Axiomatic Characterization of Committee Scoring Rules

Piotr Skowron, Piotr Faliszewski, Arkadii Slinko

Abstract

We provide an axiomatic characterization of committee scoring rules, analogues of the scoring rules for the multiwinner elections. We show that committee scoring rules are characterized by the set of four standard axioms, anonymity, neutrality, consistency and continuity, and by two axioms specific to multiwinner rules, committee neutrality and committee dominance. In the course of our proof, we develop several new notions. In particular, we introduce and characterize decision rules, a notion that may be useful on its own.

1 Introduction

Committee scoring rules, recently introduced by Elkind et al. [15], are multiwinner rules that—on the intuitive level—extend the idea of single-winner scoring protocols to the multiwinner setting. In this paper we confirm this intuition by providing their axiomatic characterization in the style of Young's characterization for the single-winner case [51] (see the work of Merlin [37] for another presentation of Young's result, and the survey of Chebotarev and Shamis [11] for a comprehensive list of related characterizations).

In a multiwinner election, we are given a set of candidates, a collection of voters with preferences over these candidates, and an integer k. The goal is to choose a committee (i.e., a size-k subset of candidates), while respecting these given preferences. Multiwinner elections are very interesting for a number of reasons, and, in particular, due to their very varied applications. Indeed, we can use multiwinner election rules whenever our goal is to select a subset of objects. For example, we may use multiwinner elections to choose a country's parliament, to pick a list of items a search engine should present to the users, to short-list a group of candidates for a prize, to decide which set of products a company should offer to its customers, or even as part of a genetic algorithm [16]. There are many other applications and we point the reader to the works of Lu and Boutilier [34, 35], Elkind et al. [15], and Skowron et al. [45] for a more detailed discussion with an AI focus.

Since there are many different applications of multiwinner elections, there are also many different multiwinner voting rules. For example, there are rules based on the extension of the single-winner Condorcet principle[1] (see, e.g., the works of Fishburn [22], Kaymak and Sanver [27], Ratliff [40], and Barberá and Coelho [5]), there are rules based on approval ballots, where voters express which candidates are and which are not acceptable for them (see, e.g., the works of Kilgour [30], Kilgour and Marshall [32], and Aziz et al. [3]), and there are numerous other rules based on ordinal preference orders, including committee scoring rules. Indeed, even committee scoring rules are very diverse and include simple "best k" rules such as SNTV or k-Borda, more involved rules, such as the Chamberlin–Courant rule [10] that focus on providing proportional representation, or even more involved ones, such as the variants of the OWA-based rules of Skowron et al. [45] and Aziz et al. [4]. Unfortunately, so far our understanding of multiwinner voting is quite limited, especially as compared to the single-winner setting. By providing a Young-style characterization of committee scoring rules, we hope to somewhat rectify this situation.

Committee scoring rules extend their single-winner variants as follows. A single-winner scoring rule is a function that given a position of a candidate in a vote (that is, in the ranking of candidates

[1] Condorcet principle says that if for some candidate c it holds that for every other candidate d a majority of voters prefers c to d, then c should be the unique winner of the given single-winner election.

provided by the voter) outputs a score. The score of a candidate in an election is the sum of the scores he or she gets from all the votes, and the candidates with the highest score win. (For example, the Borda score of candidate c in vote v is the number of candidates that v ranks below c; the Borda rule elects candidates with the highest total Borda score.) The difference in the definition of committee scoring rules is that instead of single candidates we elect size-k committees and, so, we need a different notion of a position. Specifically, we say that the position of a committee in a given vote is the set of the positions of its members in this vote. A committee scoring function assigns scores to such committee positions (with m candidates there are $\binom{m}{k}$ such committee positions) and the total score of a committee is the sum of the scores it gets from the voters. The committee(s) with the highest score win. (Formally, we consider committee scoring rules as rules that rank committees based on a given profile of votes. That is, a committee scoring rule ranks the committees from the one with the highest score to the one with the lowest one. This means that we view committee scoring rules as social welfare functions, generalized to the multiwinner setting. In this respect, our approach is closer to that of Merlin [37] than to that taken originally by Young [51].)

Given the above description, it is quite natural to expect that committee scoring rules are, indeed, the natural generalization of single-winner scoring rules to the multiwinner setting (as much as one can judge naturalness of various notions). Nonetheless, we would like to obtain a formal justification for this intuition and, to this end, we consider Young's characterization of single-winner scoring rules [51]. (We explain the meaning of the axioms below.)

Theorem A (Young's Characterization of Single-Winner Scoring Rules). *A single-winner voting rule is a scoring rule if and only if it is symmetric, consistent, and continuous.*

Informally put, a voting rule is symmetric if it treats all the voters and all the candidates in a uniform way; it is consistent if for each two elections with the same winner, this candidate also wins if we put these two elections together; and it is continuous if it is always possible to ensure a candidate's victory in a given election by appending to it sufficiently many elections where this candidate is a unique winner. We do show that a very similar characterization—where symmetry, consistency, and continuity are adapted to their natural multiwinner variants—does hold. The only difference is that in our characterization we also use two new axioms, committee neutrality and committee dominance, that deal with issues specific to multiwinner elections. Roughly put, committee neutrality says that if a committee A is better than committee B in some election, then it is also better after we permute members of A and B in some of the votes (but without changing the positions of committees A and B in these votes). Committee dominance ensures a certain monotonicity constraint that Elkind et al. [15] put on committee scoring rules. (Young disregards monotonicity considerations, but if one were to use a typical definition of a single-winner scoring rules where the score assigned to a higher position has to be greater or equal to the scores assigned to lower positions, then one would have to extend Young's characterization in a similar spirit.) Informally stated, our main result is as follows.

Theorem B (Characterization of Committee Scoring Rules). *A multiwinner voting rule is a committee scoring rule if and only if it is symmetric, consistent, continuous, committee neutral, and satisfies committee dominance.*

Including the committee neutrality axiom is inevitable, but also very natural. On one hand, committee scoring rules do satisfy it, and, on the other hand, it is trivially satisfied for single-winner scoring rules (if we take their winners to be single-member committees, then there is nothing to permute).

We believe that committee scoring rules form a very important class of multiwinner rules and, thus, deserve a detailed research program. Our characterization is a natural step of this program. In addition to providing a formal argument that committee scoring rules are a natural extension of single-winner scoring rules to the multiwinner setting, the result is useful for the following reasons:

1. Having an axiomatic characterization is very helpful for understanding the nature of committee scoring rules. (Indeed, due to Young's [51] work and the research that followed, scoring rules are among the best-understood and most widely studied single-winner voting rules.)

2. Many rules are much easier to characterize within the class of committee scoring rules than in the unrestricted setting. For example, Faliszewski et al. [18] characterize the Bloc rule among committee scoring rules. Using our results, one obtains an unrestricted characterization.

3. Our proof approach requires introducing several new notions and ideas. In particular, we discuss decision rules, which given two committees can tell which one is better (but we do not impose the transitivity requirement on such rules). While we use them for technical reasons, we envision that they could find an application of their own.

2 Definitions and Notations

For each positive integer t, by $[t]$ we mean the set $\{1, \ldots, t\}$, and by $[t]_k$ we mean the set of all k-element subsets of $[t]$. For each set X and each $k \in \mathbb{N}$, by $S_k(X)$ we denote the set of all k-element subsets of X (so, in particular, we have that $S_k([t]) = [t]_k$). For a given set X, we let $\Pi_{>}(X)$ and $\Pi_{\geq}(X)$ denote, respectively, the set of all linear orders and the set of all weak orders over X.

2.1 Multiwinner Elections

Let $A = \{a_1, \ldots, a_m\}$ be the set of all the candidates, and let $N = \{1, 2, \ldots\}$ be the set of all the possible voters. We refer to the members of $S_k(A)$ as size-k committees, or, simply, as committees, when k is clear from the context. For each subset $V \subseteq N$, by $\mathcal{P}(V)$ we denote the set of all $|V|$-element tuples of elements from $\Pi_{>}(A)$, indexed by the elements of V. We refer to the elements of $\mathcal{P}(V)$ as preference profiles for the set of voters V. We set $\mathcal{P} = \{P \in \mathcal{P}(V) : V \subseteq N\}$ to be the set of all possible preference profiles. For each preference profile $P \in \mathcal{P}(V)$ by $\mathrm{Vot}(P)$ we denote the set of all the voters in P (in particular, we have that for each $P \in \mathcal{P}(V)$ it holds that $\mathrm{Vot}(P) = V$). For each profile P and each voter $v \in \mathrm{Vot}(P)$, by $P(v)$ we denote the preference order of v in P.

Our proof approach crucially relies on using what we call *k-decision rules*. A *k-decision rule* f_k,

$$f_k \colon \mathcal{P} \to \Big(S_k(A) \times S_k(A) \to \{-1, 0, 1\}\Big),$$

is a function that for each preference profile $P \in \mathcal{P}$ provides a mapping, $f_k(P)$, that compares size-k committees (for each two size-k committees C_1 and C_2, we require that $f_k(P)(C_1, C_2) = -f_k(P)(C_2, C_1)$). For each preference profile $P \in \mathcal{P}$, a given $k \in \mathbb{N}$, and each two k-element subsets $C_1, C_2 \in S_k(A)$, we write $C_1 \succ_P C_2$ if $f_k(P)(C_1, C_2) = 1$ (the intended meaning is that according to the decision rule f_k applied to profile P, committee C_1 is preferred over committee C_2). Analogously, we write $C_1 =_P C_2$ if $f_k(P)(C_1, C_2) = 0$ (with the intended meaning that for profile P, rule f_k says that both committees are equally good). We write $C_1 \succeq_P C_2$ if $C_1 \succ_P C_2$ or $C_1 =_P C_2$. Sometimes, when P is a more involved expression, we write $C_1 \succeq[P]\, C_2$ instead of $C_1 \succeq_P C_2$ and $C_1 =[P]\, C_2$ instead of $C_1 =_P C_2$. (Note that for decision rules we do not require the relations $\succ_P$ and $\succeq_P$ to be transitive; we will add this requirement for multiwinner election rules.) By $C_1 \prec_P C_2$ we mean $C_2 \succ_P C_1$, and by $C_1 \preceq_P C_2$ we mean $C_2 \succeq_P C_1$.

Since the just-introduced notation does not specify the rule f_k explicitly, the reader may be worried about possible confusion. While, on one hand, including f_k in the notation would make the discussion more explicit, it would also make the notation even more heavy than it already is.[2] Since

[2] As the reader will see in Section A.1, our notation is quite explicit and—unfortunately—heavy in a number of places. We experimented with several approaches and the current one is a result of many weeks of work on improving the clarity and readability of the manuscript.

most of the time we use the notation with respect to a single k-decision rule (fixed at the beginning of our proof), we believe that the benefits of the abbreviated notation outweigh the drawbacks.

A *k-winner election rule* f_k is a k-decision rule that additionally satisfies the transitivity requirement, i.e., it is a k-decision rule such that for each profile P and each three committees C_1, C_2, and C_3 it holds that

$$C_1 \succeq_P C_2 \land C_2 \succeq_P C_3 \implies C_1 \succeq_P C_3.$$

A multiwinner election rule f is a family $(f_k)_{k \in \mathbb{N}}$ of k-winner rules, with one k-winner rule for each committee size k. We remark that often multiwinner rules are defined to simply return the set of winning committees, whereas in our case they—effectively—return weak orders over all possible committees of a given size. Our definition is a bit more general and provides a multiwinner analog of the notion of a social welfare function.

2.2 Committee Scoring Rules

For a preference order $\pi \in \Pi_{>}(A)$, by $\mathrm{pos}_\pi(a)$ we denote the position of a in π (the top-ranked candidate has position 1 and the bottom-ranked candidate has position m). A score function assigns a number of points to each position in a preference order. For example, we define the Borda score function, $\beta \colon [m] \to \mathbb{N}$, so that $\beta(i) = m - i$. Similarly, for each $t \in [m]$, we define the t-Approval score function, α_t, so that if $i \leq t$ then $\alpha_t(i) = 1$ and otherwise $\alpha_t(i) = 0$.

We extend the notion of a position of a candidate to the case of committees in the following way. For each preference order $\pi \in \Pi_{>}(A)$ and each committee $C \in S_k(A)$, by $\mathrm{pos}_\pi(C)$ we mean the set $\mathrm{pos}_\pi(C) = \{\mathrm{pos}_\pi(a) : a \in C\}$. By a *committee scoring function for committees of size k*, we mean a function λ, $\lambda : [m]_k \to \mathbb{R}$, that for each element of $[m]_k$, interpreted as a position of a committee, assigns a score. (An intuitive understanding is that the score measures the satisfaction of a voter from a committee that he or she ranks at the given position.) By a *committee scoring function* we mean a family of committee scoring functions for all possible committee sizes.

We require that committee scoring functions satisfy the following dominance relation. Consider committee size k, and let I and J be two sets from $[m]_k$ (i.e., two committee positions) such that $I = \{i_1, \ldots, i_k\}$, $J = \{j_1, \ldots, j_k\}$, and it holds that $i_1 < \cdots < i_k$ and $j_1 < \cdots < j_k$. We say that I dominates J if for each $t \in [k]$ we have $i_t \leq j_t$ (note that this notion might be referred to as "weak dominance" as well, since a set dominates itself). If I dominates J, then we require that $\lambda(I) \geq \lambda(J)$. For each set of voters $V \subseteq N$ and each preference profile $P \in \mathcal{P}(V)$, by $\mathrm{score}_\lambda(C, P)$ we denote the total score that the voters from V assign to committee C. Formally, we have that $\mathrm{score}_\lambda(C, P) = \sum_{v \in \mathrm{Vot}(P)} \lambda(\mathrm{pos}_{P(v)}(C))$.

A multiwinner election rule is a *committee scoring rule* if there exists a committee scoring function λ such that for each two equal-size committees C_1 and C_2, we have that $C_1 \succ_P C_2$ if and only if $\mathrm{score}_\lambda(C_1, P) > \mathrm{score}_\lambda(C_2, P)$, and $C_1 =_P C_2$ if and only if $\mathrm{score}_\lambda(C_1, P) = \mathrm{score}_\lambda(C_2, P)$. Committee scoring rules were introduced by Elkind et al. [15] and were later studied by Faliszewski et al. [18, 17] (very related notions were considered by Skowron et al. [45] and by Aziz et al. [3, 4]). Below we present some examples of committee scoring rules:

1. The single non-transferable vote rule (the SNTV rule) uses scoring function $\lambda_{\mathrm{SNTV}}(i_1, \ldots, i_k) = \sum_{t=1}^{k} \alpha_1(i_t)$. In other words, for a given voter it assigns score 1 to every committee that contains the highest-ranked candidate, thus SNTV selects k candidates which are ranked on top by most voters.

2. The Bloc rule uses function $\lambda_{\mathrm{Bloc}}(i_1, \ldots, i_k) = \sum_{t=1}^{k} \alpha_k(i_t)$, i.e., the score a committee gets from a single vote is the number of committee members this vote has among its top k positions.

3. The k-Borda rule uses function $\lambda_{k\text{-Borda}}(i_1, \ldots, i_k) = \sum_{t=1}^{k} \beta(i_t)$, i.e., the score a committee gets from a single vote is the sum of the Borda scores of the committee members.

4. The Borda-based Chamberlin–Courant rule (due to Chamberlin and Courant [10]) is defined through function $\lambda_{\beta\text{-CC}}(i_1, \ldots, i_k) = \beta(\min(i_1, \ldots, i_k))$. Intuitively, under the Chamberlin–Courant rule the highest-ranked member of a committee is treated as the representative for the given voter, and this voter assigns to the committee the Borda score of his or her representative.

Elkind et al. [15] and Faliszewski et al. [18, 17] discuss a number of subclasses of committee scoring rules, showing that each of the above rules is, in some well-defined way, rather different from the other ones. Indeed, using their nomenclature (not described here), SNTV is the only nontrivial separable representation-focused rule, Bloc is the only nontrivial weakly separable top-k-counting rule, k-Borda is separable (but neither representation-focused nor top-k-counting), and the Borda-based Chamberlin–Courant rule is representation-focused (but neither separable nor top-k-counting). Elkind et al. [15] and Faliszewski et al.[18, 17] analyze these subclasses of committee scoring rules and show that they differ quite significantly both in terms of axiomatic properties and in terms of computational properties of the rules included in them (we provide more discussion in Appendix B). Yet, in this work we deal with the class of committee scoring rules as a whole and do not consider its internal structure.

3 Axioms for Our Characterization

In this section we describe the axiomatic properties that we use in our characterization of committee scoring rules. Apart from committee-neutrality and committee dominance, these properties are natural, straightforward generalizations of the respective notions from the world of single-winner elections. We formulate them for the case of k-decision rules (for a given value of k), but since k-winner rules are a type of k-decision rules, the properties apply to k-winner rules as well. For each of our properties $\mathfrak{P}$, we say that a multiwinner election rule $f = \{f_k\}_{k \in \mathbb{N}}$ satisfies $\mathfrak{P}$ if f_k satisfies $\mathfrak{P}$ for each $k \in \mathbb{N}$.

We start by recalling the definitions of anonymity and neutrality, two properties that are very natural requirements for fair election rules. Intuitively, anonymity says that neither of the voters is specifically privileged nor discriminated against, whereas neutrality says the same for the candidates.

Definition 1 (Anonymity)**.** *We say that a k-decision rule f_k is* anonymous *if for each two (not necessarily different) sets of voters $V, V' \subseteq N$ such that $|V| = |V'|$, for each bijection $\rho : V \to V'$ and for each two preference profiles $P_1 \in \mathcal{P}(V)$ and $P_2 \in \mathcal{P}(V')$ such that $P_1(v) = P_2(\rho(v))$ for each $v \in V$, it holds that $f_k(P_1) = f_k(P_2)$.*

Let σ be a permutation of the set of candidates A. For a committee C, by $\sigma(C)$ we mean the committee $\{\sigma(c) \colon c \in C\}$. For a linear order $\pi \in \Pi_{>}(A)$, by $\sigma(\pi)$ we denote the linear order such that for each two candidates a and b we have $a \ \pi \ b \iff \sigma(a) \ \sigma(\pi) \ \sigma(b)$. For a given k-decision rule f_k and profile P, by $\sigma(f_k(P))$ we mean the function such that for each two size-k committees C_1 and C_2 it holds that $\sigma(f_k(P))(\sigma(C_1), \sigma(C_2) = f_k(P)(C_1, C_2)$.

Definition 2 (Neutrality)**.** *A k-decision rule f_k is* neutral *if for each permutation σ of A and each two preference profiles P_1, P_2 over the same voter set V, such that $P_1(v) = \sigma(P_2(v))$ for each $v \in V$, it holds that $f_k(P_1) = \sigma(f_k(P_2))$.*

A rule that is anonymous and neutral is called *symmetric*. We note that our definition of anonymity resembles the ones used by Young [51] or Merlin [37] rather than the traditional ones, as presented by May [36] or Arrow [1]. The difference comes from the fact that we (just like Young and Merlin) need to consider elections with differing voter sets. Indeed, the next axiom describes a situation where we merge two elections with disjoint voter sets.

Definition 3 (Consistency)**.** *A k-decision rule f_k is* consistent *if for each two profiles P and P' over disjoint sets of voters, $V \subset N$ and $V' \subset N$, $V \cap V' = \emptyset$, and each two committees $C_1, C_2 \in S_k(A)$,* *(i) if $C_1 \succ_P C_2$ and $C_1 \succeq_{P'} C_2$, then it holds that $C_1 \succ_{P+P'} C_2$ (where $P + P'$ denotes the profile that consists of all the voters from P and P') and (ii) if $C_1 \succeq_P C_2$ and $C_1 \succeq_{P'} C_2$, then it holds that $C_1 \succeq_{P+P'} C_2$.*

In some sense, consistency is the most essential element of Young's characterization, that distinguishes single-winner scoring rules from the other single-winner rules. (Though we mention that Kemeny rules satisfies the reinforcement axiom, which is very similar to consistency, but different.[3] We point the reader to the work of Young and Levenglick for an axiomatic characterization of the Kemeny's rule [52].)

Remark 1. *In our proofs, we often use the consistency axiom in the following way. Let C_1 and C_2 be two committees and let P and Q be two profiles over disjoint voter sets, such that $C_1 \succ_{P+Q} C_2$ and $C_1 =_P C_2$. Using consistency, we can conclude that $C_1 \succ_Q C_2$. Indeed, if, for example, it were the case that $C_2 \succ_Q C_1$ then by consistency (as applied to merging profiles P and Q) we would have to conclude that $C_2 \succ_{P+Q} C_1$ and we assumed the opposite.*

The next axiom is specific to the multiwinner setting and, to the best of our knowledge, is new to this paper. Informally speaking, it says that each voter views each committee as a coherent body rather than as a set of independent members (and, thus, the value of a committee does not change if we swap the positions of some committee members).

Definition 4 (Committee Neutrality)**.** *A k-decision rule f_k is* committee-neutral *if for each two committees $C_1, C_2 \in S_k(A)$, each voter set V, and each two preference profiles $P_1, P_2 \in \mathcal{P}(V)$ such that P_1 and P_2 agree on all but a single vote v, the following holds: If it is possible to obtain $P_1(v)$ from $P_2(v)$ by permuting candidates within set $C_1 \setminus C_2$, within set $C_2 \setminus C_1$, and within set $C_1 \cap C_2$, then it holds that $C_1 \succ_{P_1} C_2 \iff C_1 \succ_{P_2} C_2$ and $C_1 =_{P_1} C_2 \iff C_1 =_{P_2} C_2$.*

Our next axiom regards the dominance relation between committee positions, and specifies a basic monotonicity condition (it can also be viewed as a form of Pareto dominance).

Definition 5 (Committee Dominance)**.** *A k-decision rule f_k has the* committee dominance *property if for each two committees $C_1, C_2 \in S_k(A)$ and each profile P where for every vote $v \in P$, $\mathrm{pos}_{P(v)}(C_1)$ dominates $\mathrm{pos}_{P(v)}(C_2)$, it holds that $C_1 \succeq_P C_2$.*

The definition of committee scoring rules requires that if λ is a committee scoring function (for committee size k), and I and J are two committee positions such that I dominates J, then $\lambda(I) \geq \lambda(J)$. In our characterization, we use the committee dominance axiom to ensure that this property indeed holds.

Finally, we define the continuity property, which ensures that if a certain set of voters V prefers C_1 over C_2, then for each set of voters V', disjoint from V, there exists some (possibly large) number n, such that if we clone V exactly n times and add such a profile to V', then in this final profile C_1 is preferred to C_2 (note that when we speak of cloning voters, we implicitly assume that the decision rule is anonymous and that the identities of the cloned voters do not matter). Thus, continuity might be viewed as a kind of "large enough majority always gets its choice" principle.

Definition 6 (Continuity)**.** *An anonymous k-decision rule f_k is* continuous *if for each two committees $C_1, C_2 \in S_k(A)$ and each two profiles P_1 and P_2 where $C_1 \succ_{P_2} C_2$, there exists a number $n \in \mathbb{N}$ such that for the profile Q that consists of the profile P_1 and of n copies of the profile P_2, it holds that $C_1 \succ_Q C_2$.*

[3] In essence, if we were to express the reinforcement axiom in our language, it would be defined in the same way as our consistency axiom, except that it would only by applicable to profiles P and P' such that $f_k(P) = f_k(P')$. That is, reinforcement puts a condition on the result of a decision rule, provided that the rule is applied to some profile $P + P'$ such that the result of comparing each two committees is the same both under P and under P'. Our variant of consistency puts a condition on the result of a decision rule, provided that the rule is applied to some profile $P + P'$ such that there is at least one pair of committees whose comparison under P and P' gives the same result. In effect, our consistency axiom is far stronger than the reinforcement axiom.

For more discussion on continuity, we refer the reader to the original work of Young [51]. Briefly put, the role of the continuity axiom is to avoid certain tie-breaking issues (Young's characterization without continuity defines rules that are scoring rules that break ties using other scoring rules, that break ties using yet other scoring rules, and so on).

4 Main Result

We now present our main result, the axiomatic characterization of committee scoring rules.

Theorem 1. *A multiwinner election rule f is symmetric, committee-neutral, consistent, continuous, and has the committee dominance property if and only if it is a committee scoring rule.*

The proof is rather involved and we divide it into two main parts. The full proof is provided in the appendix: in the main text we only include its high level idea and the sketch of the first part of the proof. We do so for several reasons. First, we have restricted space. Second, the first part contains completely new ideas, whereas the second part implements a complicated inductive step inspired by Young's original approach (yet we emphasize that our reasoning there is new and contains complex new technical arguments and conceptual ideas). Third, we believe that the notion of committee-pairs scoring rule has potential that has not yet been unleashed and, thus, we want to promote the notion. Fourth, it is much easier to present the first part of the proof as a closed unit.

In the first part, in Section A.1, we show that preference profiles can be represented as elements of a certain linear space and we prove a variant of Theorem 1 for the case of decision rules (through Theorem 2 and Corollary 1), i.e., for the case where we do not require transitivity. Naturally, in this case the theorem does not speak of committee scoring rules (since the outcomes of committee scoring rules always have to be transitive), but, instead, we introduce a somewhat more general notion of *committee-pairs scoring rules*. A committee-pairs scoring rule is defined through a *committee-pairs scoring function*, which assigns a score (possibly negative) to a pair of committee positions. A committee-pairs scoring rule, given two committees, C_1 and C_2, and a profile P, computes for each voter $v \in \mathrm{Vot}(P)$ the committee-pairs score of $(\mathrm{pos}_{P(v)}(C_1), \mathrm{pos}_{P(v)}(C_2))$. Then it sums up these scores and, if the value is positive, it declares C_1 to be the preferred over C_2; if the sum is zero, it declares the committees to be equally good; and if the sum is negative, it declares C_2 to be preferred over C_1.

The main technical conclusion from the analysis provided for decision rules is that the set of profiles for which two arbitrary committees, C_1 and C_2, are judged as equally good, is a linear subspace of dimension $m! - 1$ in the linear space of all preference profiles of dimension $m!$ (intuitively—for the case of anonymous rules—every profile can be seen as an $m!$-dimensional vector specifying how many times each possible vote repeats). The second important conclusion, provided in Lemma 4, says that we only need to characterize the subspace where C_1 and C_2 are judged as equally good. If a given (nontrivial) scoring function correctly identifies committees as equivalent in the subspace of profiles P such that $C_1 =_P C_2$, then it must correctly identify the profiles P' where $C_1 \succ_{P'} C_2$.

We use the results from Section A.1 in two ways. First, Corollary 1 is an interesting characterization in its own right. Second, Theorem 2 and Lemma 4 jointly provide a very strong technical tool for the second part of the proof.

The second part of the proof is included in Sections A.2 and A.3. In Section A.2, we prove Theorem 1 for the case where f is used to recognize in which profiles a certain committee C_1 is preferred over some other committee C_2, such that $|C_1 \cap C_2| = k - 1$. If $|C_1 \cap C_2| = k - 1$ then there are only two candidates, let us refer to them as c_1 and c_2, such that $C_1 = (C_1 \cap C_2) \cup \{c_1\}$, and $C_2 = (C_1 \cap C_2) \cup \{c_2\}$. Thus, this case closely resembles the single-winner setting, studied by Young [51] and Merlin [37]. For each two candidates c_1 and c_2, Young and Merlin show a base of the linear space of preference profiles that satisfies the following two properties:

(i) For each preference profile in the base, the scores of c_1 and c_2 are equal according to every possible scoring function.

(ii) Candidates c_1 and c_2 are "symmetric" and, thus, every neutral and anonymous voting rule has to judge them as equally good.

These observation allow one to use geometric arguments to note that the set of profiles in which c_1 is preferred over c_2 can be separated from the set of profiles in which c_2 is preferred over c_1 by a hyperplane. The coefficients of the linear equation that specifies this hyperplane define a single-winner scoring rule, and this scoring rule is exactly the voting rule that one started with. In Section A.2 we use the same geometric arguments, but the construction of the appropriate base is more sophisticated. Indeed, finding this base is the core technical part of Section A.2.

In Section A.3 we extend the result from Section A.2 to the case of any two committees (irrespective of the size of their intersection), concluding the proof. Here, finding appropriate base seems even harder and, consequently, we use a different technique. To deal with committees C_1 and C_2 that have fewer than $k-1$ elements in common, we form a third committee, C_3, whose intersections with C_1 and C_2 have more elements than the intersection of C_1 and C_2. Then, using an inductive argument, we conclude that the space of profiles P where $C_1 =_P C_3$ is $(m!-1)$-dimensional, and that the same holds for the space of profiles P such that $C_2 =_P C_3$. An intersection of two linear spaces with this dimension has dimension at least $m! - 2$ and, so, we have an at-least-$(m-2)!$ dimensional space of profiles P such that $C_1 =_P C_2$. Using combinatorial tricks, we find a profile P' which does not belong to the space but for which $C_1 =_{P'} C_2$ still holds. This gives us our $(m-1)!$-dimensional space. By applying results from the first part of the proof, this suffices to conclude that the committee scoring function that we found for committees that differ in at most one element works for all other committees as well.

The whole discussion, i.e., this and the next section, and the appendix, is divided into small subsections, each with a title describing its main outcome. These section titles are intended to help the reader navigate through the proof, but otherwise one can read the text as a continuous piece. In particular, all the notations, conventions, and definitions carry over from one subsection to the next, and so on.

4.1 First Part of the Proof: Decision Rules

In this section we provide the proof for the variant of our main result for k-decision rules. Some proofs of our lemmas in this section are omited from the main text, but the reader can find them in the appendix. Recall that the outcomes of k-decision rules do not need to be transitive. That is, for a k-decision rule f_k it is possible to have a profile P and three committees such that $C_1 \succ_P C_2$, $C_2 \succ_P C_3$, and $C_3 \succ_P C_1$. Omitting transitivity allows us to derive necessary technical results.

Setting up the Decision Rule. Let us fix some arbitrary committee size k, and let f_k be some symmetric, consistent, committee-neutral, and continuous k-decision rule. Our goal in this part of the proof is to show that this rule must be a committee-pairs scoring rule (we will define this notion soon), and to obtain a number of technical tools for the second part of the proof provided in the appendix.

First Domain Change Anonymity of f_k allows us to use a more convenient domain for representing preference profiles. Specifically, we can view f_k as a function:

$$f_k : \mathbb{N}^{m!} \to (S_k(A) \times S_k(A) \to \{-1, 0, 1\}),$$

with the domain $\mathbb{N}^{m!}$ instead of $\mathcal{P}$ (recall the definition of a k-decision rule in Section 2.1). Each element P of $\mathbb{N}^{m!}$, referred to as a *voting situation*, describes how many voters cast each of the $m!$

possible different votes. We view voting situations as vectors of nonnegative integers, indexed with preference orders from $\Pi_{>}(A)$ (that is, for $\pi \in \Pi_{>}(A)$ and voting situation P, by $P(\pi)$ we mean the number of voters in P with preference order π). This will be helpful in our further analysis because it allows us to perform algebraic operations on voting situations, such as adding them or multiplying them by constants. For a voting situation P and a constant $c \in \mathbb{N}$, cP is the voting situation that corresponds to P in which each voter was replaced by c copies of him or her. Similarly, for two voting situations P and Q, $P + Q$ is the voting situation obtained by merging P and Q.

Independence of Committee Comparison from Irrelevant Swaps We observe that for each two committees C_1 and C_2, the result of their comparison according to f_k depends only on the positions on which they are ranked by the voters (and not on the positions of candidates not in $C_1 \cup C_2$).

Lemma 1. *Let C_1 and C_2 be two size-k committees, let P be a voting situation, and let a and b be two candidates such that $a, b \notin C_1 \cup C_2$. Let v be a vote in P. By $P[v, a \leftrightarrow b]$ we mean the voting situation obtained from P by swapping a and b in v. It holds that $C_1 \succeq_P C_2 \iff C_1 \succeq_{P[v,a\leftrightarrow b]} C_2$.*

Proof. We assume that $C_1 \succeq_P C_2$ and we will show that $C_1 \succeq_{P[v,a\leftrightarrow b]} C_2$. For the sake of contradiction, we assume that this is not the case and that $C_2 \succ_{P[v,a\leftrightarrow b]} C_1$ holds.

We rename the candidates so that $C_1 \setminus C_2 = \{a_1, \dots, a_\ell\}$ and $C_2 \setminus C_1 = \{b_1, \dots, b_\ell\}$, and we define σ to be a permutation (over the set of candidates) that for each $x \in [\ell]$ swaps a_x and b_x, but leaves all the other candidates intact. That is, $\sigma(a_1) = b_1, \sigma(b_1) = a_1, \dots, \sigma(a_\ell) = b_\ell, \sigma(b_\ell) = a_\ell$, and for each candidate $c \notin \{a_1, \dots, a_\ell, b_1, \dots, b_\ell\}$ it holds that $\sigma(c) = c$. Since $C_1 \succeq_P C_2$, by neutrality we have that $C_2 \succeq_{\sigma(P)} C_1$. Due to our assumptions, it holds that $C_2 \succ_{P[v,a\leftrightarrow b]} C_1$ and, by consistency, we get that:

$$C_2 \succ [\sigma(P) + P[v, a \leftrightarrow b]] \, C_1. \tag{1}$$

We write $v[a \leftrightarrow b]$ to denote the vote (and, by our convention, a single-vote voting situation) obtained from v by swapping candidates a and b. Let $Q = v[a \leftrightarrow b] + \sigma(v)$ be a voting situation that consists of $v[a \leftrightarrow b]$ and $\sigma(v)$. We observe that $\sigma(P) - \sigma(v) = \sigma(P[v, a \leftrightarrow b] - v[a \leftrightarrow b])$. This is because $P[v, a \leftrightarrow b] - v[a \leftrightarrow b]$ is the same as $P - v$. Since we have that:

$$\sigma(P) + P[v, a \leftrightarrow b] - Q \quad = \quad (\sigma(P) - \sigma(v)) + (P[v, a \leftrightarrow b] - v[a \leftrightarrow b]),$$

and the two voting situations in the final sum are symmetric with respect to σ and committees C_1 and C_2, by symmetry of f_k we have that:

$$C_2 = [\sigma(P) + P[v, a \leftrightarrow b] - Q] \, C_1. \tag{2}$$

Thus, by using consistency—as applied to equations (1) and (2) in the way described in Remark 1—we get that $C_2 \succ_Q C_1$. Let δ be a permutation that is identical to σ, except that it also swaps a and b. We note that:

$$\delta(v[a \leftrightarrow b]) = \sigma(v) \quad \text{and} \quad \delta(\sigma(v)) = v[a \leftrightarrow b].$$

That is, $\delta(Q) = Q$, and, by symmetry, it must be the case that $C_2 =_Q C_1$. However, this is a contradiction with our previous conclusion that $C_2 \succ_Q C_1$. This completes the proof. □

Committee-Pairs Scoring Rules Before proceeding further, we need to define *committee-pairs scoring rules*, a useful class of k-decision rules. These rules are somehow similar to committee scoring rules, with the difference that the scores of two committees cannot be computed independently. Instead, for each pair of committee positions (I_1, I_2) we define a numerical value, the score that a voter assigns to the pair of committees (C_1, C_2) under the condition that C_1 and C_2 stand in this voter's preference order on positions I_1 and I_2, respectively (this score can be negative). If the total score of a pair of committees (C_1, C_2) is positive, then C_1 is preferred over C_2; if it is negative, then C_2 is preferred over C_1; if it is equal to zero, then C_1 and C_2 are seen as equally good.

Definition 7 (Committee-pairs scoring rules)**.** *Let $d : [m]_k \times [m]_k \to \mathbb{R}$ be a* committee-pairs scoring function*, that is, a function that for each pair of committee positions (I_1, I_2), where $I_1, I_2 \in [m]_k$, returns a score value (possibly negative), such that for each I_1 and I_2, it holds that $d(I_1, I_2) = -d(I_2, I_1)$. For each preference profile $P \in \mathcal{P}$ and for each pair of committees (C_1, C_2), we define the score:*

$$\mathrm{score}_d(C_1, C_2, P) = \sum_{v \in \mathrm{Vot}(P)} d(\mathrm{pos}_{P(v)}(C_1), \mathrm{pos}_{P(v)}(C_2)).$$

A k-decision rule is a committee-pairs scoring rule *if there exists a committee-pairs scoring function d such that for each voting situation P and each two committees C_1 and C_2 it holds that: (i) $C_1 \geq_P C_2 \iff \mathrm{score}_d(C_1, C_2, P) \geq 0$, and (ii) $C_1 =_P C_2 \iff \mathrm{score}_d(C_1, C_2, P) = 0$.*

Naturally, we can speak of applying committee-pairs scoring rules to voting situations instead of applying them to preference profiles. For a voting situation $P \in \mathcal{P}$, the score of committee pair (C_1, C_2) is:

$$\mathrm{score}_d(C_1, C_2, P) = \sum_{\pi \in \Pi_>(A)} P(\pi) \cdot d(\mathrm{pos}_\pi(C_1), \mathrm{pos}_\pi(C_2)).$$

(Recall that for $\pi \in \Pi_>(A)$, $P(\pi)$ is the number of voters with preference order π within P.)

Putting the Focus on Two Fixed Committees Recall that we have assumed f_k to be symmetric, consistent, committee-neutral and continuous. Now, we fix a pair of size-k committees, C_1 and C_2, and define f_{C_1,C_2} to be the rule that acts on voting situations in the same way as f_k does, but with the difference that it only distinguishes whether (i) C_1 is preferred over C_2, or (ii) C_2 is preferred over C_2, or (iii) C_1 and C_2 are seen as equally good. In other words, we set $f_{C_1,C_2}(P)$ to be -1, 0 or 1 depending on $f_k(P)$ ranking C_1 lower than, equivalent to, or higher than C_2, respectively (f_k can be viewed as the collection of rules $f_{C'_1,C'_2}$, one for each possible pair of committees C'_1 and C'_2).

We define function $f_{k,|C_1 \cap C_2|}$ so that for each voting situation P, relation $f_{k,|C_1 \cap C_2|}(P)$ is the restriction of $f_k(P)$ to pairs of committees C'_1, C'_2 such that $|C'_1 \cap C'_2| = |C_1 \cap C_2|$.

Defining the Distinguished Profiles For each two committee positions I_1 and I_2 such that $|I_1 \cap I_2| = |C_1 \cap C_2|$, we consider a single-vote voting situation $v(C_1 \to I_1, C_2 \to I_2)$, where C_1 and C_2 are ranked on positions I_1 and I_2, respectively, and all the other candidates are ranked arbitrarily, but in some fixed, predetermined order.

Let us consider two cases. First, let us assume that for each two committee positions I_1 and I_2 such that $|I_1 \cap I_2| = |C_1 \cap C_2|$, it holds that C_1 is equivalent to C_2 relative to $v(C_1 \to I_1, C_2 \to I_2)$ (i.e., that $C_1 =[v(C_1 \to I_1, C_2 \to I_2)]\ C_2$ holds for each two committee positions I_1, I_2 such that $|I_1 \cap I_2| = |C_1 \cap C_2|$). By committee-neutrality and by Lemma 1, we infer that for each single-vote voting situation v we have $C_1 =_v C_2$ (because the set of positions shared by C_1 and C_2 always has the same cardinality). Further, by consistency, we conclude that f_{C_1,C_2} is trivial, i.e., for every voting situation P it holds that $C_1 =_P C_2$. By neutrality, we get that $f_{k,|C_1 \cap C_2|}$ also is trivial (i.e., it declares equivalent each two committees whose intersection has $|C_1 \cap C_2|$ candidates). Of course, in this case f_k is a committee-pairs scoring rule (with trivial scoring function).

If the above case does not hold, then there are some two committee positions, I^*_1 and I^*_2, such that $|I^*_1 \cap I^*_2| = |C_1 \cap C_2|$ and C_1 is not equivalent to C_2 relative to $v(C_1 \to I^*_1, C_2 \to I^*_2)$. Without loss of generality we assume that $C_1 >[v(C_1 \to I^*_1, C_2 \to I^*_2)]\ C_2$. A big chunk of the discussion that follows is aimed at showing that in this case we also can conclude that f_k is a committee-pairs scoring rule.

For each two committee positions I_1 and I_2 such that $|I_1 \cap I_2| = |I^*_1 \cap I^*_2| = |C_1 \cap C_2|$, and for each two nonnegative integers x and y, we define voting situation:

$$P^{y(C_1 \to I^*_1, C_2 \to I^*_2)}_{x(C_1 \to I_1, C_2 \to I_2)} = y \cdot \big(v(C_1 \to I^*_1, C_2 \to I^*_2)\big) + x \cdot \big(v(C_1 \to I_1, C_2 \to I_2)\big).$$

That is, in this voting situation there are y voters that rank C_1 and C_2 on positions, respectively, I_1^* and I_2^*, and there are x voters that rank C_1 and C_2 on positions, respectively, I_1 and I_2.

Deriving the Components for the Committee-Pairs Scoring Function for f_k We now proceed toward defining a committee-pairs scoring function for f_k. To this end, we define the value Δ_{I_1,I_2} as:

$$\Delta_{I_1,I_2} = \begin{cases} \sup\left\{\frac{y}{x} \colon C_2 \succ\left[P^{y(C_1 \to I_1^*, C_2 \to I_2^*)}_{x(C_1 \to I_2, C_2 \to I_1)}\right] C_1, \quad x, y \in \mathbb{N}\right\} & \text{for } C_1 \succ[v(C_1 \to I_1, C_2 \to I_2)]\ C_2 \\ -\inf\left\{\frac{y}{x} \colon C_2 \succ\left[P^{y(C_1 \to I_2^*, C_2 \to I_1^*)}_{x(C_1 \to I_2, C_2 \to I_1)}\right] C_1, \quad x, y \in \mathbb{N}\right\} & \text{for } C_2 \succ[v(C_1 \to I_1, C_2 \to I_2)]\ C_1 \\ 0 & \text{for } C_1 =[v(C_1 \to I_1, C_2 \to I_2)]\ C_2. \end{cases} \tag{3}$$

This definition certainly might seem unintuitive at first. However, we will show that the values Δ_{I_1,I_2}, for all possible I_1 and I_2, in essence, define a committee-pairs scoring function for f_k. The next few lemmas should build an intuition for the nature of these values and for how this committee-pairs scoring rule operates. As a quick sanity check, the reader can verify that $\Delta_{I_1^*,I_2^*} = 1$.[4]

Let us argue that the values Δ_{I_1,I_2} are well defined. To this end, let us fix some committee positions I_1 and I_2 (such that $|I_1 \cap I_2| = |I_1^* \cap I_2^*|$). First, due to continuity of f_k, we see that the appropriate sets in Equation (3) are non-empty. For example, if $C_1 \succ[v(C_1 \to I_1, C_2 \to I_2)]\ C_2$ and, thus, $C_2 \succ[v(C_1 \to I_2, C_2 \to I_1)]\ C_1$, then continuity of f_k ensures that there exists (possibly large) x such that $C_2 \succ\left[P^{(C_1 \to I_1^*, C_2 \to I_2^*)}_{x(C_1 \to I_2, C_2 \to I_1)}\right] C_1$. This proves that the set from the first condition of (3) is nonempty. An analogous reasoning proves the same fact for the set from the second condition in (3). Second, we claim that the value Δ_{I_1,I_2} needs to be finite. This is evident for the case where we take the infimum over the set of positive rational numbers. For the case where we take the supremum, this follows from Lemma 2, below.

Lemma 2. *For each two committee positions I_1 and I_2 (such that $|I_1 \cap I_2| = |I_1^* \cap I_2^*|$) it holds that $\Delta_{I_2,I_1} = -\Delta_{I_1,I_2}$.*

The next lemma shows that Δ_{I_1,I_2} provides a threshold value for proportions of voters in distinguished profiles with respect to the relation between C_1 and C_2.

Lemma 3. *Let I_1 and I_2 be two committee positions such that $|I_1 \cap I_2| = |I_1^* \cap I_2^*|$. For each $x, y \in \mathbb{N}$ it holds that:*

1. *if $C_1 \succ[v(C_1 \to I_1, C_2 \to I_2)]\ C_2$ and $\frac{y}{x} < \Delta_{I_1,I_2}$, then $C_2 \succ\left[P^{y(C_1 \to I_1^*, C_2 \to I_2^*)}_{x(C_1 \to I_2, C_2 \to I_1)}\right] C_1$,*
2. *if $C_2 \succ[v(C_1 \to I_1, C_2 \to I_2)]\ C_1$ and $\frac{y}{x} > -\Delta_{I_1,I_2}$, then $C_2 \succ\left[P^{y(C_1 \to I_2^*, C_2 \to I_1^*)}_{x(C_1 \to I_2, C_2 \to I_1)}\right] C_1$.*

Putting Together the Committee-Pairs Scoring Function for f_k We are ready to define a committee-pairs scoring function d for f_k. For each two committee positions, I_1 and I_2, we set:

$$d(I_1, I_2) = \Delta_{I_1,I_2},$$

which allows us to rewrite Definition 7 as follows (considering P to be a voting situation):

$$\mathrm{score}_d(C_1, C_2, P) = \sum_{\pi \in \Pi_{>}(A)} P(\pi) \cdot \Delta_{\mathrm{pos}_{P,v}(C_1), \mathrm{pos}_{P,v}(C_2)}.$$

Note that by Lemma 2, d indeed satisfies the conditions put on a committee-pairs scoring rule.

The next theorem shows that d is a committee-pairs scoring function for f_{C_1,C_2}. Based on this result, we will argue that it works for all pairs of committees.

---

[4] Briefly put, we first note that for each positive integer z, we have $C_1 =[P^{z(C_1 \to I_1^*, C_2 \to I_2^*)}_{z(C_1 \to I_2^*, C_2 \to I_1^*)}]\ C_2$. Further, due to consistency of f_k (used as in Remark 1), we observe that $C_1 \succ[P^{y(C_1 \to I_1^*, C_2 \to I_2^*)}_{x(C_1 \to I_2^*, C_2 \to I_1^*)}]\ C_2$ whenever $y > x$ and that $C_2 \succ[P^{y(C_1 \to I_1^*, C_2 \to I_2^*)}_{x(C_1 \to I_2^*, C_2 \to I_1^*)}]\ C_1$ whenever $y < x$. We conclude that $\Delta_{I_1^*,I_2^*} = \sup\left\{\frac{y}{x} : y < x,\ \text{for } x, y \in \mathbb{N}_+\right\} = 1$.

Theorem 2. *For each voting situation P it holds that (i) if* $\mathrm{score}_d(C_1, C_2, P) > 0$*, then* $C_1 \succ_P C_2$*, and (ii) if* $\mathrm{score}_d(C_1, C_2, P) = 0$*, then* $C_1 =_P C_2$*.*

We have proven Theorem 2 for the function f_{C_1,C_2}. From neutrality it follows that the thesis holds for $f_{k,|C_1 \cap C_2|}$. Now, let us argue that it also holds for f_k. Indeed, for the thesis of Theorem 2 we needed to define a committee-pairs scoring function d only for such pairs (I_1, I_2) that $|I_1 \cap I_2| = |C_1 \cap C_2|$. Since the choice of C_1 and C_2 was arbitrary, for each value of i, $0 \le i \le k$, we can take arbitrary sets $C_1 = C_{1,i}$ and $C_2 = C_{2,i}$ such that $|C_{1,i} \cap C_{2,i}| = i$, repeat the reasoning from this section, and obtain the thesis of Theorem 2 for a committee-pairs scoring function d defined for such pairs (I_1, I_2) that $|I_1 \cap I_2| = i$. Repeating this for each i allows us to derive the full-fledged function d for pairs (I_1, I_2) of arbitrary size of intersection, and consequently, to prove Theorem 2 for f_k.

Young-Style Characterization of Committee-Pairs Scoring Rules This way we have proven an analogous result to our main Theorem 1 for decision rules.

Corollary 1. *A decision rule is symmetric, committee-neutral, consistent, and continuous if and only if it is a committee-pair scoring rule.*

We conclude this section by arguing that this corollary is likely to be far more important than simply being a by product on our way to characterizing committee scoring rules. Indeed, we believe that there are natural committee-pairs scoring rules that are not committee scoring rules, and that still are very useful. For example, let us consider a committee pairs scoring rule (for committee size k) defined through the following committee pairs scoring function d:

$$d(I_1, I_2) = \begin{cases} 1 & \text{if all the positions in } I_1 \setminus I_2 \text{ precede all the positions in } I_2 \setminus I_1, \\ -1 & \text{if all the positions in } I_2 \setminus I_1 \text{ precede all the positions in } I_1 \setminus I_2, \\ 0 & \text{otherwise.} \end{cases}$$

If we take committee size to be $k = 1$, then given a preference profile P, the committee-pairs scoring rule defined by d simply returns a weighted equivalent of P's majority graph. A majority graph for a preference profile is a directed graph where each candidate is a vertex, and for each two candidates c and d there is a directed edge from c to d exactly if more voters prefer c to d.

For committee sizes $k > 1$, our committee-pairs scoring rule would give us some generalization of the majority graph for committees. Indeed, in some sense, every committee-pairs scoring rule can be seen as providing some sort of generalization of the majority graph. Naturally, given a majority graph (or some its generalization provided by a committee-pairs scoring rule), it is not at all obvious which candidate (or which committee) should be a winner. However, there is a whole theory of tournament solution concepts that one can employ to figure this out (we point the reader to Laslier's textbook for an overview [33]). While deeper study of committee-pairs scoring rules and decision rules is not in the scope of this paper, we believe it is a very interesting direction for future research.

5 Conclusions

We have provided an axiomatic characterization of committee scoring rules, a new class of multiwinner voting rules recently introduced by Elkind et al. [15]. Committee scoring rules form a remarkably general class of multiwinner systems that consists of many nontrivial rules with a variety of applications. Thus, our characterization constitutes a fundamental framework for further axiomatic studies of this fascinating class and makes an important step towards their understanding. We mention that various properties of committee scoring rules, and the internal structure of the class, were already studied by Elkind et al. [15] and Faliszewski et al. [18, 17]. However, they mostly focused on specific rules and on subclasses of the whole class.

Our main theorem required developing a set of useful tools and new concepts, such as decision rules. We believe that they are an interesting notion that deserves further study.

References

[1] K. J. Arrow. *Social Choice and Individual Values*. Wiley, New York, 2nd edition, 1963.

[2] B. Asplach. Johnson graphs are hamilton-connected. *Ars Mathematica Contemporanea*, 6(1):21–23, 2013.

[3] H. Aziz, M. Brill, V. Conitzer, E. Elkind, R. Freeman, , and T. Walsh. Justified representation in approval-based committee voting. In *Proceedings of the 29th Conference on Artificial Intelligence (AAAI-2015)*, 2015.

[4] H. Aziz, S. Gaspers, J. Gudmundsson, S. Mackenzie, N. Mattei, and T. Walsh. Computational aspects of multi-winner approval voting. In *Proceedings of the 14th International Conference on Autonomous Agents and Multiagent Systems*, May 2015.

[5] S. Barberá and D. Coelho. How to choose a non-controversial list with k names. *Social Choice and Welfare*, 31(1):79–96, 2008.

[6] S. Barberà and D. Coelho. How to choose a non-controversial list with k names. *Social Choice and Welfare*, 31(1):79–96, 2008.

[7] S. Barber and B. Dutta. Implementability via protective equilibria. *Journal of Mathematical Economics*, 10(1):49–65, 1982.

[8] N. Betzler, A. Slinko, and J. Uhlmann. On the computation of fully proportional representation. *Journal of Artificial Intelligence Research*, 47:475–519, 2013.

[9] Felix Brandt, Markus Brill, and Paul Harrenstein. Tournament solutions. In F. Brandt, V. Conitzer, U. Endriss, J. Lang, and A. D. Procaccia, editors, *Handbook of Computational Social Choice*, chapter 3. Cambridge University Press, 2016.

[10] B. Chamberlin and P. Courant. Representative deliberations and representative decisions: Proportional representation and the borda rule. *American Political Science Review*, 77(3):718–733, 1983.

[11] P. Chebotarev and E. Shamis. Characterizations of scoring methods for preference aggregation. *Annals of Operations Research*, 80:299–332, 1998.

[12] S. Ching. A simple characterization of plurality rule. *Journal of Economic Theory*, 71(1):298–302, 1996.

[13] A. Copeland. A "reasonable" social welfare function. Mimeographed notes from a Seminar on Applications of Mathematics to the Social Sciences, University of Michigan, 1951.

[14] B. Debord. An axiomatic characterization of Borda's k-choice function. *Social Choice and Welfare*, 9(4):337–343, 1992.

[15] E. Elkind, P. Faliszewski, P. Skowron, and A. Slinko. Properties of multiwinner voting rules. In *Proceedings of the 13th International Conference on Autonomous Agents and Multiagent Systems (AAMAS-2014)*, May 2014. Also presented in FIT-2013.

[16] P. Faliszewski, J. Sawicki, R. Schaefer, and M. Smolka. Multiwinner voting in genetic algorithms for solving illposed global optimization problems. In *Proceedings of the 19th International Conference on the Applications of Evolutionary Computation*, 2016. To appear.

[17] P. Faliszewski, P. Skowron, A. Slinko, and N. Talmon. Committee scoring rules: Axiomatic classification and hierarchy. In *Proceedings of the 25th International Joint Conference on Artificial Intelligence (IJCAI-2016)*, 2016.

[18] P. Faliszewski, P. Skowron, A. Slinko, and N. Talmon. Multiwinner analogues of the plurality rule: Axiomatic and algorithmic views. In *Proceedings of the 30th Conference on Artificial Intelligence (AAAI-2016)*, 2016.

[19] D.S. Felsenthal and Z. Maoz. Normative properties of four single-stage multi-winner electoral procedures. *Behavioral Science*, 37:109–127, 1992.

[20] P. Fishburn. *The Theory of Social Choice,*. Princeton University Press, 1973.

[21] P. Fishburn. Condorcet social choice functions. *SIAM Journal on Applied Mathematics*, 33(3):469–489, 1977.

[22] P. Fishburn. Majority committees. *Journal of Economic Theory*, 25(2):255–268, 1981.

[23] P. Fishburn and W. Gehrlein. Borda's rule, positional voting, and condorcet's simple majority principle. *Public Choice*, 28(1):79–88.

[24] R. Freeman, M. Brill, and V. Conitzer. On the axiomatic characterization of runoff voting rules. In *Proceedings of the 28th Conference on Artificial Intelligence (AAAI-2014)*, pages 675–681, 2014.

[25] P. Gärdenfors. Positionalist voting functions. *Theory and Decision*, 4(1):1–24, 1973.

[26] A. Gibbard. Manipulation of voting schemes. *Econometrica*, 41(4):587–601, 1973.

[27] B. Kaymak and R. Sanver. Sets of alternatives as condorcet winners. *Social Choice and Welfare*, 20(3):477–494, 2003.

[28] B. Kaymak and R. Sanver. Sets of alternatives as condorcet winners. *Social Choice and Welfare*, 20(3):477–494, 2003.

[29] M. G. Kendall. A new measure of rank correlation. *Biometrika*, 30(1/2):81–93, 1938.

[30] D. Kilgour. Approval balloting for multi-winner elections. In J. Laslier and R. Sanver, editors, *Handbook on Approval Voting*, pages 105–124. Springer, 2010.

[31] M. Kilgour. Approval balloting for multi-winner elections. In *Handbook on Approval Voting*. Springer, 2010. Chapter 6.

[32] M. Kilgour and E. Marshall. Approval balloting for fixed-size committees. In *Electoral Systems, Studies in Choice and Welfare*, volume 12, pages 305–326, 2012.

[33] J. Laslier. *Tournament Solutions and Majority Voting*. Springer, 1997.

[34] T. Lu and C. Boutilier. Budgeted social choice: From consensus to personalized decision making. In *Proceedings of the 22nd International Joint Conference on Artificial Intelligence (IJCAI-2011)*, pages 280–286, 2011.

[35] T. Lu and C. Boutilier. Value directed compression of large-scale assignment problems. In *Proceedings of the 29th Conference on Artificial Intelligence (AAAI-2015)*, pages 1182–1190, 2015.

[36] K. O. May. A set of independent necessary and sufficient conditions for simple majority decision. *Econometrica*, 20(4):680–684, 1952.

[37] V. Merlin. The axiomatic characterization of majority voting and scoring rules. *Mathematical Social Sciences*, 41(161):87–109, 2003.

[38] Peter C. P. Fishburn. Axioms for approval voting: Direct proof. *Journal of Economic Theory*, 19(1):180–185, 1978.

[39] A. Procaccia, J. Rosenschein, and A. Zohar. On the complexity of achieving proportional representation. *Social Choice and Welfare*, 30(3):353–362, April 2008.

[40] T. Ratliff. Some startling inconsistencies when electing committees. *Social Choice and Welfare*, 21(3):433–454, 2003.

[41] J. T. Richelson. A characterization result for the plurality rule. *Journal of Economic Theory*, 19(2):548–550, 1978.

[42] M. Satterthwaite. Strategy-proofness and Arrow's conditions: Existence and correspondence theorems for voting procedures and social welfare functions. *Journal of Economic Theory*, 10(2):187–217, 1975.

[43] M. Sertel. Characterizing approval voting. *Journal of Economic Theory*, 45(1):207–211, 1988.

[44] P. Skowron. What do we elect committees for? a voting committee model for multi-winner rules. In *Proceedings of the 24th International Joint Conference on Artificial Intelligence (IJCAI-2015)*, pages 1141–1148, 2015.

[45] P. Skowron, P. Faliszewski, and J. Lang. Finding a collective set of items: From proportional multirepresentation to group recommendation. In *Proceedings of the 29th Conference on Artificial Intelligence (AAAI-2015)*, 2015.

[46] P. Skowron, P. Faliszewski, and A. Slinko. Achieving fully proportional representation: Approximability result. *Artificial Intelligence*, 222:67–103, 2015.

[47] J. Smith. Aggregation of preferences with variable electorate. *Econometrica*, 41(6):1027–1041, 1973.

[48] J. Smith. Aggregation of preferences with variable electorate. *Econometrica*, 41(6):1027–1041, 1973.

[49] R. Yager. On ordered weighted averaging aggregation operators in multicriteria decisionmaking. *IEEE Transactions on Systems, Man and Cybernetics*, 18(1):183–190, 1988.

[50] H. Young. An axiomatization of Borda's rule. *Journal of Economic Theory*, 9(1):43–52, 1974.

[51] H. Young. Social choice scoring functions. *SIAM Journal on Applied Mathematics*, 28(4):824–838, 1975.

[52] H. Young and A. Levenglick. A consistent extension of Condorcet's election principle. *SIAM Journal on Applied Mathematics*, 35(2):285–300, 1978.

Piotr Skowron
Department of Computer Science
University of Oxford
United Kingdom
Email: p.k.skowron@gmail.com

Piotr Faliszewski
Wydział Informatyki, Elektroniki i Telekomunikacji
AGH University of Science and Technology
Krakow, Poland
Email: faliszew@agh.edu.pl

Arkadii Slinko
Department of Mathematics
University of Auckland
New Zealand
Email: a.slinko@auckland.ac.nz

A Proof of the Main Result

In this section we provide the full proof of the main theorem. For the sake of consistency in this text we also include the high-level comments from the main text.

Theorem 1. *A multiwinner election rule f is symmetric, committee-neutral, consistent, continuous, and has the committee dominance property if and only if it is a committee scoring rule.*

The proof is rather involved and we divide it into two main parts. In the first part, in Section A.1, we show that preference profiles can be represented as elements of a certain linear space and we prove a variant of Theorem 1 for the case of decision rules (through Theorem 2 and Corollary 1), i.e., for the case where we do not require transitivity. Naturally, in this case the theorem does not speak of committee scoring rules (since the outcomes of committee scoring rules always have to be transitive), but, instead, we introduce a somewhat more general notion of *committee-pairs scoring rules*. A committee-pairs scoring rule is defined through a *committee-pairs scoring function*, which assigns a score (possibly negative) to a pair of committee positions. A committee-pairs scoring rule, given two committees, C_1 and C_2, and a profile P, computes for each voter $v \in \mathrm{Vot}(P)$ the committee-pairs score of $(\mathrm{pos}_{P(v)}(C_1), \mathrm{pos}_{P(v)}(C_2))$. Then it sums up these scores and, if the value is positive, it declares C_1 to be the preferred over C_2; if the sum is zero, it declares the committees to be equally good; and if the sum is negative, it declares C_2 to be preferred over C_1.

The main technical conclusion from the analysis provided for decision rules is that the set of profiles for which two arbitrary committees, C_1 and C_2, are judged as equally good, is a linear subspace of dimension $m! - 1$ in the linear space of all preference profiles of dimension $m!$ (intuitively—for the case of anonymous rules—every profile can be seen as an $m!$-dimensional vector specifying how many times each possible vote repeats). The second important conclusion, provided in Lemma 4, says that we only need to characterize the subspace where C_1 and C_2 are judged as equally good. If a given (nontrivial) scoring function correctly identifies committees as equivalent in the subspace of profiles P such that $C_1 =_P C_2$, then it (or its negation) must correctly identify the profiles P' where $C_1 >_{P'} C_2$.

We use the results from Section A.1 in two ways. First, Corollary 1 is an interesting characterization in its own right. Second, Theorem 2 and Lemma 4 jointly provide a very strong technical tool for the second part of the proof.

The second part of the proof is included in Sections A.2 and A.3. In Section A.2, we prove Theorem 1 for the case where f is used to recognize in which profiles a certain committee C_1 is preferred over some other committee C_2, such that $|C_1 \cap C_2| = k - 1$. If $|C_1 \cap C_2| = k - 1$ then there are only two candidates, let us refer to them as c_1 and c_2, such that $C_1 = (C_1 \cap C_2) \cup \{c_1\}$, and $C_2 = (C_1 \cap C_2) \cup \{c_2\}$. Thus, this case closely resembles the single-winner setting, studied by Young [51] and Merlin [37]. For each two candidates c_1 and c_2, Young and Merlin show a base of the linear space of preference profiles that satisfies the following two properties:

(i) For each preference profile in the base, the scores of c_1 and c_2 are equal according to every possible scoring function.

(ii) Candidates c_1 and c_2 are "symmetric" and, thus, every neutral and anonymous voting rule has to judge them as equally good.

These observation allow one to use geometric arguments to note that the set of profiles in which c_1 is preferred over c_2 can be separated from the set of profiles in which c_2 is preferred over c_1 by a hyperplane. The coefficients of the linear equation that specifies this hyperplane define a single-winner scoring rule, and this scoring rule is exactly the voting rule that one started with. In Section A.2 we use the same geometric arguments, but the construction of the appropriate base is more sophisticated. Indeed, finding this base is the core technical part of Section A.2.

In Section A.3 we extend the result from Section A.2 to the case of any two committees (irrespective of the size of their intersection), concluding the proof. Here, finding appropriate base seems even harder and, consequently, we use a different technique. To deal with committees C_1 and C_2 that have fewer than $k-1$ elements in common, we form a third committee, C_3, whose intersections with C_1 and C_2 have more elements than the intersection of C_1 and C_2. Then, using an inductive argument, we conclude that the space of profiles P where $C_1 =_P C_3$ is $(m!-1)$-dimensional, and that the same holds for the space of profiles P such that $C_2 =_P C_3$. An intersection of two linear spaces with this dimension has dimension at least $m! - 2$ and, so, we have an at-least-$(m-2)!$ dimensional space of profiles P such that $C_1 =_P C_2$. Using combinatorial tricks, we find a profile P' which does not belong to the space but for which $C_1 =_{P'} C_2$ still holds. This gives us our $(m-1)!$-dimensional space. By applying results from the first part of the proof, this suffices to conclude that the committee scoring function that we found for committees that differ in at most one element works for all other committees as well.

A.1 First Part of the Proof: Decision Rules

We start our analysis by considering k-decision rules. Recall that the outcomes of k-decision rules do not need to be transitive. That is, for a k-decision rule f_k it is possible to have a profile P and three committees such that $C_1 \succ_P C_2$, $C_2 \succ_P C_3$, and $C_3 \succ_P C_1$. Omitting transitivity allows us to derive necessary technical results.

The whole discussion, i.e., this and the next section, is divided into small subsections, each with a title describing its main outcome. These section titles are intended to help the reader navigate through the proof, but otherwise one can read the text as a continuous piece. In particular, all the notations, conventions, and definitions carry over from one subsection to the next, and so on.

Setting up the Decision Rule. Let us fix some arbitrary committee size k, and let f_k be some symmetric, consistent, committee-neutral, and continuous k-decision rule. Our goal in this part of the proof is to show that this rule must be a committee-pairs scoring rule (we will define this notion soon), and to obtain a number of technical tools for the second part of the proof.

First Domain Change Anonymity of f_k allows us to use a more convenient domain for representing preference profiles. Specifically, we can view f_k as a function:

$$f_k : \mathbb{N}^{m!} \to (S_k(A) \times S_k(A) \to \{-1, 0, 1\}),$$

with the domain $\mathbb{N}^{m!}$ instead of $\mathcal{P}$ (recall the definition of a k-decision rule in Section 2.1). Each element P of $\mathbb{N}^{m!}$, referred to as a *voting situation*, describes how many voters cast each of the $m!$ possible different votes. We view voting situations as vectors of nonnegative integers, indexed with preference orders from $\Pi_>(A)$ (that is, for $\pi \in \Pi_>(A)$ and voting situation P, by $P(\pi)$ we mean the number of voters in P with preference order π). This will be helpful in our further analysis because it allows us to perform algebraic operations on voting situations, such as adding them or multiplying them by constants. For a voting situation P and a constant $c \in \mathbb{N}$, cP is the voting situation that corresponds to P in which each voter was replaced by c copies of him or her. Similarly, for two voting situations P and Q, $P + Q$ is the voting situation obtained by merging P and Q.

Given a voting situation P, when we speak of "some vote v in P," we mean "some preference order that occurs within P." We implicitly treat each vote v (i.e., each preference order) as a voting situation that contains this vote only. When we say that we modify some vote within some voting situation P, we mean modifying only one copy of this vote, and not all the votes that have the same preference order. This is different from the case of preference profiles where we distinguished votes from voters—we recall that we used the symbol v to denote voters and $P(v)$ to denote the vote of v within P. However, we think that this will not lead to any confusion and our arguments will become clearer.

Independence of Committee Comparison from Irrelevant Swaps We observe that for each two committees C_1 and C_2, the result of their comparison according to f_k depends only on the positions on which they are ranked by the voters (and not on the positions of candidates not in $C_1 \cup C_2$).

Lemma 1. *Let C_1 and C_2 be two size-k committees, let P be a voting situation, and let a and b be two candidates such that $a, b \notin C_1 \cup C_2$. Let v be a vote in P. By $P[v, a \leftrightarrow b]$ we mean the voting situation obtained from P by swapping a and b in v. It holds that $C_1 \succeq_P C_2 \iff C_1 \succeq_{P[v,a\leftrightarrow b]} C_2$.*

Proof. This proof is provided in the main text of the manuscript. □

Committee-Pairs Scoring Rules Before proceeding further, we need to define *committee-pairs scoring rules*, a useful class of k-decision rules. These rules are somehow similar to committee scoring rules, with the difference that the scores of two committees cannot be computed independently. Instead, for each pair of committee positions (I_1, I_2) we define a numerical value, the score that a voter assigns to the pair of committees (C_1, C_2) under the condition that C_1 and C_2 stand in this voter's preference order on positions I_1 and I_2, respectively (this score can be negative). If the total score of a pair of committees (C_1, C_2) is positive, then C_1 is preferred over C_2; if it is negative, then C_2 is preferred over C_1; if it is equal to zero, then C_1 and C_2 are seen as equally good.

Definition 7 (Committee-pairs scoring rules)**.** *Let $d : [m]_k \times [m]_k \to \mathbb{R}$ be a* committee-pairs scoring function*, that is, a function that for each pair of committee positions (I_1, I_2), where $I_1, I_2 \in [m]_k$, returns a score value (possibly negative), such that for each I_1 and I_2, it holds that $d(I_1, I_2) = -d(I_2, I_1)$. For each preference profile $P \in \mathcal{P}$ and for each pair of committees (C_1, C_2), we define the score:*

$$\mathrm{score}_d(C_1, C_2, P) = \sum_{v \in \mathrm{Vot}(P)} d(\mathrm{pos}_{P(v)}(C_1), \mathrm{pos}_{P(v)}(C_2)).$$

A k-decision rule is a committee-pairs scoring rule *if there exists a committee-pairs scoring function d such that for each voting situation P and each two committees C_1 and C_2 it holds that: (i) $C_1 \succeq_P C_2 \iff \mathrm{score}_d(C_1, C_2, P) \geq 0$, and (ii) $C_1 =_P C_2 \iff \mathrm{score}_d(C_1, C_2, P) = 0$.*

Naturally, we can speak of applying committee-pairs scoring rules to voting situations instead of applying them to preference profiles. For a voting situation $P \in \mathcal{P}$, the score of committee pair (C_1, C_2) is:

$$\mathrm{score}_d(C_1, C_2, P) = \sum_{\pi \in \Pi_>(A)} P(\pi) \cdot d(\mathrm{pos}_\pi(C_1), \mathrm{pos}_\pi(C_2)).$$

(Recall that for $\pi \in \Pi_>(A)$, $P(\pi)$ is the number of voters with preference order π within P.)

Putting the Focus on Two Fixed Committees Recall that we have assumed f_k to be symmetric, consistent, committee-neutral and continuous. Now, we fix a pair of size-k committees, C_1 and C_2, and define f_{C_1,C_2} to be the rule that acts on voting situations in the same way as f_k does, but with the difference that it only distinguishes whether (i) C_1 is preferred over C_2, or (ii) C_2 is preferred over C_2, or (iii) C_1 and C_2 are seen as equally good. In other words, we set $f_{C_1,C_2}(P)$ to be -1, 0 or 1 depending on $f_k(P)$ ranking C_1 lower than, equivalent to, or higher than C_2, respectively (f_k can be viewed as the collection of rules $f_{C'_1,C'_2}$, one for each possible pair of committees C'_1 and C'_2).

We define function $f_{k,|C_1 \cap C_2|}$ so that for each voting situation P, relation $f_{k,|C_1 \cap C_2|}(P)$ is the restriction of $f_k(P)$ to pairs of committees C'_1, C'_2 such that $|C'_1 \cap C'_2| = |C_1 \cap C_2|$.

Defining the Distinguished Profiles For each two committee positions I_1 and I_2 such that $|I_1 \cap I_2| = |C_1 \cap C_2|$, we consider a single-vote voting situation $v(C_1 \to I_1, C_2 \to I_2)$, where C_1 and C_2 are ranked on positions I_1 and I_2, respectively, and all the other candidates are ranked arbitrarily, but in some fixed, predetermined order.

Let us consider two cases. First, let us assume that for each two committee positions I_1 and I_2 such that $|I_1 \cap I_2| = |C_1 \cap C_2|$, it holds that C_1 is equivalent to C_2 relative to $v(C_1 \to I_1, C_2 \to I_2)$ (i.e., that $C_1 =[v(C_1 \to I_1, C_2 \to I_2)]\ C_2$ holds for each two committee positions I_1, I_2 such that $|I_1 \cap I_2| = |C_1 \cap C_2|$). By committee-neutrality and by Lemma 1, we infer that for each single-vote voting situation v we have $C_1 =_v C_2$ (because the set of positions shared by C_1 and C_2 always has the same cardinality). Further, by consistency, we conclude that f_{C_1,C_2} is trivial, i.e., for every voting situation P it holds that $C_1 =_P C_2$. By neutrality, we get that $f_{k,|C_1 \cap C_2|}$ also is trivial (i.e., it declares equivalent each two committees whose intersection has $|C_1 \cap C_2|$ candidates). Of course, in this case f_k is a committee-pairs scoring rule (with trivial scoring function).

If the above case does not hold, then there are some two committee positions, I_1^* and I_2^*, such that $|I_1^* \cap I_2^*| = |C_1 \cap C_2|$ and C_1 is not equivalent to C_2 relative to $v(C_1 \to I_1^*, C_2 \to I_2^*)$. Without loss of generality we assume that $C_1 \succ[v(C_1 \to I_1^*, C_2 \to I_2^*)]\ C_2$. A big chunk of the discussion that follows is aimed at showing that in this case we also can conclude that f_k is a committee-pairs scoring rule.

For each two committee positions I_1 and I_2 such that $|I_1 \cap I_2| = |I_1^* \cap I_2^*| = |C_1 \cap C_2|$, and for each two nonnegative integers x and y, we define voting situation:

$$P^{y(C_1 \to I_1^*, C_2 \to I_2^*)}_{x(C_1 \to I_1, C_2 \to I_2)} = y \cdot \left(v(C_1 \to I_1^*, C_2 \to I_2^*)\right) + x \cdot \left(v(C_1 \to I_1, C_2 \to I_2)\right).$$

That is, in this voting situation there are y voters that rank C_1 and C_2 on positions, respectively, I_1^* and I_2^*, and there are x voters that rank C_1 and C_2 on positions, respectively, I_1 and I_2.

Deriving the Components for the Committee-Pairs Scoring Function for f_k We now proceed toward defining a committee-pairs scoring function for f_k. To this end, we define the value Δ_{I_1,I_2} as:

$$\Delta_{I_1,I_2} = \begin{cases} \sup\left\{\frac{y}{x} \colon C_2 \succ\left[P^{y(C_1 \to I_1^*, C_2 \to I_2^*)}_{x(C_1 \to I_2, C_2 \to I_1)}\right] C_1, \quad x, y \in \mathbb{N}\right\} & \text{for } C_1 \succ[v(C_1 \to I_1, C_2 \to I_2)]\ C_2 \\ -\inf\left\{\frac{y}{x} \colon C_2 \succ\left[P^{y(C_1 \to I_2^*, C_2 \to I_1^*)}_{x(C_1 \to I_2, C_2 \to I_1)}\right] C_1, \quad x, y \in \mathbb{N}\right\} & \text{for } C_2 \succ[v(C_1 \to I_1, C_2 \to I_2)]\ C_1 \\ 0 & \text{for } C_1 =[v(C_1 \to I_1, C_2 \to I_2)]\ C_2. \end{cases} \quad (3)$$

This definition certainly might seem unintuitive at first. However, we will show that the values Δ_{I_1,I_2}, for all possible I_1 and I_2, in essence, define a committee-pairs scoring function for f_k. The next few lemmas should build an intuition for the nature of these values and for how this committee-pairs scoring rule operates. As a quick sanity check, the reader can verify that $\Delta_{I_1^*,I_2^*} = 1$.[5]

Let us argue that the values Δ_{I_1,I_2} are well defined. To this end, let us fix some committee positions I_1 and I_2 (such that $|I_1 \cap I_2| = |I_1^* \cap I_2^*|$). First, due to continuity of f_k, we see that the appropriate sets in Equation (3) are non-empty. For example, if $C_1 \succ[v(C_1 \to I_1, C_2 \to I_2)]\ C_2$ and, thus, $C_2 \succ[v(C_1 \to I_2, C_2 \to I_1)]\ C_1$, then continuity of f_k ensures that there exists (possibly large) x such that $C_2 \succ\left[P^{(C_1 \to I_1^*, C_2 \to I_2^*)}_{x(C_1 \to I_2, C_2 \to I_1)}\right] C_1$. This proves that the set from the first condition of (3) is nonempty. An analogous reasoning proves the same fact for the set from the second condition in (3). Second, we claim that the value Δ_{I_1,I_2} needs to be finite. This is evident for the case where we take the infimum over the set of positive rational numbers. For the case where we take the supremum, this follows from Lemma 2, below.

[5] Briefly put, we first note that for each positive integer z, we have $C_1 =[P^{z(C_1 \to I_1^*, C_2 \to I_2^*)}_{z(C_1 \to I_2^*, C_2 \to I_1^*)}]\ C_2$. Further, due to consistency of f_k (used as in Remark 1), we observe that $C_1 \succ[P^{y(C_1 \to I_1^*, C_2 \to I_2^*)}_{x(C_1 \to I_2^*, C_2 \to I_1^*)}]\ C_2$ whenever $y > x$ and that $C_2 \succ[P^{y(C_1 \to I_1^*, C_2 \to I_2^*)}_{x(C_1 \to I_2^*, C_2 \to I_1^*)}]\ C_1$ whenever $y < x$. We conclude that $\Delta_{I_1^*,I_2^*} = \sup\left\{\frac{y}{x} : y < x, \text{ for } x, y \in \mathbb{N}_+\right\} = 1$.

Lemma 2. *For each two committee positions I_1 and I_2 (such that $|I_1 \cap I_2| = |I_1^* \cap I_2^*|$) it holds that $\Delta_{I_2,I_1} = -\Delta_{I_1,I_2}$.*

Proof. We assume, without loss of generality, that $C_1 \succ [v(C_1 \to I_1, C_2 \to I_2)]\, C_2$.[6] Let us consider two sets:

$$U = \left\{ \frac{y}{x} \colon C_2 \succ \left[P^{y(C_1 \to I_1^*, C_2 \to I_2^*)}_{x(C_1 \to I_2, C_2 \to I_1)} \right] C_1, \quad x, y \in \mathbb{N} \right\}$$

(U is the set that we take supremum of in Equation (3)), and:

$$L = \left\{ \frac{y}{x} \colon C_2 \succ \left[P^{y(C_1 \to I_2^*, C_2 \to I_1^*)}_{x(C_1 \to I_1, C_2 \to I_2)} \right] C_1, \quad x, y \in \mathbb{N} \right\}$$

(thus, L is the set that we take infimum of in Equation (3), for Δ_{I_2,I_1}). We will show that $\sup U = \inf L$. First, we show that $\sup U \le \inf L$. For the sake of contradiction, let us assume that this is not the case, i.e., that there exists $\frac{y}{x} \in U$ and $\frac{y'}{x'} \in L$ such that $\frac{y}{x} > \frac{y'}{x'}$. Since $\frac{y}{x} \in U$ and $\frac{y'}{x'} \in L$, we get that:

$$C_2 \succ \left[P^{y(C_1 \to I_1^*, C_2 \to I_2^*)}_{x(C_1 \to I_2, C_2 \to I_1)} \right] C_1 \quad \text{and} \quad C_2 \succ \left[P^{y'(C_1 \to I_2^*, C_2 \to I_1^*)}_{x'(C_1 \to I_1, C_2 \to I_2)} \right] C_1.$$

Let us consider voting situation S:

$$S = y' \cdot P^{y(C_1 \to I_1^*, C_2 \to I_2^*)}_{x(C_1 \to I_2, C_2 \to I_1)} + y \cdot P^{y'(C_1 \to I_2^*, C_2 \to I_1^*)}_{x'(C_1 \to I_1, C_2 \to I_2)}.$$

By consistency, we have that $C_2 \succ_S C_1$. However, let us count the number of voters in S that rank committees C_1 and C_2 on particular positions. There are yy' voters that rank C_1 and C_2 on positions I_1^* and I_2^*, respectively, and there are yy' voters that rank C_1 and C_2 on positions I_2^* and I_1^*. Due to neutrality and consistency, these voters cancel each other out. (Formally, if S' were a voting situation limited to these voters only, we would have $C_1 =_{S'} C_2$. This so due to symmetry of f_k and the fact that for permutation σ that swaps all the members of $C_1 \setminus C_2$ with all the members of $C_2 \setminus C_1$, we have $S' = \sigma(S')$.) Next, there are $x'y$ voters that ranks C_1 and C_2 on positions I_1, and I_2, and there are xy' voters that rank C_1 and C_2 on positions I_2 and I_1, respectively. Since we assumed that $\frac{y}{x} > \frac{y'}{x'}$, we have that $x'y > xy'$. So, xy' of the voters from each of the two just-mentioned groups cancel each other out (in the same sense as above), and we are left with considering $x'y - xy' > 0$ voters that rank C_1 and C_2 on positions I_1 and I_2. Thus, we conclude that $C_1 \succ_S C_2$. However, this contradicts the fact that $C_2 \succ_S C_1$ and we conclude that $\sup U \le \inf L$.

Next, we show that $\sup U \ge \inf L$. To this end, we will show that there are no values $\frac{y}{x}$ and $\frac{y'}{x'}$ such that $\sup U < \frac{y}{x} < \frac{y'}{x'} < \inf L$. For the sake of contradiction let us assume that this is not the case and that such values exist. It must be the case that $\frac{y}{x}$ is not in U and, so, we have:

$$C_1 \succeq \left[P^{y(C_1 \to I_1^*, C_2 \to I_2^*)}_{x(C_1 \to I_2, C_2 \to I_1)} \right] C_2. \tag{4}$$

Since $\frac{y}{x}$ also cannot be in L, we have:

$$C_1 \succeq \left[P^{y(C_1 \to I_2^*, C_2 \to I_1^*)}_{x(C_1 \to I_1, C_2 \to I_2)} \right] C_2. \tag{5}$$

By neutrality (applied to (5) with permutation σ that swaps candidates from $C_1 \setminus C_2$ with those from $C_2 \setminus C_1$), we have that:

$$C_2 \succeq \left[P^{y(C_1 \to I_1^*, C_2 \to I_2^*)}_{x(C_1 \to I_2, C_2 \to I_1)} \right] C_1. \tag{6}$$

By putting together Equations (4) and (6), and by noting that the same reasoning can be repeated for $\frac{y'}{x'}$ instead of $\frac{y}{x}$, we conclude that it must be the case that:

$$C_1 = \left[P^{y(C_1 \to I_1^*, C_2 \to I_2^*)}_{x(C_1 \to I_2, C_2 \to I_1)} \right] C_2 \quad \text{and} \quad C_1 = \left[P^{y'(C_1 \to I_1^*, C_2 \to I_2^*)}_{x'(C_1 \to I_2, C_2 \to I_1)} \right] C_2.$$

[6] This assumption is without loss of generality because the condition from the statement of the lemma, $\Delta_{I_2,I_1} = -\Delta_{I_1,I_2}$, is symmetric; if it held that $C_2 \succ [v(C_1 \to I_1, C_2 \to I_2)]\, C_1$ then we could simply swap I_2 and I_1, and we would prove that $\Delta_{I_1,I_2} = -\Delta_{I_2,I_1}$.

After applying the permutation σ that swaps all the members of $C_1 \setminus C_2$ with all the members of $C_2 \setminus C_1$ to the voting situation from the first of the above two conditions, from the neutrality we get that:

$$C_1 =\left[P^{y(C_1 \to I_2^*, C_2 \to I_1^*)}_{x(C_1 \to I_1, C_2 \to I_2)}\right] C_2 \quad \text{and} \quad C_1 =\left[P^{y'(C_1 \to I_1^*, C_2 \to I_2^*)}_{x'(C_1 \to I_2, C_2 \to I_1)}\right] C_2. \tag{7}$$

We now define voting situation Q:

$$Q = x' \cdot P^{y(C_1 \to I_2^*, C_2 \to I_1^*)}_{x(C_1 \to I_1, C_2 \to I_2)} + x \cdot P^{y'(C_1 \to I_1^*, C_2 \to I_2^*)}_{x'(C_1 \to I_2, C_2 \to I_1)}.$$

From Equation (7) (and consistency), we get that $C_1 =_Q C_2$. In Q there is the same number of voters who rank C_1 and C_2 on positions I_1 and I_2 as those that rank them on positions I_2 and I_1, respectively (so these voters cancel out). On the other hand, there are yx' voters who rank C_1 and C_2 on positions I_2^* and I_1^*, and $y'x$ voters who rank these committees on positions I_1^* and I_2^*, respectively. Since $yx' < y'x$, we get that $C_1 \succ_Q C_2$, which contradicts our earlier observation that $C_1 =_Q C_2$. We conclude that it must be the case that $\sup U \geq \inf L$.

Finally, since we have shown that $\sup U \leq \inf L$ and $\sup U \geq \inf L$, we have that $\sup U = \inf L$. This proves that $\Delta_{I_2,I_1} = -\Delta_{I_1,I_2}$. □

The next lemma shows that Δ_{I_1,I_2} provides a threshold value for proportions of voters in distinguished profiles with respect to the relation between C_1 and C_2.

Lemma 3. *Let I_1 and I_2 be two committee positions such that $|I_1 \cap I_2| = |I_1^* \cap I_2^*|$. For each $x, y \in \mathbb{N}$ it holds that:*

1. *if $C_1 \succ[v(C_1 \to I_1, C_2 \to I_2)]\ C_2$ and $\frac{y}{x} < \Delta_{I_1,I_2}$, then $C_2 \succ\left[P^{y(C_1 \to I_1^*, C_2 \to I_2^*)}_{x(C_1 \to I_2, C_2 \to I_1)}\right] C_1$,*
2. *if $C_2 \succ[v(C_1 \to I_1, C_2 \to I_2)]\ C_1$ and $\frac{y}{x} > -\Delta_{I_1,I_2}$, then $C_2 \succ\left[P^{y(C_1 \to I_2^*, C_2 \to I_1^*)}_{x(C_1 \to I_2, C_2 \to I_1)}\right] C_1$.*

Proof. Let us start with proving the first statement. We assume that $C_1 \succ[v(C_1 \to I_1, C_2 \to I_2)]\ C_2$ and, for the sake of contradiction, that:

$$C_1 \succeq \left[P^{y(C_1 \to I_1^*, C_2 \to I_2^*)}_{x(C_1 \to I_2, C_2 \to I_1)}\right] C_2. \tag{8}$$

From the definition of Δ_{I_1,I_2} we infer that there exist two numbers $x', y' \in \mathbb{N}$, such that $\frac{y}{x} < \frac{y'}{x'} \leq \Delta_{I_1,I_2}$ and:

$$C_2 \succ \left[P^{y'(C_1 \to I_1^*, C_2 \to I_2^*)}_{x'(C_1 \to I_2, C_2 \to I_1)}\right] C_1. \tag{9}$$

Let us consider a voting situation that is obtained from $P^{y(C_1 \to I_1^*, C_2 \to I_2^*)}_{x(C_1 \to I_2, C_2 \to I_1)}$ (i.e., from the voting situation that appears in (8)) by swapping positions of C_1 and C_2, i.e., let us consider voting situation $P^{y(C_1 \to I_2^*, C_2 \to I_1^*)}_{x(C_1 \to I_1, C_2 \to I_2)}$. Naturally, in such a voting situation C_2 is weakly preferred over C_1:

$$C_2 \succeq \left[P^{y(C_1 \to I_2^*, C_2 \to I_1^*)}_{x(C_1 \to I_1, C_2 \to I_2)}\right] C_1. \tag{10}$$

By Equations (9), (10), and consistency of f_k, we see that in the voting situation:

$$P = x' \cdot P^{y(C_1 \to I_2^*, C_2 \to I_1^*)}_{x(C_1 \to I_1, C_2 \to I_2)} + x \cdot P^{y'(C_1 \to I_1^*, C_2 \to I_2^*)}_{x'(C_1 \to I_2, C_2 \to I_1)}$$

committee C_2 is strictly preferred over C_1 (i.e., $C_2 \succ_P C_1$). Let us now count the voters in P. There are xx' who put C_1 and C_2 on positions I_1 and I_2, respectively, and there are xx' voters who put C_1 and C_2 on positions I_2 and I_1, respectively. By the same arguments as used in the proof of Lemma 2, these voters cancel each other out. Next, there are $y'x$ voters who put C_1 and C_2 on positions I_1^* and I_2^*, respectively, and there are $x'y$ voters who put C_1 and C_2 on positions I_2^* and I_1^*, respectively. Since $y'x > yx'$. we conclude that $C_1 \succ_P C_2$ (again, using the same reasoning as we used in Lemma 2 for similar arguments). This is a contradiction with our earlier observation that $C_2 \succ_P C_1$. This completes the proof of the first part of the lemma.

The proof of the second statement is similar and we provide it for the sake of completeness. We assume that $C_2 \succ [v(C_1 \to I_1, C_2 \to I_2)]\ C_1$ and, for the sake of contradiction, that:

$$C_1 \succeq \left[P^{y(C_1 \to I_2^*, C_2 \to I_1^*)}_{x(C_1 \to I_2, C_2 \to I_1)}\right]\ C_2. \tag{11}$$

From the definition of Δ_{I_1,I_2} we know that there must be two numbers $x', y' \in \mathbb{N}$, such that $\frac{y}{x} > \frac{y'}{x'} \geq -\Delta_{I_1,I_2}$ and:

$$C_2 \succ [P^{y'(C_1 \to I_2^*, C_2 \to I_1^*)}_{x'(C_1 \to I_2, C_2 \to I_1)}]\ C_1. \tag{12}$$

If we swap the positions of committees C_1 and C_2 in the voting situation used in Equation (11), then by neutrality we have that:

$$C_2 \succeq \left[P^{y(C_1 \to I_1^*, C_2 \to I_2^*)}_{x(C_1 \to I_1, C_2 \to I_2)}\right]\ C_1 \tag{13}$$

We now form voting situation:

$$Q = x' \cdot P^{y(C_1 \to I_1^*, C_2 \to I_2^*)}_{x(C_1 \to I_1, C_2 \to I_2)} + x \cdot P^{y'(C_1 \to I_2^*, C_2 \to I_1^*)}_{x'(C_1 \to I_2, C_2 \to I_1)}$$

By Equations (12), (13), and consistency of f_k we have that $C_2 \succ_Q C_1$. However, counting voters again leads to a contradiction. Indeed, we have xx' voters who put C_1 and C_2 and positions I_1 and I_2, respectively, and xx' voters who put C_1 and C_2 on positions I_2 and I_1, respectively. These voters cancel each other out. Then we have $y'x$ voters who put C_1 and C_2 on positions I_2^* and I_1^*, respectively, and we have $x'y$ voters who put C_1 and C_2 on positions I_1^* and I_2^*, respectively. Since $y'x < x'y$, we have that $C_1 \succ_Q C_2$, which is a contradiction with our previous conclusion that $C_2 \succ_Q C_1$. This proves the second part of the lemma. □

Putting Together the Committee-Pairs Scoring Function for f_k We are ready to define a committee-pairs scoring function d for f_k. For each two committee positions, I_1 and I_2, we set:

$$d(I_1, I_2) = \Delta_{I_1,I_2},$$

which allows us to rewrite Definition 7 as follows (considering P to be a voting situation):

$$\mathrm{score}_d(C_1, C_2, P) = \sum_{\pi \in \Pi_{>}(A)} P(\pi) \cdot \Delta_{\mathrm{pos}_{P,v}(C_1), \mathrm{pos}_{P,v}(C_2)}.$$

Note that by Lemma 2, d indeed satisfies the conditions put on a committee-pairs scoring rule.

The next theorem shows that d is a committee-pairs scoring function for f_{C_1,C_2}. Based on this result, we will argue that it works for all pairs of committees.

Theorem 2. *For each voting situation P it holds that (i) if* $\mathrm{score}_d(C_1, C_2, P) > 0$*, then* $C_1 \succ_P C_2$*, and (ii) if* $\mathrm{score}_d(C_1, C_2, P) = 0$*, then* $C_1 =_P C_2$.

Proof. We start by proving part (i). Let P be some voting situation such that $\mathrm{score}_d(C_1, C_2, P) > 0$. For the sake of contradiction we assume that $C_2 \succeq_P C_1$.

The idea of the proof is to perform a sequence of transformations of P so that the result according to f_k does not change, but, eventually, each voter puts committees C_1 and C_2 either on positions I_1^*, I_2^* or the other way round. Let s be the total number of transformations we perform, and let P_i be the voting situation that we obtain after the i-th transformation. We ensure that for each voting situation P_i it holds that $\mathrm{score}_d(C_1, C_2, P_i) > 0$ and $C_2 \succeq_{P_i} C_1$. In particular, for the final voting situation P_s we will have $C_2 \succeq_{P_s} C_1$, $\mathrm{score}_d(C_1, C_2, P_s) > 0$, and each voter will put committees C_1 and C_2 on positions I_1^* and I_2^* or the other way round. However, from $\mathrm{score}_d(C_1, C_2, P_s) > 0$ we will conclude that there must be more voters who put C_1 and C_2 on positions I_1^* and I_2^* than on positions I_2^* and I_1^*. This will be a contradiction with $C_2 \succeq_{P_s} C_1$.

We now describe the transformations. We set $P_0 = P$. We perform the i-th transformation in the following way. If for each voter in P_{i-1}, committees C_1 and C_2 stand on positions I_1^* and I_2^* (or the other way round), we finish our sequence of transformations. Otherwise, we take a preference order of an arbitrary voter from P_{i-1}, for whom the set of committee positions of C_1 and C_2 is not $\{I_1^*, I_2^*\}$. Let us denote this voter by v_i. Let z denote the number of voters in P_{i-1} who rank C_1 and C_2 on the same positions as v_i, including v_i (so $z \geq 1$). Let I_1 and I_2 denote the positions of the committees C_1 and C_2 in the preference order of v_i, respectively. Let $\epsilon = \mathrm{score}_d(C_1, C_2, P_{i-1})/2z > 0$.

Case 1: If $C_1 =[v(C_1 \to I_1, C_2 \to I_2)]\ C_2$, then we obtain P_i by removing from P_{i-1} all z voters with the same preference order as v_i. By consistency of f_k, it follows that in the resulting voting situation P_i it still holds that $C_2 \succeq_{P_i} C_1$ (this is, in essence, the same canceling out of voters that we already used in Lemmas 2 and 3). Also, since $\mathrm{score}_d(C_1, C_2, v(C_1 \to I_1, C_2 \to I_2)) = \Delta_{I_1,I_2} = 0$ it still holds that $\mathrm{score}_d(C_1, C_2, P_i) > 0$.

Case 2: If $C_1 \succ[v(C_1 \to I_1, C_2 \to I_2)]\ C_2$, then let x and y be such integers that $\Delta_{I_1,I_2} - \epsilon < \frac{y}{x} < \Delta_{I_1,I_2}$ (recall that ϵ is defined just above Case 1, and that z is the number of voters with the same preference order as v_i). We define two new voting situations:

$$R_{i-1} = z \cdot P^{y(C_1 \to I_1^*, C_2 \to I_2^*)}_{x(C_1 \to I_2, C_2 \to I_1)} \quad \text{and} \quad Q_{i-1} = x \cdot P_{i-1} + R_{i-1}.$$

From Lemma 3 it follows that $C_2 \succ_{R_{i-1}} C_1$ and, by consistency, we get that $C_2 \succ_{Q_{i-1}} C_1$. Let us now calculate the value of $\mathrm{score}_d(C_1, C_2, R_{i-1})$. R_{i-1} consists of zx voters who rank C_1 and C_2 on positions I_2 and I_1 (and who contribute $zx\Delta_{I_2,I_1} = -zx\Delta_{I_1,I_2}$ to the value of $\mathrm{score}_d(C_1, C_2, R_{i-1})$) and of zy voters who rank C_1 and C_2 on positions I_1^* and I_2^*, respectively (who contribute value $zy\Delta_{I_1^*,I_2^*} = zy$). That is, we have $\mathrm{score}_d(C_1, C_2, R_{i-1}) = -zx\Delta_{I_1,I_2} + zy$. Further, by definition of ϵ, we have that $\mathrm{score}_d(C_1, C_2, P_{i-1}) = 2z\epsilon$. In effect, we have that:

$$\begin{aligned}
\mathrm{score}_d(C_1, C_2, Q_{i-1}) &= x \cdot \mathrm{score}_d(C_1, C_2, P_{i-1}) + \mathrm{score}_d(C_1, C_2, R_{i-1}) \\
&= 2zx\epsilon + (-zx\Delta_{I_1,I_2} + zy) \\
&= 2zx\epsilon + zx(-\Delta_{I_1,I_2} + \frac{y}{x}) \geq 2zx\epsilon - zx\epsilon > 0.
\end{aligned}$$

The first inequality (in the final row) follows from the fact that we assumed $\Delta_{I_1,I_2} - \epsilon < \frac{y}{x} < \Delta_{I_1,I_2}$. We now move on to Case 3, where we also build voting situation Q_{i-1} with a similar property, and then describe how to obtain P_i from Q_{i-1}'s.

Case 3: If $C_2 \succ[v(C_1 \to I_1, C_2 \to I_2)]\ C_1$, then our reasoning is very similar to that from Case 2. Let x and y be such integers that $-\Delta_{I_1,I_2} < \frac{y}{x} < -\Delta_{I_1,I_2} + \epsilon$. We define two voting situations:

$$R_{i-1} = z \cdot P^{y(C_1 \to I_2^*, C_2 \to I_1^*)}_{x(C_1 \to I_2, C_2 \to I_1)} \quad \text{and} \quad Q_{i-1} = x \cdot P_{i-1} + R_{i-1}.$$

From Lemma 3 it follows that $C_2 \succ_{R_{i-1}} C_1$, and, thus, from consistency, we get that $C_2 \succ_{Q_{i-1}} C_1$. Further, using similar analysis as in Case 2, we get that:

$$\begin{aligned}
\mathrm{score}_d(C_1, C_2, Q_{i-1}) &= x \cdot \mathrm{score}_d(C_1, C_2, P_{i-1}) + zx\Delta_{I_2,I_1} - zy \\
&= 2zx\epsilon + zx(-\Delta_{I_1,I_2} - \frac{y}{x}) \geq 2zx\epsilon - zx\epsilon > 0.
\end{aligned}$$

The first inequality (in the final row) follows from the assumption that $-\Delta_{I_1,I_2} < \frac{y}{x} < -\Delta_{I_1,I_2} + \epsilon$. Below we describe how to obtain P_i from Q_{i-1} (for both Cases 2 and 3).

In Cases 2 and 3, in the voting situation Q_{i-1} exactly zx voters have C_1 and C_2 on positions I_2 and I_1, respectively (for both cases, these voters are introduced in voting situation R_{i-1}). Further, there are exactly zx voters who rank C_1 and C_2 on positions I_1 and I_2, respectively (these are the cloned-x-times voters that were originally in P_{i-1}). We define P_i as Q_{i-1} with these $2zx$ voters removed. Since we removed the same number of voters who rank C_1 and C_2 on positions I_2 and I_1, respectively, as the number of voters who rank these committees on positions I_1 and I_2, respectively, we conclude that $\mathrm{score}_d(C_1, C_2, P_i) = \mathrm{score}_d(C_1, C_2, Q_{i-1}) > 0$ and that $C_2 \succeq_{P_i} C_1$.

We note that after the just-described transformation none of the voters has C_1 and C_2 on positions I_1 and I_2, respectively, and that we only added number of voters that rank C_1 and C_2 on positions I_1^* and I_2^* (or the other way round). Hence, if we perform such transformations for all possible pairs of committee positions I_1 and I_2, we will obtain our final voting situation, P_s, for which it holds that (i) $\mathrm{score}_d(C_1, C_2, P_s) > 0$, (ii) $C_2 \succeq_{P_s} C_1$, and (iii) in P_s each voter ranks C_1 and C_2 on positions I_1^* and I_2^* (or the other way round). Given (i) and (iii) we conclude that in P_s there are more votes in which C_1 stands on position I_1^* and C_2 stands on position I_2^* than there are voters where the opposite holds. However, this implies that $C_1 \succ_{P_s} C_2$ and contradicts the fact that $C_2 \succeq_{P_s} C_1$. This completes the proof of the first part of the theorem.

Next, we consider part (ii) of the theorem. Let P be some voting situation such that $\mathrm{score}_d(C_1, C_2, P) = 0$. For the sake of contradiction we assume that $C_2 \neq_P C_1$, and, without loss of generality, we make it $C_2 \succ_P C_1$. Since for the voting situation $v(C_1 \to I_1^*, C_2 \to I_2^*)$ it holds that $\mathrm{score}_d(C_1, C_2, v(C_1 \to I_1^*, C_2 \to I_2^*)) > 0$, then for each $n \in \mathbb{N}$, in voting situation:

$$Q_n = nP + v(C_1 \to I_1^*, C_2 \to I_2^*),$$

we have $\mathrm{score}_d(C_1, C_2, Q_n) > 0$, and—from part (i) of the theorem—we get that $C_1 \succ_{Q_n} C_2$. On the other hand, continuity requires that there exists some value of n such that $C_2 \succ_{Q_n} C_1$. This gives a contradiction and completes the proof. □

We have proven Theorem 2 for the function f_{C_1,C_2}. From neutrality it follows that the thesis holds for $f_{k,|C_1 \cap C_2|}$. Now, let us argue that it also holds for f_k. Indeed, for the thesis of Theorem 2 we needed to define a committee-pairs scoring function d only for such pairs (I_1, I_2) that $|I_1 \cap I_2| = |C_1 \cap C_2|$. Since the choice of C_1 and C_2 was arbitrary, for each value of i, $0 \leq i \leq k$, we can take arbitrary sets $C_1 = C_{1,i}$ and $C_2 = C_{2,i}$ such that $|C_{1,i} \cap C_{2,i}| = i$, repeat the reasoning from this section, and obtain the thesis of Theorem 2 for a committee-pairs scoring function d defined for such pairs (I_1, I_2) that $|I_1 \cap I_2| = i$. Repeating this for each i allows us to derive the full-fledged function d for pairs (I_1, I_2) of arbitrary size of intersection, and consequently, to prove Theorem 2 for f_k.

Young-Style Characterization of Committee-Pairs Scoring Rules This way we have proven an analogous result to our main Theorem 1 for decision rules.

Corollary 1. *A decision rule is symmetric, committee-neutral, consistent, and continuous if and only if it is a committee-pair scoring rule.*

As in interlude, we argue that this corollary is likely to be far more important than simply being a by product on our way to characterizing committee scoring rules. Indeed, we believe that there are natural committee-pairs scoring rules that are not committee scoring rules, and that still are very useful. For example, let us consider a committee pairs scoring rule (for committee size k) defined through the following committee pairs scoring function d:

$$d(I_1, I_2) = \begin{cases} 1 & \text{if all the positions in } I_1 \setminus I_2 \text{ precede all the positions in } I_2 \setminus I_1, \\ -1 & \text{if all the positions in } I_2 \setminus I_1 \text{ precede all the positions in } I_1 \setminus I_2, \\ 0 & \text{otherwise.} \end{cases}$$

If we take committee size to be $k = 1$, then given a preference profile P, the committee-pairs scoring rule defined by d simply returns a weighted equivalent of P's majority graph. A majority graph for a preference profile is a directed graph where each candidate is a vertex, and for each two candidates c and d there is a directed edge from c to d exactly if more voters prefer c to d.

For committee sizes $k > 1$, our committee-pairs scoring rule would give us some generalization of the majority graph for committees. Indeed, in some sense, every committee-pairs scoring rule can be seen as providing some sort of generalization of the majority graph. Naturally, given a majority graph (or some its generalization provided by a committee-pairs scoring rule), it is not at all obvious which candidate (or which committee) should be a winner. However, there is a whole theory of tournament solution concepts that one can employ to figure this out (we point the reader to Laslier's textbook for an overview [33]). While deeper study of committee-pairs scoring rules and decision rules is not in the scope of this paper, we believe it is a very interesting direction for future research.

Getting Ready for the Second Domain Change We have almost completed the first part of the proof. However, we still need to derive one more technical tool (Lemma 4 below). However, to achieve this goal, we need to change our domain from $\mathbb{N}^{m!}$ to $\mathbb{Q}^{m!}$, and before we make this change, we need to introduce several new notions. (While the correctness of our first domain relied on the decision rule being symmetric, this second domain change, as noted by Young [51], uses our further axioms.)

We distinguish one specific voting situation, $e = \langle 1, 1, \ldots, 1\rangle$, called the *null profile*, describing the setting where each possible vote is cast exactly once. It immediately follows that under each symmetric k-decision rule f_k, each two committees are ranked equally in e, i.e., for each two committees C'_1, C'_2 we have $C'_1 =_e C'_2$.

Definition 8 (Independence of Symmetric Profiles)**.** *A symmetric k-decision rule f_k is* independent of symmetric profiles *if for every voting situation $P \in \mathbb{N}^{m!}$ and for every $\ell \in \mathbb{N}$, we have that $f_k(P + \ell e) = f_k(P)$.*

Definition 9 (Homogeneity)**.** *A symmetric k-decision rule f_k is* homogeneous *if for every voting situation $P \in \mathbb{N}^{m!}$ and for every $\ell \in \mathbb{N}$, we have $f_k(\ell P) = f_k(P)$.*

Intuitively, independence of symmetric profiles says that if we add one copy of each possible vote then they will all cancel each other out. Homogeneity says that the result of an election depends only on the proportions of votes with particular preference voters and not on their exact numbers. One can verify that each symmetric and consistent k-decision rule satisfies both independence of symmetric profiles and homogeneity (indeed, the requirement in the definition of homogeneity is a special case of the requirement from the definition of consistency).[7]

Second Domain Change Now we are ready to perform our second domain change, to extend our domain from $\mathbb{N}^{m!}$ to $\mathbb{Q}^{m!}$. To this end, we use the following result, originally stated for single-winner rules but it can be adapted to the multiwinner setting in a straightforward way.

Theorem 3 (Young [51], Merlin [37])**.** *Let f_k be k-decision rule, $f_k\colon \mathbb{N}^{m!} \to (S_k(A) \times S_k(A) \to \{-1, 0, 1\})$ that is symmetric, independent of symmetric profiles and homogeneous. There exists a unique extension of f_k to the domain $\mathbb{Q}^{m!}$ (which we also denote by f_k), satisfying for each $\ell \in \mathbb{N}$, and $P \in \mathbb{N}^{m!}$ the following two conditions:*

1. $f_k(P - \ell e) = f_k(P)$,
2. $f_k\left(\frac{P}{\ell}\right) = f_k(P)$.

[7] Indeed, the reader may ask why do we introduce independence of symmetric profiles and homogeneity, when what we require from them already follows from consistency. The reason is that, we believe, these two properties better explain—on the intuitive level—why the second domain change is correct.

From now on, when we speak of voting situations, we mean voting situations from our new domain, $\mathbb{Q}^{m!}$. We note that within our new domain, the score of a pair (C_1, C_2) of committees within voting situations P under committee-pairs scoring function d can still be expressed as:

$$\mathrm{score}_d(C_1, C_2, P) = \sum_{\pi \in \Pi_>(A)} P(\pi) \cdot d(\mathrm{pos}_\pi(C_1), \mathrm{pos}_\pi(C_2)).$$

Indeed, for committee-pairs scoring rules, this definition gives the unique extension that Theorem 3 speaks of.

Completing the First Part of the Proof Throughout most of the discussion so far, C_1 and C_2 were two fixed committees. However, since their choice was arbitrary, all the results we have established so far hold for all pairs of committees. From now on we no longer assume C_1 and C_2 to be these two fixed committees.

Since $\mathbb{Q}^{m!}$ is a linear space over the field of rational numbers, from Theorem 2 we infer that for each two committees C_1 and C_2, the space of voting situations P such that $C_1 =_P C_2$ is a hyperplane in the $m!$-dimensional vector space of all voting situations. This is so, because if we treat a voting situation P as a vector of $m!$ variables, then condition $\mathrm{score}_d(C_1, C_2, P) = 0$ turns out to simply be a single linear equation. Hence, the space of voting situations P such that $C_1 =_P C_2$ is a hyperplane and has dimension $m! - 1$.

Corollary 2. *The set $\{P \in \mathbb{Q}^{m!} \colon C_1 =_P C_2\}$ is a hyperplane in the vector space of all voting situations.*

From now on, we assume that our k-decision rule f_k is transitive, that is, we require that for each voting situation P and each three committees C'_1, C'_2, and C'_3 it holds that:

$$(C'_1 \succeq_P C'_2) \text{ and } (C'_2 \succeq_P C'_3) \text{ implies } (C'_1 \succeq_P C'_3).$$

In other words, from now on we require f_k to be a k-winner election rule.

Lemma 4. *Let f_k be a symmetric, consistent, committee-neutral, committee-dominant, continuous k-winner election rule, and let $\lambda \colon [m]_k \to \mathbb{R}$ be a committee scoring function. If it holds that for each two committees C_1 and C_2 and each voting situation P it holds that the committee scores of C_1 and C_2 are equal (according to λ) if and only C_1 and C_2 are equivalent according to f_k, then it holds that: For each two committees C_1 and C_2 and each voting situation P, if the committee score of C_1 is greater than that of C_2 (according to λ) then C_1 is preferred over C_2 according to f_k (i.e., $C_1 \succ_P C_2$).*

Proof. Based on λ, we build a committee-pairs scoring function g as follows. For each two committee positions I_1 and I_2, we have $g(I_1, I_2) = \lambda(I_1) - \lambda(I_2)$. The score of a committee pair (C_1, C_2) in voting situation P under committee-pairs score function g is given by:

$$\mathrm{score}_g(C_1, C_2, P) = \sum_{\pi \in \Pi_>(A)} P(\pi) \cdot g(\mathrm{pos}_\pi(C_1), \mathrm{pos}_\pi(C_2)).$$

Let us fix $x \in [k-1]$ and two arbitrary committees C^*_1 and C^*_2 such that $|C^*_1 \cap C^*_2| = x$. We note that, by the assumptions of the theorem, if it holds that:

$$\mathrm{score}_g(C^*_1, C^*_2, P) = 0 \iff C^*_1 =_P C^*_2,$$

then, by Corollary 3, $H = \{P \in \mathbb{Q}^{m!} \colon C^*_1 =_P C^*_2)\}$ is an $(m! - 1)$-dimensional hyperplane. More so, this is the same hyperplane as the following two (where d is the committee-pairs scoring function from the thesis of Theorem 2, built for f_k):

$$\{P \in \mathbb{Q}^{m!} \colon \mathrm{score}_g(C^*_1, C^*_2, P) = 0\} \quad \text{and} \quad \{P \in \mathbb{Q}^{m!} \colon \mathrm{score}_d(C^*_1, C^*_2, P) = 0\},$$

We claim that for C^*_1 and C^*_2 one of the following conditions must hold:

1. For each voting situation P, if $\mathrm{score}_g(C_1^*, C_2^*, P) > 0$ then $C_1^* \succ_P C_2^*$.
2. For each voting situation P, if $\mathrm{score}_g(C_1^*, C_2^*, P) > 0$ then $C_2^* \succ_P C_1^*$.

Why is this so? For the sake of contradiction, let us assume that there exist two voting situations, P and Q, such that $\mathrm{score}_g(C_1^*, C_2^*, P) > 0$ and $\mathrm{score}_g(C_1^*, C_2^*, Q) > 0$, but $C_1^* \succeq_P C_2^*$ and $C_2^* \succeq_Q C_1^*$. From the fact that $\mathrm{score}_g(C_1^*, C_2^*, P) > 0$ and $\mathrm{score}_g(C_1^*, C_2^*, Q) > 0$, we see that the points P and Q lie on the same side of hyperplane H and neither of them lies on H. From $C_1^* \succeq_P C_2^*$, $C_2^* \succeq_Q C_1^*$, and from Theorem 2, we see that $\mathrm{score}_d(C_1^*, C_2^*, P) \geq 0$ and $\mathrm{score}_d(C_1^*, C_2^*, Q) \leq 0$. That is, at least one of the voting situations P and Q lies on the hyperplane, or they both lie on different sides of the hyperplane. This gives a contradiction and proves our claim.

Now, using the committee dominance axiom, we exclude the second condition. For each $i \in [m-k+1]$ we set $I_i = \{i, i+1, \ldots, i+k-1\}$. Let I and J denote, respectively, the best possible and the worst possible position of a committee, i.e., $I = I_1$ and $J = I_{m-k+1}$. For the sake of contradiction, let us assume that there exists a profile P, where $\mathrm{score}_g(C_1^*, C_2^*, P) > 0$ and $C_2^* \succ_P C_1^*$. Since there exists a profile with $\mathrm{score}_g(C_1^*, C_2^*, P) > 0$, it must be the case that $\lambda(I) > \lambda(J)$ (otherwise λ would be a constant function). Thus there must exist p such that $\lambda(I_p) > \lambda(I_{p+k-x})$. Let us consider a voting situation S consisting of a single vote where C_1^* stands on position I_p and C_2^* stands on position I_{p+k-x}. Since $\lambda(I_p) > \lambda(I_{p+k-x})$, we have that $\mathrm{score}_g(C_1^*, C_2^*, S) > 0$. By committee-dominance of f_k, it follows that $C_1^* \succeq_P C_2^*$. However, from the reasoning in the preceding paragraph (applied to profile P), we know that either $C_1^* \succ_P C_2^*$ or $C_2^* \succ_P C_1^*$. Putting these two facts together, we conclude that $C_1^* \succ_P C_2^*$. Since we have shown a single profile P such that $\mathrm{score}_g(C_1^*, C_2^*, P) > 0$ and $C_1^* \succ_P C_2^*$, by the argument from the previous paragraph, we know that for every profile P it holds that:

$$\text{If } \mathrm{score}_g(C_1^*, C_2^*, P) > 0 \text{ then } C_1^* \succ_P C_2^*.$$

Our choice of committees C_1^* and C_2^* was arbitrary and, thus, the above implication holds for all pairs of committees. This completes the proof. □

Given Lemma 4, in our following discussion we can focus our attention on the space $\{P\colon C_1 =_P C_2\}$. If we show that committees C_1 and C_2 are equivalent, according to some given k-winner election rule, if and only if the score of C_1 is equal to the score of C_2, according to some committee scoring function λ, then we can conclude that f_k is a committee scoring rule defined by committee scoring function λ. This observation concludes the first part of the proof.

A.2 Second Part of the Proof: Committees with All but One Candidate in Common

We now start the second part of the proof. The current section is independent from the results of the previous one, but we do use all the notation that was introduced and, in particular, we consider voting situations over $\mathbb{Q}^{m!}$. We will use results from Section A.1 in the next section, where we conclude the whole proof.

The Setting and Our Goal Throughout this section, we assume f_k to be a k-winner election rule that is symmetric, consistent, committee-neutral, committee-dominant, and continuous. As before, the committee size is k. Our goal is to show that as long as we consider committees that contain some $k-1$ fixed members and can differ only in the final one, f_k can be seen as a committee scoring rule. The discussion in this section is inspired by that of Young [51] and Merlin [37], but the main part of our analysis is new (in particular Lemma 8).

Position-Difference Function Let P be a voting situation in $\mathbb{Q}^{m!}$, C be some size-k committee, and I be a committee position. We define the weight of position I with respect to C within P as:

$$\text{pos-weight}_I(C, P) = \sum_{\pi \in \Pi_>(A)\colon \text{pos}_\pi(C)=I} P(\pi),$$

That is $\text{pos-weight}_I(C, P)$ is the (rational) number of votes in which committee C is ranked on position I.

For each two committees C_1, C_2 such that $|C_1 \cap C_2| = k - 1$, we define a committee position-difference function $\alpha_{C_1,C_2}\colon \mathbb{Q}^{m!} \to \mathbb{Q}^{\binom{m}{k}}$ that for each voting situation $P \in \mathbb{Q}^{m!}$ returns a vector of $\binom{m}{k}$ elements, indexed by committee positions (i.e., elements of $[m]_k$), such that for each committee position I, we have:

$$\alpha_{C_1,C_2}(P)[I] \;=\; \text{pos-weight}_I(C_1, P) - \text{pos-weight}_I(C_2, P).$$

Naturally, $\alpha_{C_1,C_2}(P)$ is a linear function of P. We also note that for each voting situation P, we have:

$$\sum_{I \in [m]_k} \alpha_{C_1,C_2}(P)[I] = 0. \tag{14}$$

To see why this is the case, we note that $\sum_{I\in[m]_k} \text{pos-weight}_I(C_1, P) = \sum_{\pi\in\Pi_>(A)} P(\pi)$ because every vote is accounted exactly once. Thus, we have that:

$$\begin{aligned}
\sum_{I\in[m]_k} \alpha_{C_1,C_2}(P)[I] &= \sum_{I\in[m]_k} \Big(\text{pos-weight}_I(C_1, P) - \text{pos-weight}_I(C_2, P)\Big) \\
&= \sum_{I\in[m]_k} \text{pos-weight}_I(C_1, P) - \sum_{J\in[m]_k} \text{pos-weight}_J(C_2, P) \\
&= \sum_{\pi\in\Pi_>(A)} P(\pi) - \sum_{\pi'\in\Pi_>(A)} P(\pi') = 0.
\end{aligned}$$

Position-difference functions will be important technical tools that we will soon use in the proof (in particular, in Lemma 8 we will show that if $\alpha_{C_1,C_2}(P) = \langle 0, \dots, 0\rangle$ then $C_1 =_P C_2$). However, we need to provide some more tools first.

Johnson Graphs and Hamiltonian Paths We will need the following graph-theoretic results to build certain votes and preference profiles in our following analysis. We mention that the graphs that Lemmas 5 and 6 speak of are called Johnson graphs. Lemma 5 has been shown before (we found the result in the work of Asplach [2] and could not trace an earlier reference[8]), and we provide the proof for the sake of completeness.

Lemma 5. *Let p and j be integers such that $1 \le j \le p$. Let $G(j, p)$ be a graph constructed in the following way. We associate j-element subsets of $\{1, \dots p\}$ with vertices and we say that two vertices are connected if the corresponding subsets differ by exactly one element (they have $j - 1$ elements in common). Such a graph contains a Hamiltonian path, i.e., a path that visits each vertex exactly once, that starts from the set $\{1, \dots, j\}$ and ends in the set $\{p - j + 1, \dots, p\}$.*

Proof. We prove this lemma by induction over j and p. For $j = 1$ and for each $p \ge 1$, it is easy to see that the required path exists (in this case, the graph is simply a full clique). This provides the induction base. For the inductive step, we assume that there are two numbers, p' and j', such

[8]We suspect the results might have been known before the work of Asplach. Indeed, similar results appear in the form of algorithms that output all size-k subsets of a given set in the order so that each two consecutive sets differ in only one element. Yet, we need the specific variants provided in Lemmas 5 and 6 that finish the Hamiltonian path on a specific vertex. Asplach [2] does not mention directly that his proofs provide this property, but close inspection shows that this is the case.

that for each p and j ($j \leq p$) such that $p < p'$ and $j < j'$ it holds that graph $G(j, p)$ contains a Hamiltonian path satisfying the constraints from the lemma. We will prove that such a path also exists for $G(j', p')$.

We partition the set of vertices of $G(j', p')$ into $p'-j+1$ groups $V(j', p', 1), \dots, V(j', p', p'-j+1)$, where for each $x \in \{1, \dots, p' - j + 1\}$, group $V(j', p', x)$ consists of all sets of j elements (vertices of the graph) such that x is the lowest among them.

We build our Hamiltonian path for $G(j', p')$ as follows. We start with the vertex $\{1, \dots, j'\}$. By our inductive hypothesis, we know that there is a path that starts with $\{1, \dots, j'\}$, traverses all vertices in $V(j', p', 1)$, and ends in $\{1, p' - j' + 2, \dots, p'\}$. From $\{1, p' - j' + 2, \dots, p'\}$ we can go, over a single edge, to $\{2, p' - j' + 2, \dots, p'\}$. Starting with this vertex, by our inductive hypothesis, we can traverse all the vertices of $V(j', p', 2)$. Then, over a single edge, we can move to some vertex from $V(j', p', 3)$, traverse all the vertices there, and so on. By repeating this procedure, we will eventually reach some vertex in the set $V(j', p', p' - j' + 1)$. However, $V(j', p', p' - j' + 1)$ contains exactly one vertex, $\{p' - j' + 1, \dots, p'\}$. This means that we have found the desired Hamiltonian path. □

Lemma 6. *Let r, p and j be integers such that $1 \leq r \leq p$ and $1 \leq j \leq p-1$. Let $\tilde{G}(j, p, r)$ be a graph constructed in the following way: (i) A j-element subset of $\{1, \dots, p\}$ is a vertex of $\tilde{G}(j, p, r)$ exactly if it contains at least one element smaller than r. (ii) There is an edge between two vertices if they differ in exactly one element (i.e., if they have $j - 1$ elements in common). Such a graph contains a Hamiltonian path.*

Proof. The proof is very similar to the previous one. We partition the set of vertices of $\tilde{G}(j, p, r)$ into $r - 1$ groups $V(j, p, 1), \dots, V(j, p, r - 1)$, where for each $x \in \{1, \dots, r - 1\}$, group $V(j, p, x)$ consists of all the sets (i.e., all the vertices) such that x is their smallest member.

We build our Hamiltonian path for $\tilde{G}(j, p, r)$ as follows. We start with the vertex $\{1, \dots, j\}$. By Lemma 5, we can continue the path from $\{1, \dots, j\}$, traverse all vertices in $V(j, p, 1)$, and end in $\{1, p - j + 2, \dots, p\}$. From $\{1, p - j + 2, \dots, p\}$ we can go, over a single edge, to $\{2, p - j + 2, \dots, p\}$, and we can traverse all vertices in $V(j, p, 2)$. Then we can go, over a single edge, to some vertex from $V(j, p, 3)$, and we can continue in the same way as in the proof of Lemma 5. □

The Codomain of α_{C_1,C_2} Let us consider two distinct committees C_1 and C_2. Using Lemma 5, we establish the dimension of the codomain of function α_{C_1,C_2}. This result will be useful in the proof of Lemma 8.

Lemma 7. *Let C_1 and C_2 be two distinct committees. The codomain of the function α_{C_1,C_2} has dimension $\binom{m}{k} - 1$.*

Proof. From Equation (14), we get that the dimension of the codomain of function α_{C_1,C_2} is at most $\binom{m}{k} - 1$. Now, let us consider graph $G = G(k, m)$ from Lemma 5 and the Hamiltonian path specified in this lemma. Note that we can understand each vertex in G as a committee position. For each edge (I, I') on our Hamiltonian path, consider a single vote where C_1 stands on position I and C_2 stands on position I'. For such a vote, α_{C_1,C_2} returns a vector with all 0s except a single 1 on position I and a single -1 on position I'. It is easy to observe that there are $\binom{m}{k} - 1$ such votes and that so constructed vectors are linearly independent. □

C_1-C_2-Symmetric Profiles The final tool that we need to provide before we prove Lemma 8 is the definition of C_1-C_2-symmetric profiles. Suppose σ is a permutation of A. Then we can extend its action to linear orders and voting situations in the natural way.

Definition 10. *Let C_1 and C_2 be two size-k committees. We say that a voting situation P is C_1-C_2-*symmetric *if there exists a permutation of the set of candidates σ and a sequence of committees $F_1, F_2, \dots, F_x$ such that:*

1. $P = \sigma(P)$,

2. $C_1 = F_1 = F_x$ and $C_2 = F_2$,

3. for each $i \in [x-1]$ it holds that $\sigma(F_i) = F_{i+1}$.

If a voting situation P is C_1-C_2-*symmetric* then we know that $C_1 =_P C_2$. Why is this the case? For the sake of contradiction let us assume that $C_1 \neq_P C_2$, and, without loss of generality, that $C_1 \succ_P C_2$. From $C_1 \succ_P C_2$ (which translates to $F_1 \succ_P F_2$) by symmetry of f_k we infer that $F_2 \succ_{\sigma(P)} F_3$, thus that $F_2 \succ_P F_3$. By the same arguments, we get that $F_1 \succ_P F_2 \succ_P F_3 \succ_P \ldots \succ_P F_x$. Thus, we get that $C_1 \succ_P C_1$, a contradiction.

Further, we observe that for each C_1-C_2-symmetric voting situation P it holds that $\alpha_{C_1,C_2}(P) = \langle 0, \ldots, 0 \rangle$. Indeed, using notation from Definition 10, we note that since $\sigma(C_1) = C_2$ and since $\sigma(P) = P$, for each (fractional) vote in P where committee C_1 stands on some position I we can uniquely assign a (fractional) vote in P where committee C_2 stands on the same position I. This shows that $\alpha_{C_1,C_2}(P) \geq \langle 0, \ldots, 0 \rangle$. By an analogous argument (using the fact that $\sigma^{(-1)}(C_2) = C_1$ and $\sigma^{(-1)}(P) = P$) we infer that $\alpha_{C_1,C_2}(P) \leq \langle 0, \ldots, 0 \rangle$ and, so, we conclude that $\alpha_{C_1,C_2}(P) = \langle 0, \ldots, 0 \rangle$.

Inferring Committee Equivalence Using α_{C_1,C_2} We are ready to present our main technical tool, Lemma 8. On the intuitive level, it says that information provided by the α_{C_1,C_2} function for a profile P is sufficient to distinguish whether C_1 is equivalent to C_2 with respect to P.

Lemma 8. *For each two committees $C_1, C_2 \in S_k(A)$ such that $|C_1 \cap C_2| = k - 1$ and for each voting situation $P \in \mathbb{Q}^{m!}$, if $\alpha_{C_1,C_2}(P) = \langle 0, \ldots, 0 \rangle$, then $C_1 =_P C_2$.*

Proof. The kernel of a linear function is the space of all vectors for which this function returns the zero vector. In particular, the kernel of α_{C_1,C_2}, denoted $\ker(\alpha_{C_1,C_2})$, is the space of all voting situations P such that $\alpha_{C_1,C_2}(P) = \langle 0, \ldots, 0 \rangle$. Since the domain of function α_{C_1,C_2} has dimension $m!$ and, by Lemma 7, its codomain has dimension $\binom{m}{k} - 1$, the kernel of α_{C_1,C_2} has dimension $m! - \binom{m}{k} + 1$. We will construct a base of this kernel that will consists of C_1-C_2-symmetric voting situations only. Since for each C_1-C_2-symmetric voting situation P it holds that $C_1 =_P C_2$ and $\alpha_{C_1,C_2}(P) = \langle 0, \ldots, 0 \rangle$, by consistency of f_k and linearity of α_{C_1,C_2} we will get the thesis of the theorem.

We prove the statement by a two-dimensional induction on k (committee size) and m (size of the set of candidates). As a base for the induction we will show that the property holds for $k = 1$ and all values of m. For the inductive step we will show that from the fact that the property holds for committee size $j - 1$ and for $p - 1$ candidates it follows that the property also holds for committee size j and for p candidates. This will allow as to conclude that the property holds for all values of m and k with $m \geq k$.

For $k = 1$ and for an arbitrary value of m, the problem collapses to the single-winner setting. It has been shown by Young [51] (and by Merlin [37]) that for each two candidates c_1 and c'_1, there exists a base of $\ker(\alpha_{\{c_1\},\{c'_1\}})$ that consists of $m! - (m - 1)$ voting situations which are $\{c_1\}$-$\{c'_1\}$-symmetric. This gives us the base for the induction.

Let us now prove the inductive step. We want to show that the statement is satisfied for $A_p = \{a_1, a_2, \ldots a_p\}$, $C_{1,j} = \{a_1, a_2, \ldots, a_j\}$ and $C_{2,j} = \{a'_1, a_2, \ldots, a_j\}$, where we set $a'_1 = a_{j+1}$. (We note that since f_k is symmetric, the exact names of the candidates we use here are irrelevant, and we picked these for notational convenience.) From the sets A_p, $C_{1,j}$ and $C_{2,j}$ we take out element a_j and get $A_{p-1} = \{a_1, a_2, \ldots, a_{j-1}, a_{j+1}, \ldots a_p\}$, $C_{1,(j-1)} = \{a_1, a_2, \ldots, a_{j-1}\}$ and $C_{2,(j-1)} = \{a'_1, a_2, \ldots, a_{j-1}\}$. Let V_{j-1} be a base of $\ker(\alpha_{C_{1,(j-1)},C_{2,(j-1)}})$ that consists of $C_{1,(j-1)}$-$C_{2,(j-1)}$-symmetric voting situations. We know that it exists from the induction hypothesis. We also know that it consists of $(p-1)! - \binom{p-1}{j-1} + 1$ voting situations. We now build the desired base for $\ker(\alpha_{C_{1,j},C_{2,j}})$ using V_{j-1} as the starting point. Our base has to consist of $p! - \binom{p}{j} + 1$ linearly independent, $C_{1,j}$-$C_{2,j}$-symmetric voting situations.

First, for each voting situation $P \in V_{j-1}$ and for each $r \in \{1, \dots p\}$ we create a voting situation P_r as follows. We take each vote v in P and we put a_j in the r-th position of v, pushing the candidates on positions $r, r+1, r+2, \dots$ back by one position, but keeping their relative order unchanged. There are $p! - p\binom{p-1}{j-1} + p$ such vectors and it is easy to see that they are linearly independent. Let us refer to the set of these vectors as B_1. Naturally, the vectors from B_1 do not span the whole space $\ker(\alpha_{\{a_1,\dots,a_j\},\{a'_1,\dots,a_j\}})$; there is simply to few of them. However, there is also a certain structural reason for this and understanding this reason will help us further in the proof. Let $\mathrm{lin}(B_1)$ denote the set of linear combinations of voting situations from B_1. For each $r \in \{1, \dots p\}$ and each $T \in \mathrm{lin}(B_1)$, let $T(a_j \to r)$ denote the voting situation that consists of all votes from T which have a_j on the r-th position. We can see that for each $r \in \{1, \dots p\}$ and each $T \in \mathrm{lin}(B_1)$, it holds that $\alpha_{C_{1,j},C_{2,j}}(T(a_j \to r)) = \langle 0, \dots, 0\rangle$ (the reason for this is that $T(a_j \to r)$ is, in essence, a linear combination of voting situations from V_{j-1}, with a_j inserted at position r) . This property certainly does not hold for all the voting situations in $\ker(\alpha_{\{a_1,\dots,a_j\},\{a'_1,\dots,a_j\}})$.

We now form the second part of our base, denoted B_2 and consisting of $p\binom{p-1}{j-1} \cdot \frac{j-1}{j} - (p-1)$ voting situations ($C_{1,j}$-$C_{2,j}$-symmetric and linearly independent from each other and all the voting situations in B_1). We start constructing each voting situation in B_2 by constructing its distinctive vote. To construct a distinctive vote, we first select the position for candidate a_j; we consider each position from $\{1, \dots p\}$. Let us fix $r \in \{1, \dots p\}$ as the position that we picked. Next, we select a set of j positions for the candidates from $\{a_1, \dots, a_{j-1}, a'_1\}$. To do that, we first construct the following graph. We associate all sets of $j-1$ positions such that r is greater[9] than at least one of them with vertices (for a fixed r there are $\binom{p-1}{j-1} - \binom{p-r}{j-1}$ such vertices; we choose $j-1$ positions out of $p-1$ still available, but we omit the situations where all these $j-1$ positions are greater than r). We say that two vertices are connected if the corresponding sets differ by exactly one element. From Lemma 6 it follows that such a graph contains a Hamiltonian path. Now, for each edge (X, X') on the considered Hamiltonian path we do the following. Let $B = X \cap X'$, and let b and b' be the two elements such that $b < b'$ and $\{b, b'\} = (X \setminus B) \cup (X' \setminus B)$. (In other words, b and b' are the two elements on which X and X' differ.) Note that $|B| = j-2$. We form a distinctive vote by putting candidate a_j on position r, candidates $a_2, \dots, a_{j-1}$ on the positions from B (in some arbitrary order), a_1 on position b, a'_1 on position b', and all the other candidates on the remaining positions (in some arbitrary order).

How many distinctive votes have we constructed? There are p possible values for the position of a_j, and for each such position we consider a graph. If the position of a_j is r, then the graph has $\binom{p-1}{j-1} - \binom{p-r}{j-1}$ vertices. Thus, altogether, there number of vertices is:[10]

$$\sum_{r=1}^{p}\left(\binom{p-1}{j-1} - \binom{p-r}{j-1}\right) = p\binom{p-1}{j-1} - \sum_{r=1}^{p}\binom{p-r}{j-1}$$
$$= p\binom{p-1}{j-1} - \binom{p}{j} = p\binom{p-1}{j-1} - \frac{p}{j}\binom{p-1}{j-1} = p\binom{p-1}{j-1}\frac{j-1}{j}.$$

(An intuitive way to obtain the same result is as follows. Let us fix the value r uniformly at random. The vertices for the graph for this value of r are size-$(j-1)$ subsets of $p-1$ positions, except those subsets that contain only elements greater than r. By symmetry, on the average the number of subsets that we omit is a $\frac{1}{j}$ fraction of all the subsets. Since we have all the graphs for all values of r, altogether we have $p\binom{p-1}{j-1}\frac{j-1}{j}$ vertices.) One of the graphs is empty (it is the one that is constructed for $r = 1$, because there is no element in $\{1, \dots, p\}$ lower than $r = 1$). Thus we have $p-1$ non-empty graphs. As a result, the total number of edges in the considered Hamiltonian paths

[9] There is a possible point of confusion here. By "greater" we mean greater as a number. So, for example, position 7 is greater than position 5 (even though we would say that a candidate ranked on position 5 is ranked higher than candidate ranked on position 7).

[10] The second equality follows from a standard property of binomial coefficients: for $m, n \in \mathbb{N}$ we have $\sum_{k=0}^{n}\binom{k}{m} = \binom{n+1}{m+1}$.

is $p\binom{p-1}{j-1}\frac{j-1}{j} - (p-1)$. Every edge corresponds to a distinctive vote, so this is also the number of distinctive votes constructed.

For each distinctive vote v constructed, we build a voting situation as follows:

Case 1. If a_1 and a'_1 are both ranked ahead of a_j, then we let τ be permutation $\tau := (a_1, a_j, a'_1)$ (i.e., we let τ be the identity permutation except that $\tau(a_1) = a_j$, $\tau(a_j) = a'_1$, $\tau(a'_1) = a_1$) and we let the voting situation consist of three votes, v, $\tau(v)$, and $\tau^{(2)}(v)$:

$$\begin{aligned} v\colon & \cdots \succ a_1 \succ \cdots \succ a'_1 \succ \cdots \succ a_j \succ \cdots \\ \tau(v)\colon & \cdots \succ a_j \succ \cdots \succ a_1 \succ \cdots \succ a'_1 \succ \cdots \\ \tau^{(2)}(v)\colon & \cdots \succ a'_1 \succ \cdots \succ a_j \succ \cdots \succ a_1 \succ \cdots \end{aligned}$$

Note that permutation τ and the sequence $F_1 = \{a_1, \ldots, a_j\}$, $F_2 = \{a_2, \ldots, a_j, a'_1\}$, $F_3 = \{a_1, \ldots, a_{j-1}, a'_1\}$, $F_4 = \{a_1, \ldots, a_j\}$ witness that this voting situation is $C_{1,j}$-$C_{2,j}$-symmetric.

Case 2. If it is not the case that a_1 and a'_1 are both ranked ahead of a_j in the distinctive vote v, then we know that there is some other candidate $a \in \{a_2, \ldots, a_{j-1}\}$ ranked ahead of a_j. This is due to our construction of distinctive votes—we always put a_j on position r and make sure that there is some candidate ranked on a position ahead of r. If all the candidates $a_2, \ldots, a_{j-1}$ were ranked behind a_j, then it would have to be the case that both a_1 and a'_1 are ranked ahead of a_j. Since it is not the case that both a_1 and a'_1 are ranked ahead of a_j, there must be some other candidate from $\{a_2, \ldots, a_{j-1}\}$ that is. We call this candidate a. We let ρ be permutation $\rho := (a_1, a'_1)(a, a_j)$ (i.e., we let ρ be the identity permutation, except that it swaps a_1 with a'_1 and a with a_j). We form a voting situation that consists of v and $\rho(v)$:

$$\begin{aligned} v\colon & \cdots \succ a \succ \cdots \succ a_j \succ \cdots \succ a_1 \succ \cdots \succ a'_1 \succ \cdots \\ \rho(v)\colon & \cdots \succ a_j \succ \cdots \succ a \succ \cdots \succ a'_1 \succ \cdots \succ a_1 \succ \cdots \end{aligned}$$

Permutation ρ and the sequence $F_1 = \{a_1, \ldots, a_j\}$, $F_2 = \{a_2, \ldots, a_j, a'_1\}$, $F_3 = \{a_1, \ldots, a_j\}$ witness that this is a $C_{1,j}$-$C_{2,j}$-symmetric voting situation.

Set B_2 consists of all the voting situations constructed from the distinctive votes.

For each $r \in \{1, \ldots, p\}$, each set of $j-1$ positions R from $\{1, \ldots, p\} \setminus \{r\}$, and each voting situation P, we define $\gamma_{r,R}(P)$ to be the total (possibly fractional) number of votes from P that have a_j on the r-th position and that have candidates from $\{a_1, a_2, \ldots, a_{j-1}\}$ on positions from R. We define $\gamma'_{r,R}(P)$ analogously, for the votes where a_j is on position r and candidates $a'_1, a_2, \ldots, a_{j-1}$ take positions from R. We define $\alpha_{r,R}(P)$ to be $\gamma_{r,R}(P) - \gamma'_{r,R}(P)$. For example, for each $P \in B_1$ we have $\alpha_{r,R}(P) = 0$.

Let us consider voting situations from B_2 which were created from a single Hamiltonian path in one of the graphs. The distinctive votes for all these voting situations have a_j on the same position; we denote this position by r. For each such voting situation P, each non-distinctive vote belonging to P has a_j on a position ahead of position r. Further, we see that there exist exactly two sets R_1 and R_2 such that $\alpha_{r,R_1}(P) \neq 0$ and $\alpha_{r,R_2}(P) \neq 0$. These are the sets that correspond to the vertices connected by the edge from which the distinctive vote for P was created (for one of them, let us say R_1, we have $\alpha_{r,R_1}(P) = 1$, and for the other we have $\alpha_{r,R_2}(P) = -1$; to see that this holds, recall that a_j is ranked on positions ahead of r in non-distinctive votes and, thus, it suffices to consider the distinctive vote only).

Now we are ready to explain why the vectors from $B_1 \cup B_2$ are linearly independent. For each nontrivial linear combination L of the vectors from $B_1 \cup B_2$ we will show that L cannot be equal to the zero vector. For the sake of contradiction let us assume that $L = \langle 0, \ldots, 0 \rangle$. We start with showing that all coefficients of vectors from B_2 in L are equal to zero. Again, for the sake of contradiction let us assume that this is not the case. Let B'_2 consist of those vectors from B_2 that appear in L

with a non-zero coefficient. Let r be the largest position of a_j in some vote in B'_2 (by "largest position" we mean largest numerically, i.e., for each vote v that occurs in some voting situation from B'_2 it holds that $\mathrm{pos}_v(a_j) \le r$). Let $B'_{2,r}$ be the set of all voting situations from B'_2 that have some votes which have a_j on position r. Each voting situation in $B'_{2,r}$ consists of either two or three votes. However, the votes belonging to those voting situations which have a_j on position r must be distinctive votes (all non-distinctive votes for voting situations in B_2 have a_j on positions ahead of r). Each such distinctive vote is built from an edge of a single Hamiltonian path (they come from the same Hamiltonian path because otherwise they would not have a_j on the same position). Let S be a voting situation in $B'_{2,r}$ that has a distinctive vote built from the latest edge on the path, among the edges that contributed voting situations to $B'_{2,r}$ (to make this notion meaningful, we orient the path in one of the two possible ways). Let R_1 and R_2 be the sets of $j-1$ positions that form this edge. By the reasoning from the previous paragraph we have that $\alpha_{r,R_1}(S) \neq 0$, $\alpha_{r,R_2}(S) \neq 0$, and one of the following two conditions must hold (depending on the orientation of the Hamiltonian path that we chose):

1. For each voting situation Q' in B'_2 other than S we have $\alpha_{r,R_1}(Q') = 0$.
2. For each voting situation Q' in B'_2 other than S we have $\alpha_{r,R_2}(Q') = 0$.

Further, for each $Q \in B_1$ we have $\alpha_{r,R_1}(Q) = \alpha_{r,R_2}(Q) = 0$. Thus, since α_{r,R_1} and α_{r,R_2} are linear functions, we have that either $\alpha_{r,R_1}(L) \neq 0$ or $\alpha_{r,R_2}(L) \neq 0$. Thus, L cannot be a zero-vector, which gives a contradiction.

We have shown that all coefficients of vectors from B_2 used to form L are equal to zero. Thus L must be a linear combination of vectors from B_1. However, the vectors from B_1 are linearly independent, which means that if L is $\langle 0, \dots, 0 \rangle$, then the coefficients of all the vectors from B_1 are zeros. Thus we conclude that the vectors from $B_1 \cup B_2$ are linearly independent.

It remains to show that $B_1 \cup B_2$ indeed forms a base of the kernel of $\alpha_{C_{1,j},C_{2,j}}$. Since vectors in B_1 and B_2 are linearly independent, it suffices to check that the cardinality of $B_1 \cup B_2$ is equal to the dimension of $\ker(\alpha_{C_{1,j},C_{2,j}})$. The number of vectors in $B_1 \cup B_2$ is equal to:

$$\underbrace{\left(p! - p\binom{p-1}{j-1} + p\right)}_{|B_1|} + \underbrace{\left(p\binom{p-1}{j-1} \cdot \frac{j-1}{j} - p + 1\right)}_{|B_2|} = p! - \frac{p}{j}\binom{p-1}{j-1} + 1 = p! - \binom{p}{j} + 1$$

This completes our induction. The proof works for arbitrary committees C_1 and C_2 due to symmetry of f_k. □

We are almost ready to show that for committees that differ by one candidate only, f_k is a committee scoring rule, and to derive its committee scoring function. However, before we do that we need to change the domain once again. We will also need some notions from topology.

Topological Definitions For every set S in some Euclidean space $\mathbb{R}^n$, by $\mathrm{int}(S)$ we mean the interior of S, i.e., the largest (in terms of inclusion) open set contained in S. By $\mathrm{conv}(S)$ we mean the convex hull of S, i.e., the smallest (in terms of inclusion) convex set that contains S. Finally, by $\overline{S}$ we define the closure of S, i.e., the smallest (in terms of inclusion) closed set that contains S. We use the concept of $\mathbb{Q}$-convex sets of Young [51] and we recall his two observations.

Definition 11 ($\mathbb{Q}$-convex sets)**.** *A set $S \in \mathbb{R}^n$ is $\mathbb{Q}$-convex if $S \in \mathbb{Q}^n$ and for each $s_1, s_2 \in S$ and each $x \in \mathbb{Q}$, $0 \le x \le 1$, it holds that $x \cdot s_1 + (1-x) \cdot s_2 \in S$.*

Lemma 9 (Young [51])**.** *Set $S \in \mathbb{R}^n$ is $\mathbb{Q}$-convex if and only if $S = \mathbb{Q}^n \cap \mathrm{conv}(S)$.*

Lemma 10 (Young [51])**.** *If a set S is $\mathbb{Q}$-convex, then $\overline{S} = \overline{\mathrm{conv}(S)}$; moreover, $\overline{S}$ is convex.*

Third Domain Change In the following arguments, we fix two arbitrary committees C_1 and C_2 such that $|C_1 \cap C_2| = k - 1$ and focus on them. (In other words, we consider function f_{C_1,C_2} instead of f_k.) In this case, Lemma 8 allows us to change the domain of the function.

Let us consider two voting situations P and Q such that $\alpha_{C_1,C_2}(P) = \alpha_{C_1,C_2}(Q)$. Since α_{C_1,C_2} is a linear function, we have $\alpha_{C_1,C_2}(P - Q) = \langle 0, \ldots, 0 \rangle$. Thus, by Lemma 8, we know that $C_1 =_{P-Q} C_2$. We can express Q as $Q = P + (Q - P)$ and thus, by consistency of f_{C_1,C_2}, we have that:

$$C_1 \succ_P C_2 \iff C_1 \succ_Q C_2.$$

Consequently, to answer the question "what is the relation between committee C_1 and C_2 according to f_{C_1,C_2} in voting situation P?" it suffices to know the value $\alpha_{C_1,C_2}(P)$. In effect, we can restrict the domain of f_{C_1,C_2} to an $\left(\binom{m}{k} - 1\right)$-dimensional space D:

$$D = \left\{ P \in \mathbb{Q}^{\binom{m}{k}} : \sum_{I \in [m]_k} P[I] = 0 \right\}.$$

We interpret elements of D as the values of the committee position-difference function, α_{C_1,C_2} and, so, the condition $\sum_{I \in [m]_k} P[I] = 0$ corresponds to the property of committee position-difference functions given in Equality 14. By the argument given above the definition of D, we know that from the point of view of comparing committees C_1 and C_2 using function f_{C_1,C_2}, the vector of values α_{C_1,C_2} provides the same information as a voting situation from which it is obtained. Thus, we can think of elements of D as corresponding to voting situations.

Separating Two Committees We proceed by defining two sets, $D_1, D_2 \subseteq D$, such that:

$$D_1 = \{P \in D : C_1 \succ_P C_2\} \quad \text{and} \quad D_2 = \{P \in D : C_2 \succ_P C_1\}.$$

That is, D_1 corresponds to situations where, according to f_{C_1,C_2}, committee C_1 is preferred over C_2, and D_2 corresponds to the situations where it is the other way round. From consistency of f_{C_1,C_2}, it follows that D_1 and D_2 are $\mathbb{Q}$-convex.

We note that if f_{C_1,C_2} is trivial, i.e., if it ranks all committees as equal, then it can be expressed as a trivial committee scoring function. Assume that f_{C_1,C_2} is nontrivial. In this case one of the sets D_1 and D_2 is nonempty. From neutrality it follows that so is the other one. Now, we move our analysis from $\mathbb{Q}^{\binom{m}{k}}$ to $\mathbb{R}^{\binom{m}{k}}$, by analyzing the closures of the sets D_1 and D_2.

Lemma 11. *The sets* $\mathrm{int}(\overline{D_1})$ *and* $\overline{D_2}$ *are disjoint, convex, and nonempty relative to D (i.e.,* $\mathrm{int}(\overline{D_1}) \cap D \neq \emptyset$ *and* $\overline{D_2} \cap D \neq \emptyset$*).*

Proof. This lemma follows from the results given by Young [51] and Merlin [37]. However, in their cases the proofs are hidden in the text. We include an explicit proof for the sake of completeness.

From Lemma 10, it follows that the sets $\overline{D_1}$ and $\overline{D_2}$ are convex and, thus, the interior $\mathrm{int}(\overline{D_1})$ is also convex. Now, we prove that $\overline{D_1} \cup \overline{D_2} = \overline{D}$, a fact that will be useful in our further analysis. If this is not the case, then $\overline{D} - (\overline{D_1} \cup \overline{D_2})$ is open in $\overline{D}$. Thus, there exists a point P and an $\left(\binom{m}{k} - 1\right)$-dimensional ball $\mathcal{B}$ such that $P \in \mathcal{B} \subseteq \overline{D} - (\overline{D_1} \cup \overline{D_2})$. Naturally, $C_1 =_P C_2$. Thus, for some $S \in D_1$, there exists a (small) $x \in \mathbb{Q}$, such that $Q = x \cdot S + (1 - x) \cdot P$ belongs to the ball $\mathcal{B}$. Since Q belongs to $\mathcal{B}$, it must be the case that $C_1 =_Q C_2$. However, by consistency of f_{C_1,C_2}, we have that $C_1 \succ_Q C_2$ and, so, we have $Q \in D_1$. This is a contradiction.

Next, we show that the set $\mathrm{int}(\overline{D_1})$ is nonempty, relatively to D. For the sake of contradiction, assume that $\mathrm{int}(\overline{D_1}) \cap D = \emptyset$. Then, from neutrality, it follows that also $\mathrm{int}(\overline{D_2}) \cap D = \emptyset$. Thus, D_1 and D_2 are nowhere dense in D,[11] and so are $\overline{D_1}$, $\overline{D_2}$, and $\overline{D_1} \cup \overline{D_2} = \overline{D}$. Consequently, we get

[11] A subset A of a topological space X is called nowhere dense (in X) if there is no neighborhood in X on which A is dense.

that $\overline{D}$ is nowhere dense in D, a contradiction with the density of D in $\overline{D}$[12] (density of D follows immediately from its definition).

Now we show that $\mathrm{int}(\overline{D_1})$ and D_2 are disjoint. For the sake of contradiction, let us assume that there exists $P \in \overline{D}$, such that $P \in \mathrm{int}(\overline{D_1})$ and $P \in D_2$. From Lemma 10, we get that $\mathrm{int}(\overline{D_1}) = \mathrm{int}(\overline{\mathrm{conv}(D_1)}) = \mathrm{int}(\mathrm{conv}(D_1))$. This means that $P \in \mathrm{int}(\mathrm{conv}(D_1)) \cap D_2$ and, so, $P \in \mathrm{conv}(D_1) \cap D_2$. Since $P \in D_2$, we know that $P \in \mathbb{Q}^{\binom{m}{k}}$. By Lemma 9 we know that $D_1 = \mathbb{Q}^{\binom{m}{k}} \cap \mathrm{conv}(D_1)$. Thus, since $P \in \mathbb{Q}^{\binom{m}{k}}$ and $P \in \mathrm{conv}(D_1)$, we know that $P \in D_1$. All in all, it must be the case that $P \in D_1 \cap D_2$, which is a contradiction because $D_1 \cap D_2 = \emptyset$.

Finally, for the sake of contradiction, let us assume that there exists $Q \in \overline{D}$, such that $Q \in \mathrm{int}(\overline{D_1})$ and $Q \in \overline{D_2}$. Since $Q \in \overline{D_2}$, this means that every open set containing Q must have nonempty intersection with D_2. Consequently, $\mathrm{int}(\overline{D_1})$ has nonempty intersection with D_2, which—by the previous paragraph—gives a contradiction. This completes the proof of the lemma. ◻

Recovering the Scoring Function We are finally ready to derive our committee scoring function. From the classic hyperplane separation theorem, it follows that there exists a vector $\eta \in \mathbb{R}^{\binom{m}{k}}$ such that (for $P \in D$, by $\eta \cdot P$ we mean the dot product of P and η, both treated as $\binom{m}{k}$ dimensional vectors):

1. For each voting situation $P \in \overline{D_2}$ it holds that $\eta \cdot P \leq 0$.
2. For each voting situation $P \in \mathrm{int}(\overline{D_1})$ it holds that $\eta \cdot P > 0$.

We note that Lemma 11 allows us to directly apply the hyperplane separation theorem as the sets $\mathrm{int}(\overline{D_1})$ and $\overline{D_2}$ are disjoint. This is different from Young's [51] and Merlin's [37] approach, who operate on sets with disjoint interiors, but which do not have to be disjoint on their own.

We now show that if $P \in D$ and $\eta \cdot P > 0$, then $P \in D_1$. Since $\eta \cdot P > 0$, P cannot belong to D_2, but it might be the case that $C_1 =_P C_2$. For the sake of contradiction, let us assume that this is the case. We observe that there exists an $\left(\binom{m}{k} - 1\right)$-dimensional ball $\mathcal{B}$ in D with $P \in \mathcal{B}$, such that for each $S \in \mathcal{B}$ we have $C_1 \geq_S C_2$ (this is because P does not belong to $\overline{D_2}$). Let us now consider two cases.

Case 1. If for each $S \in \mathcal{B}$ we have $C_1 =_S C_2$, then we proceed as follows. Let us take some Q such that $C_1 >_Q C_2$. There must exist some (possibly very small) x such that $S = x \cdot Q + (1-x) \cdot P \in \mathcal{B}$. However, from consistency we would get that $C_1 >_S C_2$, a contradiction.

Case 2. If there exists $Q \in \mathcal{B}$ such that $C_1 >_Q C_2$, then we observe that there exists $0 < \epsilon < 1$ such that $S = \frac{P-\epsilon Q}{1-\epsilon} \in \mathcal{B}$. Since $S \in \mathcal{B}$, we have that $C_1 \geq_S C_2$. Further, we have that $P = \epsilon Q + (1-\epsilon)S$. By consistency of f_{C_1,C_2} we get that $C_1 >_P C_2$. However, this is a contradiction.

Next, we show that if $\eta \cdot P < 0$, then $P \in D_2$. For the sake of contradiction, let us assume that there is P such that $\eta \cdot P < 0$ but $C_1 \geq_P C_2$. Then there exists such ϵ that if $|Q - P| < \epsilon$ then $\eta \cdot Q < 0$ (and so $Q \notin \mathrm{int}(\overline{D_1})$). Thus there exists a ball $\mathcal{B}$ in D with $P \in \mathcal{B}$, such that $\mathcal{B} \cap \mathrm{int}(\overline{D_1}) = \emptyset$. Thus, $\mathcal{B} \cap D_1 = \emptyset$. We infer that some point S in $\mathcal{B}$ could be represented as a linear combination of P and some point from D_1. From consistency we would get that $C_1 >_S C_2$, a contradiction.

Remark 2. *We have shown that for each $P \in D$, (a) $\eta \cdot P > 0$ implies that $P \in D_1$ (and, so, $C_1 >_P C_2$), and (b) $\eta \cdot P < 0$ implies that $P \in D_2$ (and, so, $C_2 >_P C_1$). From symmetry, the same vector η works for each pair of committees C_1 and C_2 such that $|C_1 \cap C_2| = k - 1$.*

Now we will use continuity to prove that if $\eta \cdot P = 0$ then $C_1 =_P C_2$. For the sake of contradiction let us assume that this is not the case, i.e., that there exists a voting situation $P \in D$ such that $\eta \cdot P = 0$

[12] A subset A of a topological space X is dense in X if for every point x in X, each neighborhood of x contains at least one point from A (i.e., A has non-empty intersection with every non-empty open subset of X).

but $C_1 \neq_P C_2$. Without loss of generality, let us assume that $C_1 \succ_P C_2$. Let Q be a voting situation such that $\eta \cdot Q < 0$ and so $C_2 \succ_Q C_1$. For each x it holds that $\eta \cdot (xP + Q) < 0$ and so $C_2 \succ_{xP+Q} C_1$. However, this contradicts continuity of f_k.

From the vector η, we retrieve a committee scoring function λ. For each committee position $I \in [m_k]$ we set $\lambda(I) = \eta[I]$. Now, we can see that for each two committees C_1, C_2, and for each voting situation $P \in \mathbb{Q}^{m!}$ it holds that (see the comment below for explanation what Q is):

$$\begin{aligned}\mathrm{score}_\lambda(C_1, P) - \mathrm{score}_\lambda(C_2, P) &= \sum_{I\in[m]_k} (\lambda(I) \cdot \text{pos-weight}_I(C_1, P) - \lambda(I) \cdot \text{pos-weight}_I(C_2, P)) \\ &= \sum_{I\in[m]_k} \lambda(I) \cdot \alpha_{C_1,C_2}(P)[i] = \sum_{I\in[m]_k} \eta[I] \cdot \alpha_{C_1,C_2}(P) = \eta(I) \cdot Q,\end{aligned}$$

where $Q \in D$ is the representation of P in the space D. From the above inequality we see that $\mathrm{score}_\lambda(C_1, P) > \mathrm{score}_\lambda(C_2, P)$ implies that $C_1 \succ_P C_2$ and that $\mathrm{score}_\lambda(C_1, P) = \mathrm{score}_\lambda(C_2, P)$ implies that $C_1 =_P C_2$. From neutrality we get that the same committee scoring function λ works for every two committees, C_1' and C_2', with $|C_1' \cap C_2'| = k - 1$

There is one more issue we need to deal with. So far, we gave no argument as to why λ should satisfy the dominance property of committee scoring functions (i.e., that if I and J are two committee positions such that I dominates J, then $\lambda(I) \geq \lambda(J)$). However, to get this property it suffices to assume the committee dominance axiom for f_k.

Summarizing our discussion from this section, we get our main result, Theorem 1, for the committees C_1 and C_2, with $|C_1 \cap C_2| = k - 1$. We will continue our analysis of the remaining cases in the subsequent sections.

A.3 Putting Everything Together: Comparing Arbitrary Committees

In the previous section we have shown that if a k-winner election rule f_k is symmetric, committee-neutral, consistent, continuous, and has the committee dominance property then there exists a committee scoring function λ such that for each two committees C_1 and C_2 such that $|C_1 \cap C_2| = k-1$ and each voting situation $P \in \mathbb{Q}^{m!}$ it holds that $\mathrm{score}_\lambda(C_1, P) > \mathrm{score}_\lambda(C_2, P)$ if and only if $C_1 \succ_P C_2$, and $\mathrm{score}_\lambda(C_1, P) = \mathrm{score}_\lambda(C_2, P)$ if and only if $C_1 =_P C_2$. In this section we complete our proof by showing that this holds (through the same function λ) for every two committees.

Setting Up the Proof Let f_k be a k-winner election rule that is symmetric, committee-neutral, consistent, continuous, and has the committee dominance property. Let λ be the scoring function derived for this f_k as described at the end of the previous section. We know that for each two committees C_1 and C_2 such that $|C_1 \cap C_2| = k - 1$ and each voting situation $P \in \mathbb{Q}^{m!}$ it holds that $\mathrm{score}_\lambda(C_1, P) > \mathrm{score}_\lambda(C_2, P)$ if and only if $C_1 \succ_P C_2$, and $\mathrm{score}_\lambda(C_1, P) = \mathrm{score}_\lambda(C_2, P)$ if and only if $C_1 =_P C_2$. We will show that the same holds for all committees C_1 and C_2, irrespective of the size of their intersection. We will show this by induction over $k - |C_1 \cap C_2|$.

Let us fix some value $k' < k-1$ and let us assume that λ can be used to distinguish whether some committee C_1 is preferred over some committee C_2 whenever $|C_1 \cap C_2| > k'$. We will show that the same λ can be used to distinguish whether a committee C_1 is preferred over a committee C_2 when $|C_1 \cap C_2| = k'$.

Let C_1 and C_2 be two arbitrary committees such that $|C_1 \cap C_2| = k'$. Let us rename the candidates so that $C_1 \setminus C_2 = \{c_1, \dots, c_{k-k'}\}$, $C_1 \cap C_2 = \{c_{k-k'+1}, \dots, c_k\}$ and $C_2 \setminus C_1 = \{c_{k+1}, \dots, c_{2k-k'}\}$.

The Case Where $\mathbf{k - k'}$ Is Even If $k - k'$ is even, we consider the following two cases:

Case 1: There exists a vector of $2k - k'$ positions $\langle p_1, \dots, p_{2k-k'}\rangle$ such that:

$$\lambda(\{p_1, \dots, p_k\}) + \lambda(\{p_{k-k'+1}, \dots, p_{2k-k'}\}) \neq 2\lambda(\{p_{\frac{k-k'}{2}+1}, \dots, p_{\frac{k-k'}{2}+k}\}). \tag{15}$$

Let us consider the committee $C_3 = \{c_{\frac{k-k'}{2}+1}, \ldots, c_{\frac{k-k'}{2}+k}\}$. We consider the linear space of voting situations $P \in \mathbb{Q}^{m!}$ such that $C_1 =_P C_3$ and $C_3 =_P C_2$ (the fact that this is a linear space follows from the inductive assumption; $|C_1 \cap C_3| = |C_2 \cap C_3| > k'$). The conditions $C_1 =_P C_3$ and $C_3 =_P C_2$ are not contradictory (Consider the profile in which each vote is cast exactly once—in such profile all size-k committees are equivalent with respect to f_k). This space has dimension either $m! - 2$ or $m! - 1$. This is so, because each of the conditions $C_1 =_P C_3$ and $C_2 =_P C_3$ boils down to a single linear equation. If these equations are independent then the dimension is $m! - 2$. Otherwise, it is $m! - 1$. By transitivity of f_k we get that in each voting situation P from this space it holds that $C_1 =_P C_2$ and that the committee score of C_1 (according to λ) is equal to the committee score of C_2. Let B be a base of this space. Further, let v be a vote where each candidate c_i, $i \in \{1, \ldots, 2k - k'\}$, stands on position p_i (recall Equation (15) above), and let v' be an identical vote except that candidates from $C_1 \cup C_2$ are listed in the reverse order (i.e., c_1 is on position $p_{2k-k'}$, c_2 is on position $p_{2k-k'-1}$ and so on). Let S_b be a voting situation that consists of v and v'. The positions of C_1 and C_3 in v are:

$$\mathrm{pos}_v(C_1) = \{p_1, \ldots, p_k\} \quad \text{and} \quad \mathrm{pos}_v(C_3) = \{p_{\frac{k-k'}{2}+1}, \ldots, p_{\frac{k-k'}{2}+k}\}$$

The positions of C_1 and C_3 in v' are:

$$\mathrm{pos}_{v'}(C_1) = \{p_{k-k'+1}, \ldots, p_{2k-k'}\} \quad \text{and} \quad \mathrm{pos}_{v'}(C_3) = \{p_{\frac{k-k'}{2}+1}, \ldots, p_{\frac{k-k'}{2}+k}\}$$

Consequently, according to (15), in voting situation S_b the committee score of C_1 is not equal to that of C_3. By the inductive assumption, it must be the case that $C_1 \neq_{S_b} C_3$. This means that the voting situations in $B \cup \{S_b\}$ are linearly independent.

We now show that $C_1 =_{S_b} C_2$. Consider a permutation σ (over the candidate set) that swaps c_1 with $c_{2k-k'}$, c_2 with $c_{2k-k'-1}$, and so on. We note that $\sigma(C_1) = C_2$, $\sigma(C_2) = C_1$, and $S_b = \sigma(S_b)$. Thus, by symmetry of f_k, it must be the case that $C_1 =_{S_b} C_2$. Further, the committee scores of C_1 and C_2 are equal in S_b.

Altogether, the base $B \cup \{S_b\}$ defines an $(m!-1)$-dimensional space of voting situations P such that $C_1 =_P C_2$ and the committee scores of C_1 and C_2 are equal. From Corollary 2 we know that the set of voting situations P such that $C_1 =_P C_2$ forms a linear space of dimension $m!-1$. As a result, we get that for each voting situation P the condition $C_1 =_P C_2$ is equivalent to the condition that C_1 has the same committee score as C_2 according to λ.

The fact that $C_1 \succ_S C_2$ whenever the committee score of C_1 is greater than of C_2 follows from Lemma 4.

Case 2: For each vector of $2k - k'$ positions $\langle p_1, \ldots, p_{2k-k'} \rangle$ it holds that (note that the condition below is a negation of the condition from Case 1):

$$\lambda(\{p_1, \ldots, p_k\}) - \lambda(\{p_{\frac{k-k'}{2}+1}, \ldots, p_{\frac{k-k'}{2}+k}\}) = \lambda(\{p_{\frac{k-k'}{2}+1}, \ldots, p_{\frac{k-k'}{2}+k}\}) - \lambda(\{p_{k-k'+1}, \ldots, p_{2k-k'}\}).$$

As before, let $C_3 = \{c_{\frac{k-k'}{2}+1}, \ldots, c_{\frac{k-k'}{2}+k}\}$. Since the above equality must hold for each vector of $2k - k'$ positions, we see that if the committee score of C_1 is equal to the committee score of C_3, then the committee score of C_3 is equal to the committee score of C_2. Consequently, by the inductive assumption, we get that $C_1 =_P C_3$ implies that $C_3 =_P C_2$. Thus, by f_k's transitivity, we get that for each voting situation P, the condition $C_1 =_P C_3$ implies that $C_1 =_P C_2$. As a consequence of this reasoning, there exists an $(m! - 1)$-dimensional space of voting situations P such that $C_1 =_P C_2$ and such that C_1 has the same committee score as C_2. Similarly as in Case 1, we conclude that for each voting situation P the condition $C_1 =_P C_2$ is equivalent to the condition that C_1 has the same committee score as C_2 according to λ, and that it holds that $C_1 \succ_P C_2$ whenever the committee score of C_1 is greater than of C_2 (by Lemma 4).

The Case Where $\mathbf{k - k' \geq 3}$ and $\mathbf{k - k'}$ is Odd Similarly as before we consider two cases:

Case 1: There exists a vector of $2k - k'$ positions $\langle p_1, \ldots, p_{2k-k'} \rangle$ and a number $x \in \{1, \ldots k - k'\}$ such that:

$$\lambda(\{p_1, \ldots, p_k\}) + \lambda(\{p_{k-k'+1}, \ldots, p_{2k-k'}\}) \neq \lambda(\{p_x, \ldots, p_{k+x-1}\}) + \lambda(\{p_{k-k'+2-x}, \ldots, p_{2k-k'+1-x}\}).$$

In this case we can repeat the reasoning from Case 1 from the previous subsection (it suffices to take $C_3 = \{c_x, \ldots, c_{k+x-1}\}$).

Case 2: For each vector of $2k - k'$ positions $\langle p_1, \ldots, p_{2k-k'} \rangle$ and each number $x \in \{1, \ldots k - k'\}$ it holds that:

$$\lambda(\{p_1, \ldots, p_k\}) + \lambda(\{p_{k-k'+1}, \ldots, p_{2k-k'}\}) = \lambda(\{p_x, \ldots, p_{k+x-1}\}) + \lambda(\{p_{k-k'+2-x}, \ldots, p_{2k-k'+1-x}\}).$$

The above inequality for $x = \lfloor \frac{k-k'}{2} \rfloor$ and for $x = \lfloor \frac{k-k'}{2} \rfloor + 1$ gives, respectively (note that $k - k' - \lfloor \frac{k-k'}{2} \rfloor = \lceil \frac{k-k'}{2} \rceil$):

$$\lambda(\{p_1, \ldots, p_k\}) + \lambda(\{p_{k-k'+1}, \ldots, p_{2k-k'}\}) = \lambda(\{p_{\lfloor \frac{k-k'}{2} \rfloor} \ldots, p_{k+\lfloor \frac{k-k'}{2} \rfloor -1}\}) + \lambda(\{p_{\lceil \frac{k-k'}{2} \rceil +2} \ldots, p_{k+\lceil \frac{k-k'}{2} \rceil +1}\}),$$

and:

$$\lambda(\{p_1, \ldots, p_k\}) + \lambda(\{p_{k-k'+1}, \ldots, p_{2k-k'}\}) = \lambda(\{p_{\lfloor \frac{k-k'}{2} \rfloor +1} \ldots, p_{k+\lfloor \frac{k-k'}{2} \rfloor}\}) + \lambda(\{p_{\lceil \frac{k-k'}{2} \rceil +1} \ldots, p_{k+\lceil \frac{k-k'}{2} \rceil}\}).$$

Together, these two equalities give that:

$$\lambda(\{p_{\lfloor \frac{k-k'}{2} \rfloor} \ldots, p_{k+\lfloor \frac{k-k'}{2} \rfloor -1}\}) + \lambda(\{p_{\lceil \frac{k-k'}{2} \rceil +2} \ldots, p_{k+\lceil \frac{k-k'}{2} \rceil +1}\}) = \lambda(\{p_{\lfloor \frac{k-k'}{2} \rfloor +1} \ldots, p_{k+\lfloor \frac{k-k'}{2} \rfloor}\}) + \lambda(\{p_{\lceil \frac{k-k'}{2} \rceil +1} \ldots, p_{k+\lceil \frac{k-k'}{2} \rceil}\}).$$

Since the above equality holds for each vector of $2k-k'$ positions, after renaming the positions, we get that for each set of $k + 3$ positions $\langle q_1, \ldots, q_{k+3} \rangle$ it holds that:

$$\lambda(\{q_1, \ldots, q_k\}) + \lambda(\{q_4, \ldots, q_{k+3}\}) = \lambda(\{q_2, \ldots, q_{k+1}\}) + \lambda(\{q_3, \ldots, q_{k+2}\}).$$

After reformulation we get:

$$\lambda(\{q_1, \ldots, q_k\}) - \lambda(\{q_2, \ldots, q_{k+1}\}) = \lambda(\{q_3, \ldots, q_{k+2}\}) - \lambda(\{q_4, \ldots, q_{k+3}\}). \tag{16}$$

If k is odd, we obtain the following series of equalities (the consecutive equalities, except for the last one, are consequences of applying Equality 16 to the cyclic shifts of the list $\langle q_1, q_2, \ldots, q_{k+3} \rangle$; the last equality breaks the pattern and is a consequence of applying Equality 16 to the list $\langle q_{k+2}, q_{k+3}, q_1, q_2, \ldots, q_{k-1}, q_{k+1}, q_k \rangle$):

$$\begin{aligned}
\lambda(\{q_1, \ldots, q_k\}) - \lambda(\{q_2, \ldots, q_{k+1}\}) &= \lambda(\{q_3, \ldots, q_{k+2}\}) - \lambda(\{q_4, \ldots, q_{k+3}\}) \\
&= \lambda(\{q_5, \ldots, q_{k+3}, q_1\}) - \lambda(\{q_6, \ldots, q_{k+3}, q_1, q_2\}) \\
&= \lambda(\{q_7, \ldots, q_{k+3}, q_1, q_2, q_3\}) - \lambda(\{q_8, \ldots, q_{k+3}, q_1, q_2, q_3, q_4\}) \\
&\;\;\vdots \\
&= \lambda(\{q_{k+2}, q_{k+3}, q_1, \ldots, q_{k-2}\}) - \lambda(\{q_{k+3}, \ldots, q_1, q_{k-1}\}) \\
&= \lambda(\{q_1, \ldots, q_{k-1}, q_{k+1}\}) - \lambda(\{q_2, \ldots, q_{k+1}\}),
\end{aligned}$$

In consequence, it must be the case that $\lambda(\{q_1, \dots, q_k\}) = \lambda(\{q_1, \dots, q_{k-1}, q_{k+1}\})$. Thus, by transitivity, we get that λ is a constant function (in essence, what we have shown is that we can replace positions in the set of k positions, one by one, without changing the value of the committee scoring function). Let $C_3 = \{c_2, \dots, c_{k+1}\}$. Since λ is a constant function, then by the inductive assumption we have that for every voting situation P it holds that $C_1 =_P C_3$ and $C_3 =_P C_2$. By transitivity we get that for each voting situation P it holds that $C_1 =_P C_2$. Thus our trivial scoring function works correctly on C_1 and C_2.

Let us now assume that k is even. Now we obtain the following series of equalities (in this case all the consecutive equalities are consequences of applying Equality 16 to the cyclic shifts of the sequence $\langle q_1, q_2, \dots, q_{k+3}\rangle$):

$$\begin{aligned}\lambda(\{q_1, \dots, q_k\}) - \lambda(\{q_2, \dots, q_{k+1}\}) &= \lambda(\{q_3, \dots, q_{k+2}\}) - \lambda(\{q_4, \dots, q_{k+3}\}) \\ &= \lambda(\{q_5, \dots, q_{k+3}, q_1\}) - \lambda(\{q_6, \dots, q_{k+3}, q_1, q_2\}) \\ &= \lambda(\{q_7, \dots, q_{k+3}, q_1, q_2, q_3\}) - \lambda(\{q_8, \dots, q_{k+3}, q_1, q_2, q_3, q_4\}) \\ &\ \ \vdots \\ &= \lambda(\{q_{k+3}, q_1, \dots, q_{k-1}\}) - \lambda(\{q_1, \dots, q_k\}) \\ &= \lambda(\{q_2, \dots, q_{k+1}\}) - \lambda(\{q_3, \dots, q_{k+2}\}),\end{aligned}$$

In consequence, it is the case that:

$$\lambda(\{q_1, \dots, q_k\}) - \lambda(\{q_2, \dots, q_{k+1}\}) = \lambda(\{q_2, \dots, q_{k+1}\}) - \lambda(\{q_3, \dots, q_{k+2}\}),$$

and this holds for every sequence $\langle q_1, \dots, q_{k+2}\rangle$ of positions. Thus, we get that for each voting situation in which $\{c_1, \dots, c_k\}$ is equivalent to $\{c_2, \dots, c_{k+1}\}$, it also holds that $\{c_2, \dots, c_{k+1}\}$ is equivalent to $\{c_3, \dots, c_{k+2}\}$, it also holds that $\{c_3, \dots, c_{k+2}\}$ is equivalent to $\{c_4, \dots, c_{k+3}\}$, etc. Let $C_3 = \{c_2, \dots, c_{k+1}\}$. From the preceding reasoning we have that for each voting situation P the fact that it holds that $C_1 =_P C_3$ implies that $C_1 =_P C_2$. We conclude the proof in the same way as in the case of even $k - k'$ (Case 2). Specifically, we conclude that there exists an $(m! - 1)$-dimensional space of voting situations P such that $C_1 =_P C_2$ and such that C_1 has the same committee score as C_2. This means that for each voting situation P the condition $C_1 =_P C_2$ is equivalent to the condition that C_1 has the same committee score as C_2 according to λ, and that it holds that $C_1 \succ_P C_2$ whenever the committee score of C_1 is greater than of C_2 (by Lemma 4).

The End We have shown that if a k-winner rule is symmetric, committee-neutral, consistent, continuous, and has the committee-dominance property, then it is a committee scoring rule. On the other hand, committee scoring rules satisfy all these conditions. This completes our proof of Theorem 1.

B Related Work

Axiomatic characterizations of single-winner election rules have been studied actively for quite a long time. Indeed, the famous theorems of Arrow [1] and the related, and equally famous, results of Gibbard [26] and Satterthwaite [42] can be seen as axiomatic characterizations of the dictatorial rule[13] (however, typically these theorems are considered as impossibility results regarding design of perfect voting rules). Other famous axiomatic characterizations of single-winner rules include the characterizations of the majority rule[14] due to May [36] and Fishburn [20], several different

[13] Under the dictatorial rule, the winner is the candidate most preferred by a certain fixed voter (the dictator).
[14] The majority rule selects the one out of two candidates that is preferred by the majority of the voters.

characterizations of the Borda rule [50, 23, 47] and of the Plurality rule [41, 12], the characterization of the Kemeny rule[15] [52], a characterization of the Antiplurality rule [7], and the characterizations of the approval rule[16] due to Fishburn [38] and Sertel [43]. Freeman et al. [24] proposed an axiomatic characterization of runoff methods, i.e., methods that proceed in multiple rounds and in each round eliminate a subset of candidates (STV, a rule used, e.g., in Australia, is perhaps the best known example of such a runoff rule). Still, in terms of axiomatic properties, single-winner scoring rules appear to be the best understood single-winner rules. Some of their axiomatic characterizations were proposed by Gärdenfors [25], Smith [47] and Young [51] (we refer the reader to the survey of Chebotarev and Shamis [11] for an overview of these characterizations). Naturally, for many voting rules no axiomatic characterizations are known at all.

The state of research regarding axiomatic characterizations—or even simple axiomatic studies—for the case of multiwinner elections is far less developed. Indeed, we are aware of only one unconditional characterization of a multiwinner rule: Debord's has characterized the k-Borda rule as the only rule that satisfies neutrality, faithfulness, consistency, and the cancellation property [14] (see his work for details). Yet, there exists a very interesting line of research, where the properties of multiwinner election rules are studied. A large chunk of this literature focuses on the principle of Condorcet consistency [6, 28, 22, 40], and on approval-based multiwinner rules [31, 32, 4, 3]. Properties of other multiwinner election rules have been studied by Felsenthal and Maoz [19], Elkind et al. [15], and—in a somewhat different context—Skowron [44].

In their effort to analyze axiomatic properties of multiwinner rules, Elkind et al. [15] introduced the notion of committee scoring rules, the main focus of the current work. Committee scoring rules were later studied axiomatically and algorithmically by Faliszewski et al. [18, 17]. In these two papers, Faliszewski et al. have studied the internal structure of the class of committee scoring rules. In particular, they have identified many interesting subclasses, and found that most committee scoring rules are NP-hard to compute, but in many cases there are good approximation algorithms (the work on the complexity of committee scoring rules can be traced to the studies of the complexity of the Chamberlin–Courant rule, initiated by Procaccia, Rosenschein and Zohar [39] and continued by Lu and Boutilier [34], Betzler et al. [8], and Skowron et al. [46]). The axiomatic part of the works of Faliszewski et al., in particular, has lead to a characterization of the Bloc rule among the class of committee scoring rules. Skowron et al. [45] has studied family of multiwinner rules that are based on utility values instead of preference orders, and where these utilities are aggregated using ordered weighted average operators (OWA operators) of Yager [49]. (The same class, but for approval-based utilities, was studied by Aziz et al. [4, 3]). It is easy to express these OWA-based rules as committee scoring rules.

Finally, we mentioned a different line of work related to ours. In Section A.1 we studied decision rules which return generalizations of majority graphs. In the world of single-winner elections, majority graphs are often seen as inputs to election procedures (known as tournament solution concepts). For example, according to the Copeland method [13] the candidate with the greatest number of victories in pairwise comparisons with other candidates is a winner. The Smith set [48] is another example of such a rule: it returns the minimal (in terms of inclusion) subset of candidates, such that each member of the set is preferred by the majority of voters over each candidate outside the set. Fishburn [21] describes nine other solution concepts that explore the Condorcet principle for majority graphs. For an overview of tournament solution concepts we refer the reader to the book of Laslier [33] (and to the chapter of Brandt, Brill, and Harrenstein [9] for a more computational perspective). We believe that it would be a fascinating topic of research to explore the properties (computational or axiomatic) of the generalized tournaments generated by our decision rules.

---

[15] The Kemeny rule, given the set of rankings over the alternatives, returns a ranking that minimizes the sum of the Kendall tau [29] distances to the rankings provided by the voters.

[16] In the approval rule, each voter expresses his or her preferences by providing a set of approved candidates. A candidate that was approved by most voters is announced as the winner.