. Both questions are interesting but hard.

APPENDIX

Proof of convergence of popular markets.

The first part of the algorithm.

Let μ be the arbitrary initial matching where each agent is endowed with some house: for each $a \in A$ $\mu(a) \neq \emptyset$. Let us fix some ordering of agents $A = \{a_1, \dots, a_n\}$.

For steps $k = 1, \dots, n$ we make the following exchanges.

If in step k house $\mu(a_k)$ is the best house for agent a_k , then proceed to step $k + 1$ without changing the current matching μ . Otherwise, consider house $h \neq \mu(a_k)$ that is the best house of agent a_k . If this house h is empty, $\mu(h) = \emptyset$, then we give it to agent a_k in the new matching $\mu'(a_k) = h$. Otherwise, consider the owner of h , $\mu(h)$.

If h is the best for its owner $\mu(h)$, then proceed to the next step $k+1$ without changing the current matching μ . Otherwise, consider the best house for agent $\mu(h)$: $h' \neq h$. If $h' = \mu(a_k)$ or $\mu(h') = \emptyset$ then make the mutually beneficial two-way exchange: $\mu'(a_k) = h, \mu'(\mu(h)) = h'$. Otherwise, if $\mu(h') \notin \{a_k, \mu(h), \emptyset\}$ we make the three-way exchange: $\mu'(a_k) = h, \mu'(\mu(h)) = h', \mu'(\mu(h')) = \mu(a_k)$. This exchange is beneficial for at least two of the three agents.

After each of the above exchanges the number of agents that own their best houses goes up, and each agent gets his best house unless it is taken by some other agent. Thus after $x \leq n$ exchanges we get a new matching μ where each agent gets either his first house, his second house, or a bad house.

Denote the number of agents who get a bad house by $\beta(\mu)$. At least $x \geq 1$ agents get their first house, therefore $\beta(\mu) \leq n - x \leq n - 1$. Note that $n - \beta(\mu)$ agents get either a first house or a second house.

The second part of the algorithm.

We will make exchanges that *weakly* decrease the number of agents with a bad house $\beta(\mu)$.

Consider some agent $\mu(t)$ that gets a bad house t . If his second house s is free, we give him s : $\mu'(\mu(t)) = s$ and decrease $\beta(\mu)$ by one. Otherwise there is some agent $\mu(s)$ that owns s , and s might be his bad house or his second house (but not his first house from the definition of second house). We now study these two cases.

1. Let s be a bad house for $\mu(s)$. Denote the second house of $\mu(s)$ as h . If $h = t$ or empty, then make the two-way exchange decreasing $\beta(\mu)$ by 2. Otherwise, make the three-way exchange $\mu'(\mu(t)) = s, \mu'(\mu(s)) = h, \mu'(\mu(h)) = t$, decreasing $\beta(\mu)$ by 1, 2 or 3 depending on how the owner of h ranks t .

2. Let s be the second house for $\mu(s)$. Let f be the first house for agent $\mu(s)$. From the first part of the algorithm we know that f is also the first house of his owner $\mu(f)$. Make the following three-way exchange: $\mu'(\mu(t)) = s, \mu'(\mu(s)) = f, \mu'(\mu(f)) = t$. If t is the second house for agent $\mu(f)$, then $\beta(\mu)$ decreases by one.

Thus $\beta(\mu)$ is only constant if house s is the second house for both $\mu(t)$ and $\mu(s)$, house f is the first house for both $\mu(s)$ and $\mu(f)$, and house t is a bad house for both agents $\mu(t)$ and $\mu(f)$. Denote such exchange as *bad*. We show now that a sequence of these bad exchanges in which $\beta(\mu)$ remains constant is finite.

Current matching μ before and after a bad three-way exchange which keeps $\beta(\mu)$ a constant.

$\mu(t)$	$\mu(s)$	$\mu(f)$
s	f	f
<u>t</u>	<u>s</u>	t

$\mu(t)$	$\mu(s)$	$\mu(f)$
<u>s</u>	f	f
t	s	<u>t</u>

2.1 Let f be the first house also for agent $\mu(t)$. For convenience denote $f = f_1, s = s_1, \mu(t) = 1, \mu(s) = 2, \mu(f) = 3$. By Hall's theorem the second house for agent 3 cannot be the same as $s_1, s_3 \neq s_1$ (otherwise three agents have the same first house and the same

second house, and thus a popular matching does not exist). After the bad exchange among agents 1,2,3 the bad house t is matched to agent 3. Consider another chain of three agents that starts with the bad house t . Denote $\mu(s_3) = 4$. Note that $f_4 \neq f_1$ (otherwise four agents have the same first house, two of them have the same second house, and the other two of them also have the same second house, and thus a popular matching does not exist). Denote $\mu(f_4) = 5$. By Hall's theorem $s_5 \notin \{s_1, s_3\}$ (otherwise, similar to the previous arguments the popular matching does not exist). After the bad exchange between agents 3,4,5 the bad house is matched with agent 5, and so forth.

Current matching μ before and after two bad three-way exchanges.

1	2	3	4	5
f_1	f_1	f_1	f_4	f_4
s_1	s_1	s_3	s_3	s_5
t		t		t

1	2	3	4	5
f_1	f_1	f_1	f_4	f_4
s_1	s_1	s_3	s_3	s_5
t		t		t

In each such bad exchange two new agents enter the chain, these agents own their first and second houses. Then, we need not more than $(n + 2 - \beta(\mu))/2$ bad exchanges to reduce $\beta(\mu)$.

2.2 Let the first house f_1 for agent $\mu(t)$ be different from house f . Denote $f = f_2, s = s_1, \mu(t) = 1, \mu(s) = 2, \mu(f) = 3$. After one bad exchange agent 3 would be matched to house t .

Assume that the second house for agent 3 $s_3 \neq s_1$ – we did not meet s_3 earlier in the chain.⁷ Consider agent 4 that owns his second house s_3 . Assume that agent 4's first house f_4 was not previously in the chain: $f_4 \neq f_1, f_2$.⁸

Consider the next agent 5 and so on: we get a chain of agents such that each two neighbours have either the same first house or the same second house. Eventually we arrive to some agent k that has the same first or second house as earlier in the chain.

Let agent k be the first agent in the chain such that his first house is f_1 . Then after $(k \bmod 2)$ exchanges he gets f_1 and each other agent among $1, \dots, k$ receives either his second or first house and, by Hall's theorem, for this profile this is the unique possibility in each popular matching.

Now let agent k be the first agent in the chain such that his first house has already appeared in the chain (twice – since two neighbouring agents have the same first house) and is different from f_1 . Initially he owns his second house $\mu(k) = s_k$, and after $(k \bmod 2)$ he gets his first house and starts a series of bad exchanges along the same chain but in the opposite direction. Each agent that received his second house in the first series of exchanges now gets his first house, and vice versa. Thus agent 1 that started the chain now gets his first house f_1 (and house t is matched to $\mu(f_1)$). Each agent among $1, \dots, k$ receives either his second or first house and, by Hall's theorem, this is the unique possibility in each popular matching.

Similarly, let agent k be the first agent in the chain such that his second house has already appeared in the chain. Then after $(k \bmod 2)$ bad exchanges in one direction or agent k gets the bad house t and then again reverses the direction of bad exchanges such that agent 1 gets house f_1 , which happens after less than k exchanges.

⁷If house s_3 – was earlier in the chain (i.e. if $s_3 = s_1$), then we do not need to implement the bad exchange since by Hall's theorem in any popular matching houses f_2, s_1 must be matched to agents 2,3. In this case in any popular matching agent 1 cannot get his second house and has to get his first house. Yet, if we implement the first bad exchange and t is matched to agent 3, the next bad exchange matches s_1 to agent 3 and f_1 to agent 1, which is exactly what is prescribed by the Hall's theorem.

⁸Specifically, if $f_4 = f_1$ then by Hall's theorem in any popular matching agents 1,2,3,4 share houses f_1, f_2, s_1, s_3 and we need two bad exchanges to do that. If $f_4 = f_2$, then by Hall's theorem in each popular matching agents 3,4 are matched to f_2, s_3 , thus agent 2 is matched to s_1 and agent 1 is matched to f_1 . This is exactly what we get after three bad exchanges.

Eventually, after at most k exchanges we decrease $\beta(\mu)$. In each bad exchange we have one agent with a bad house, others have a first or a second house. In the worst case all agents have a bad house, and each agents that gets his first or his second house participates in each subsequent chain. Thus for the first agent with a bad house ($\beta = n$) we get 1 exchange, for the second agent with a bad house ($\beta = n - 1$) we get 1 exchange, for the third agent with a bad house ($\beta = n - 2$) we get 2 exchanges, and so forth. Therefore, the upper bound is $1 + 1 + 2 + \dots + (n - 1) = (n^2 - n + 2)/2$. ■

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