Committee Scoring Rules[1]

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Abstract

We present and advertise the class of committee scoring rules, recently introduced as multiwinner analogues of single-winner scoring rules. We present a hierarchy of committee scoring rules (which includes all previously studied subclasses of committee scoring rules, as well as two new subclasses) and discuss their axiomatic properties (while focusing on fixed-majority consistency and monotonicity) as well as their computational properties (with a special focus on positive results).

1 Introduction

The goal of this paper is to advertise and popularize the class of committee scoring rules, recently introduced by Elkind et al. [9] as multiwinner analogues of single-winner scoring rules. We present several examples of committee scoring rules, show situations where they can be practically useful, and discuss their axiomatic and computational properties. The paper is a fusion of the text and results from our two previous papers on the topic [13, 12] (we clearly mark which result comes from which of these papers and we omit all proofs).

We consider the standard multiwinner voting setting, where a group of voters expresses preferences over a set of candidates (in our case, by ranking them), and the task is to find a committee—a set of candidates of a given size k—that, in some sense, best represents the voters' preferences. Since there are many senses in which a committee may represent the voters' preferences, there are many multiwinner voting rules. These include, e.g., the famous single transferable vote rule (the STV rule), the Monroe rule [20], rules based on multiwinner extensions of the Condorcet principle [4, 14, 7], and a number of rules generalizing the single-winner Approval rule [17]. Yet, we focus on committee scoring rules. This means considering rules such as, e.g., SNTV, Bloc, k-Borda, the Chamberlin–Courant's rule [6], some variants of the proportional approval voting rule—PAV [17, 2], and many others.

The basic idea of Elkind et al. [9] of generalizing standard single-winner scoring rules to the multiwinner setting was to generalize the meaning of a position in a vote. To this end, they said that a position of a committee is, simply, the sorted list of the positions of its members. For example, if a committee consisted of three candidates, candidate a ranked on the top position of a vote, candidate b ranked two positions behind a in this vote, and candidate c ranked just behind b, then the position of this committee in the vote would be $(1, 3, 4)$. In consequence, a *committee scoring function* associates a score with every possible committee position, and a given committee C receives from each voter the number of points that corresponds to C's position. The committee with the highest score wins.

On the one hand, the fact that committee positions are sorted sequences of the positions of committee members means that committee scoring functions can, possibly, be very complicated (because there are exponentially many possible positions of committees). On the other hand, it also makes committee scoring rules very expressive (as witnessed by the number of committee scoring rules—such as SNTV, Bloc, k-Borda, or the Chamberlin–Courant's rule—that were introduced and studied long before the formal notion of committee scoring rules was introduced).

We aim to explore axiomatic and computational properties of committee scoring rules, and this requires partitioning them into subclasses. Indeed, the class of committee scoring

[1]This paper is a combination of two other papers by the same authors [13, 12].

rules is so general that there are only precious few properties that all the rules in the class have in common. More to the point, societies interested in using committee scoring rules are not necessarily interested in general properties of the whole class, but rather in the properties of the specific rules they consider. Indeed, already Elkind et al. [9] identified (weakly) separable committee scoring rules (such as SNTV, Bloc, and k-Borda) and representation-focused rules (such as the Chamberlin–Courant rule), and later Skowron et al. [23] and Aziz et al. [2, 3] (both teams working in a somewhat different setting than ours) introduced OWA-based rules which employ *ordered weighted average* operators, OWAs,[2] to generalize both (weakly) separable and representation-focused rules. Skowron et al. [23] and Aziz et al. [2, 3] also presented a comprehensive set of results showing computational hardness (but also approximability) for many multiwinner voting rules, most of which can be interpreted as committee scoring rules. We take on from this point and present the following results:

1. We describe the subclasses of top-k-counting rules [13] and decomposable rules [12], and present relations between all the classes of committee scoring rules discussed to date (weakly separable, representation-focused, and top-k-counting rules are all OWA-based, and all OWA-based rules are decomposable; in effect we obtain an initial hierarchy of expressiveness for committee scoring rules). Regarding classes of top-k-counting, representation-focused, and weakly separable rules, for each two of these, we show that their intersection contains exactly one nontrivial rule, and we identify this rule (in each case, it is a natural, previously-studied rule).

2. We present axiomatic properties of several classes of committee scoring rules. For example, we show that only top-k-counting rules can satisfy a certain form of majority consistency (and exactly characterize which top-k-counting rules do satisfy it); we characterize weakly separable rules as exactly those committee scoring rules that satisfy the noncrossing monotonicity property of Elkind et al. [9]; and we show that only decomposable rules can satisfy a refinement of this notion (prefix monotonicity). We also show that all nontrivial committee scoring rules have the nonimposition property (a fundamental property that says that for each possible committee there is an election where it wins uniquely).

3. We present positive algorithmic results for several classes of committee scoring rules. In particular, we show a natural $(1 - \frac{1}{e})$-approximation algorithm for a large subclass of decomposable rules, and some specialized algorithms for top-k-counting rules. (Intuitively, almost all committee scoring rules are computationally hard and, so, it is far more interesting to focus on positive algorithmic results; thus, and due to space restrictions, we omit a formal discussion of hardness results).

We believe that committee scoring rules are very practical and can be employed in a number of settings, ranging from elections of parliaments and other collective bodies (e.g., university senates), through various business settings (such as picking items a company may offer to its customers [18, 19], or picking items for users to jointly use [9, 23]), to genetic algorithms [11]. As an example of a business setting where committee scoring rule can be useful, we describe a shirt store that wants to maximize its revenue; the store has limited area, so it can only present k shirts to its customers (out of many more offered by its suppliers), and the customers have opinions regarding attractiveness of particular shirts. In a series of examples, we will argue that depending on what assumptions one makes regarding the behavior of the customers, one should use different committee scoring rules to find a set of shirts that would sell best.

[2]See the original work of Yager [24] for a general discussion of OWAs, and, e.g., the works of Kacprzyk et al. [16] or Goldsmith et al. [15] for their other applications in voting.

2 Preliminaries

For each positive integer t, we write $[t]$ to mean $\{1, \ldots, t\}$. An election is a pair $E = (C, V)$, where $C = \{c_1, \ldots, c_m\}$ is a set of candidates and $V = (v_1, \ldots, v_n)$ is a collection of voters. Each voter v_i has a preference order $\succ_i$, expressing his or her ranking of the candidates, from the most desirable one to the least desirable one. Given a voter v and a candidate c, by $\mathrm{pos}_v(c)$ we mean the position of c in v's preference order (the top-ranked candidate has position 1, the next one has position 2, and so on).

A multiwinner voting rule is a function $\mathcal{R}$ that, given an election $E = (C, V)$ and a committee size k, $1 \leq k \leq |C|$, returns a family $\mathcal{R}(E, k)$ of size-k subsets of C, i.e., the set of committees that tie as winners of the election (we use the nonunique-winner model); Later, we provide several examples of multiwinner rules.

Single-Winner Scoring Functions. Most of the multiwinner rules that we study are based on single-winner scoring functions. A single-winner scoring function for m candidates is a nonincreasing function γ, $\gamma \colon [m] \to \mathbb{R}_+$, that assigns a score value to each position in a preference order. Given a preference order $\succ_i$ and a candidate c, by the γ-score of c (given by voter v_i) we mean the value $\gamma(\mathrm{pos}_i(c))$. The two most commonly used scoring functions are the Borda scoring function, $\beta_m(i) = m - i$, and the t-approval scoring function, α_t, where $\alpha_t(i) = 1$ for $i \leq t$, and $\alpha_t(i) = 0$ otherwise.

Committee Scoring Rules. Let m and k be two positive integers, $k \leq m$ (intuitively, m is the number of candidates and k is the committee size). We write $[m]_k$ to denote the set of all length-k increasing sequences of numbers from $[m]$. Given two sequences, $I = (i_1, \ldots, i_k)$ and $J = (j_1, \ldots, j_k)$, we say that I dominates J, $I \succeq J$, if for each $t \in [k]$ we have $i_t \leq j_t$.

Let $E = (C, V)$ be an election with $C = \{c_1, \ldots, c_m\}$ and $V = (v_1, \ldots, v_n)$, and let k be a positive integer. For a committee S and voter v_i, by $\mathrm{pos}_{v_i}(S)$ we mean the sequence from $[m]_k$ obtained by sorting the set $\{\mathrm{pos}_{v_i}(c) \mid c \in S\}$ in the nondecreasing order.

Definition 1 (Elkind et al. [9]). A committee scoring function for m candidates and committee size k is a function $f_{m,k} \colon [m_k] \to \mathbb{R}_+$ such that, for each two sequences $I, J \in [m]_k$, if I dominates J then $f_{m,k}(I) \geq f_{m,k}(J)$.

Let $f = (f_{m,k})_{k \leq m}$ be a family of committee scoring functions such that for each m and k, $f_{m,k}$ is applicable to the case of m candidates and committee size k. We define the score of a committee S of size k in an election E with m candidates to be $\mathrm{score}_E(S) = \sum_{v_i \in V} f_{m,k}(\mathrm{pos}_{v_i}(S))$. The committee scoring rule $\mathcal{R}_f$ outputs those committees that have the highest score under f.

Many well-known multiwinner rules are, indeed, committee scoring rules. Below, we provide some examples:

1. Under the single nontransferable vote rule (SNTV), we output those k candidates that are ranked first by the largest number of voters; i.e., SNTV is defined through committee scoring functions $f^{\mathrm{SNTV}}_{m,k}(i_1, \ldots, i_k) = \sum_{t=1}^{k} \alpha_1(i_t) = \alpha_1(i_1)$.

2. Bloc operates under the assumption that each voter ranks the members of his or her ideal committee on top k positions, and outputs those k candidates that belong to the highest number of ideal committees. That is, Bloc uses functions $f^{\mathrm{Bloc}}_{m,k}(i_1, \ldots, i_k) = \sum_{t=1}^{k} \alpha_k(i_t)$.

3. k-Borda outputs k candidates whose sums of Borda scores are highest. That is, k-Borda uses committee scoring functions $f^{k\text{-Borda}}_{m,k}(i_1, \ldots, i_k) = \sum_{t=1}^{k} \beta_m(i_t)$.

4. Under the Chamberlin–Courant rule (the β-CC rule), the score that a voter v assigns to committee S depends only on how v ranks his or her favorite member of S

(referred to as v's representative in S). The Chamberlin–Courant rule seeks committees in which each voter ranks his or her representative as high as possible. Formally, the rule uses functions $f^{\mathrm{CC}}_{m,k}(i_1,\dots,i_k) = \beta_m(i_1)$. (We are also interested in the k-approval variant of the Chamberlin–Courant rule, α_k-CC, that uses functions $f^{\alpha_k\text{-CC}}_{m,k}(i_1,\dots,i_k) = \alpha_k(i_1)$.)

There are many other committee scoring rules and we discuss some of them throughout the paper. The trivial committee scoring rule that for every election and committee size k returns the set of all size-k subsets of candidates is defined by a family of constant functions.

Example 1. We now start our shirt store example (we will continue it throughout the paper). Let C be the set of m shirts that the store can order from its suppliers. Since the store has limited area, it can only put k different shirts on display, and it wants to pick them in a way that would maximize its revenue (i.e., the number of shirts sold). We assume that every customer ranks all the possible shirts from the best one to the worst one.[3] Let us say that a customer considers a shirt to be "good enough" if, from his or her point of view, it is among the top $3k$ shirts, and to be "very good" if it is among the top k shirts.

How should the store decide which shirts to put on display? This depends on how the customers behave. Consider a customer that ranks the available shirts on positions $i_1 < i_2 < \dots < i_k$. If this is a very picky customer that only buys a shirt if it is the very best among all possible ones (according to this customers' opinion) then the number of shirts this customer buys is given by $f^{\mathrm{SNTV}}_{m,k}(i_1,\dots,i_k) = \alpha_1(i_1)$. However, if this customer were to buy all shirts he or she considered as "very good," he or she would buy $f^{\mathrm{Bloc}}_{m,k}(i_1,\dots,i_k) = \sum_{t=1}^{k}\alpha_k(i_t)$ shirts. Depending on which type of customers the store expects to have, it should choose its selection of shirts either using SNTV or Bloc. (Surely, other types of customers are possible as well and we will discuss some of them later. It is also likely that the store would face a mixture of different types of customers, but this is beyond our study.)

3 Hierarchy of Committee Scoring Rules

In this section we describe the classes of committee scoring rules that were studied to date, describe our new classes—the class of top-k-counting rules and the class of decomposable rules—and argue how all these classes relate to each other, forming a hierarchy. We show the relations between the classes discussed in this section, with examples of notable rules, in Figure 1 (we explain the notions used in the figure throughout this section).

(Weakly) Separable Rules. We say that a family of committee scoring functions $f = (f_{m,k})_{k\le m}$ is *weakly separable* if there exists a family of (single-winner) scoring functions $(\gamma_{m,k})_{k\le m}$ with $\gamma_{m,k}\colon [m]_k \to \mathbb{R}_+$ such that for every $m \in \mathbb{N}$ and every sequence $I = (i_1,\dots,i_k) \in [m_k]$ we have:

$$f_{m,k}(i_1,\dots,i_k) = \textstyle\sum_{t=1}^{k}\gamma_{m,k}(i_t).$$

A committee scoring rule $\mathcal{R}_f$ is weakly separable if it is defined through a family of weakly separable scoring functions f. If for all m we have $\gamma_{m,1} = \dots = \gamma_{m,m}$, then we say that the function is *separable*, without the "weakly" qualification (separable rules have some axiomatic properties that other weakly separable rules lack [9]; naturally, all separable rules are also weakly separable).

The notion of (weakly) separable rules was introduced by Elkind et al. [9];[4] they pointed out that SNTV and k-Borda are separable, whereas Bloc is only weakly separable.

[3] We use this order to define natural concepts, such as a "good enough" shirt. A customer certainly knows if a shirt is good enough.

[4] In the AAMAS version of their paper, they used the term "additively separable" instead of "separable."

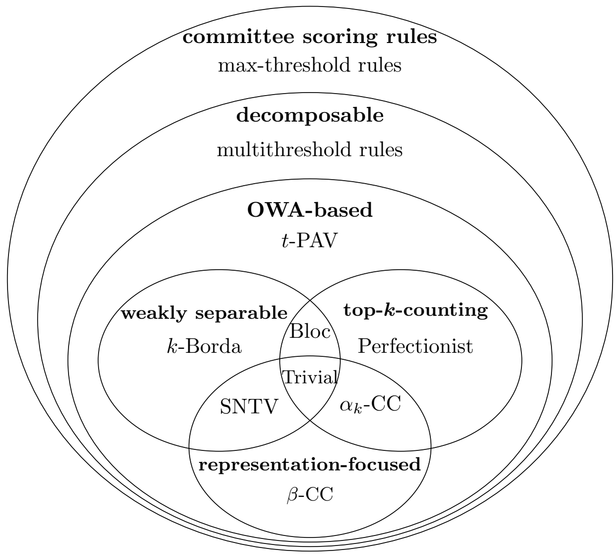

Figure 1: The hierarchy of committee scoring rules.

Representation-Focused Rules. A family of committee scoring functions $f = (f_{m,k})_{k \leq m}$ is *representation-focused* if there exists a family of (single-winner) scoring functions $(\gamma_{m,k})_{k \leq m}$ such that for every $m \in \mathbb{N}$ and every sequence $I = (i_1, \ldots, i_k) \in [m_k]$ we have:

$$f_{m,k}(i_1, \ldots, i_k) = \gamma_{m,k}(i_1).$$

A committee scoring rule $\mathcal{R}_f$ is representation-focused if it is defined through a family of representation-focused scoring functions f. The notion of representation-focused rules was introduced by Elkind et al. [9]; β-CC is the archetypal example of a representation-focused committee scoring rule. SNTV is both separable and representation-focused, and it is the only nontrivial rule with this property.

Proposition 1 (Faliszewski et al. [12])**.** *SNTV is the only nontrivial rule that is (weakly) separable and representation-focused.*

Top-k-Counting Rules. A committee scoring rule $\mathcal{R}_f$, defined by a family $f = (f_{m,k})_{k \leq m}$, is *top-k-counting* if there exists a sequence of nondecreasing functions $(g_{m,k})_{k \leq m}$, with $g_{m,k} \colon \{0, \ldots, k\} \to \mathbb{R}_+$, such that:

$$f_{m,k}(i_1, \ldots, i_k) = g_{m,k}(|\{i_t \colon i_t \leq k\}|).$$

That is, the value $f_{m,k}(i_1, \ldots, i_k)$ depends only on the number of committee members that the given voter ranks among his or her top k positions. We refer to the functions $g_{m,k}$ as the *counting functions*. Top-k-counting rules were introduced by Faliszewski et al. [13].

Remark 1. The reader may find it a bit surprising—or superfluous—to allow (as we do in the definition above) that the families of counting functions may depend on both m and k. Indeed, while dependence on k can easily be justified (see, e.g., the Perfectionist rule below), dependence on m is quite unintuitive. Nonetheless, we believe that it is important to allow it for two reasons. First, the fact that we do not have a good example of a counting function that does not depend on m does not mean we will not run into one in the future. Second, later we will show that only a certain subclass of top-k-counting rules is majority-consistent in some specific way; if we defined families of counting functions to (possibly) depend only on k, this result would no longer hold (i.e., we would not be able to say that, among committee scoring rules, only a subset of top-k-counting rules has the property in question).

It may not be immediately apparent why top-k-counting rules form a natural subclass of committee scoring rules. However, in Section 4 we present axiomatic roots from which they stem, and here we mention three notable examples of top-k-counting rules: the Bloc rule (with counting functions $g^{\mathrm{Bloc}}_{m,k}(i) = i$), the Perfectionist rule (with counting functions $g^{\mathrm{Perf}}_{m,k}$ such that $g^{\mathrm{Perf}}_{m,k}(i) = 1$ if $i = k$ and $g^{\mathrm{Perf}}_{m,k}(i) = 0$ otherwise), and the α_k-CC rule (with counting functions $g^{\mathrm{CC}}_{m,k}$ such that $g^{\mathrm{CC}}_{m,k}(0) = 0$ and $g^{\mathrm{CC}}_{m,k}(i) = 1$ for $i \geq 1$).

Bloc is the only nontrivial rule that is both weakly separable and top-k-counting, while α_k-CC is the only nontrivial rule that is both representation-focused and top-k-counting.

Proposition 2 (Faliszewski et al. [13]). *Bloc is the only nontrivial rule that is weakly separable and top-k-counting.*

Proposition 3 (Faliszewski et al. [12]). *α_k-CC is the only nontrivial rule that is representation-focused and top-k-counting.*

Example 2. Recall our shirt store example. If each customer were willing to buy at most one shirt (a fairly natural assumption), and would only buy it if he or she considered it "very good" (recall that by our assumption this means ranking the shirt among top k of all possible shirts), then the store should use α_k-CC to pick its repertoire of shirts.

OWA-Based Rules. Skowron et al. [23] introduced a class of multiwinner rules based on ordered weighted average (OWA) operators (a variant of this class was studied by Aziz et al. [2, 3]; Elkind and Ismaili [10] and Amanatidis et al. [1] use OWA operators to define different classes of multiwinner rules). While they did not directly consider elections based on preference orders, their ideas also apply to committee scoring rules.

An OWA operator Λ of dimension k is a sequence $\Lambda = (\lambda_1, \dots, \lambda_k)$ of nonnegative real numbers.

Definition 2 (Based on the idea of Skowron et al. [23], formulated by Faliszewski et al. [13]). Let $\Lambda = (\Lambda_{m,k})_{k \leq m}$ be a sequence of OWA operators such that $\Lambda_{m,k} = (\lambda^{m,k}_1, \dots, \lambda^{m,k}_k)$ has dimension k. Let $\gamma = (\gamma_{m,k})_{k \leq m}$ be a family of (single-winner) scoring functions for elections with m candidates ($\gamma_{m,k} \colon [m] \to \mathbb{N}$). Then, γ and Λ define a family $f = (f_{m,k})_{k \leq m}$ of committee scoring functions such that for each $(i_1, \dots, i_k) \in [m_k]$ we have:

$$f_{m,k}(i_1, \dots, i_k) = \textstyle\sum_{t=1}^{k} \lambda^{m,k}_t \gamma_{m,k}(i_t).$$

We refer to committee scoring rules $\mathcal{R}_f$ defined through f in this way as *OWA-based.*

It holds that weakly separable, representation-focused, and top-k-counting rules are OWA-based (for weakly separable rules we use OWA operators $(1, \dots, 1)$, for representation-focused rules we use OWA operators $(1, 0, \dots, 0)$; the argument for top-k-counting rules is due to Faliszewski et al [13] and is a bit more involved). As a corollary to the preceding propositions, we get the following observation.

Corollary 1. *All (weakly) separable and representation-focused rules are OWA-based.*

Corollary 2 (Faliszewski et al. [13]). *All top-k-counting rules are OWA-based.*

Naturally, there are also OWA-based rules that do not belong to any of the classes mentioned above. For example, by t-PAV (t-approval variant of PAV; see the work of Kilgour [17]) we mean the OWA-based rule that uses α_t as the single-winner scoring function and OWA operators of the form $\Lambda_k = (1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{k})$. (It is worth noting that Aziz et al. [2] explain axiomatically where the OWA operator for PAV comes from.)

Proposition 4 (Faliszewski et al. [12]). *For some constant t, t-PAV is neither weakly separable, representation-focused, nor top-k-counting.*

Example 3. Let us consider another scenario regarding our shirt store. Let us say that after a customer buys one "very good" shirt, he or she is willing to buy another ("very good" one) only at promotional half price, the next ("very good" one) only at third of the price, and so on. If the store were willing to offer such accumulating promotions, then the store should choose shirts to offer using k-PAV. If the store were using a different promotion scheme, e.g., if the store were offering the second shirt at half price (but would not have any promotions for the following shirts), it should use an OWA-based rule defined through α_k and OWA operators of the form $(1, \frac{1}{2}, 0, \dots, 0)$.

Decomposable Rules. Faliszewski et al. [12] introduce the following class that naturally generalizes the class of OWA-based rules.

Definition 3 (Faliszewski et al. [12]). Let $\gamma^{(t)}_{m,k} \colon [m] \to \mathbb{N}$, $t \in [k]$, be a family of (single-winner) scoring functions for elections with m candidates ($t \le k \le m$). These functions define a family of committee scoring functions $f = (f_{m,k})_{k \le m}$ such that for each $(i_1, \dots, i_k) \in [m_k]$ we have:

$$f_{m,k}(i_1, \dots, i_k) = \sum_{t=1}^{k} \gamma^{(t)}_{m,k}(i_t).$$

We refer to committee scoring rules $\mathcal{R}_f$ defined through f in this way as *decomposable.*

At first glance, decomposable rules might seem very similar to the weakly separable ones. The difference is that for fixed m and k and two different values t and t', for decomposable rules the functions $\gamma^{(t)}_{m,k}$ and $\gamma^{(t')}_{m,k}$ can be completely different. This implies that OWA-based rules are decomposable. In fact, the containment is strict.

Proposition 5 (Faliszewski et al. [12]). *The class of OWA-based rules is strictly contained in the class of decomposable rules.*

While the containment is immediate to see, proving that it is strict requires more care. We first give an intuition for a practical decomposable rule that is not OWA-based and then show that indeed such rules are not OWA-based.

Example 4. Once again we go back to the shirt store example. Recall that we assumed that a customer considers a shirt "very good" if he or she ranks it among top k ones, and considers a shirt "good enough" if he or she ranks it among top $3k$ ones. Now consider a customer who would buy two "very good" shirts (if he or she could find two such shirts), or only one "good enough" shirt (if there were no two "very good" ones). If $i_1, \dots, i_k$ are the positions (in the customer's preference order) of the shirts that the store puts on display, then the number of shirts he or she would buy is given by committee scoring function:

$$f_{m,k}(i_1, \dots, i_k) = \alpha_{3k}(i_1) + \alpha_k(i_2).$$

In this case, the store should use a rule based on this scoring function.

We refer to decomposable rules defined through committee scoring functions of the form:

$$f_{m,k}(i_1, \dots, i_k) = \lambda^{m,k}_1 \alpha_{t_{m,k,1}}(i_1) + \dots + \lambda^{m,k}_k \alpha_{t_{m,k,k}}(i_k),$$

where $\Lambda_{m,k} = (\lambda^{m,k}_1, \dots, \lambda^{m,k}_k)$ are OWA operators and $t_{m,k,1}, \dots, t_{m,k,k}$ are sequences of integers from $[k]$, as *multithreshold* rules (we put no constraints on $t_{m,k,1}, \dots, t_{m,k,k}$; both increasing and decreasing sequences are natural).

Proposition 6 (Faliszewski et al. [12])**.** *There is a multithreshold rule that is not OWA-based.*

Example 5. We continue the shirt store example. Let us say that the store does not want to maximize direct revenue, but the number of happy customers (in hope that it will lead to higher revenue in the long run). Let us say that a customer is happy if he or she finds at least two very good shirts or at least one excellent shirt (one among top $\frac{k}{10}$ of the shirts, say). Then the store should use the committee scoring function $f_{m,k}(i_1, \ldots, i_k) = \max(\alpha_{\frac{k}{10}}(i_1), \alpha_k(i_2))$.

We refer to multithreshold rules with summation replaced by the max operator as max-*threshold* rules.

Proposition 7 (Faliszewski et al. [12])**.** *There is a* max-*threshold rule that is not decomposable.*

4 Axiomatic Properties

We now focus on axiomatic properties of committee scoring rules. The choice of the properties that we study was led by the desire to understand (and separate) our classes of rules.

4.1 Nonimposition Property

Nonimposition is among the most basic properties of voting rules. A single-winner rule $\mathcal{R}$ is said to satisfy the *nonimposition property* if for each candidate there is an election where this candidate is the unique winner. We generalize this definition to the case of multiwinner rules in the following natural way.

Definition 4. Let $\mathcal{R}$ be a multiwinner rule. We say that $\mathcal{R}$ has the nonimposition property if for each candidate set C and each subset W of C there is a collection of voters V over C, such that for election $E = (C, V)$ we have $\mathcal{R}(E, |W|) = W$.

We show that all nontrivial committee scoring rules have the nonimposition property

Theorem 1 (Faliszewski et al. [12])**.** *Let $\mathcal{R}_f$ be a committee scoring rule defined by a family of committee scoring functions $f = (f_{m,k})_{k \leq m}$. The rule $\mathcal{R}_f$ satisfies the nonimposition property if and only if every committee scoring function in f is nontrivial.*

4.2 Fixed-Majority Consistency

Let us now consider the fixed-majority property introduced by Debord [8]. Intuitively, it says that if a majority of voters identify a committee (of size k) that they all like (i.e., that they all rank on top k positions) then this committee should win the election.

Definition 5 (Based on the work of Debord [8])**.** A multiwinner voting rule $\mathcal{R}$ satisfies the fixed-majority criterion for m candidates and committee size k if, for every election $E = (C, V)$ with m candidates, the following holds: if there is a committee W of size k such that more than half of the voters rank all the members of W above the non-members of W, then $\mathcal{R}(E, k) = \{W\}$. We say that $\mathcal{R}$ satisfies the fixed-majority criterion if it satisfies it for all choices of m and k (with $k \leq m$).

We ask which committee scoring rules satisfy the fixed-majority criterion. We find this question interesting for two reason. First, fixed-majority consistency is a natural, fundamental property. Second, among single-winner scoring rules, only the Plurality rule satisfies the simple majority property (i.e., guarantees that the candidate ranked first by a majority of voters wins), and fixed-majority consistency allows us to identify committee scoring rules that, in some sense, resemble the Plurality rule.

One can verify that the Bloc rule satisfies the fixed-majority criterion and that SNTV does not. This means that in the axiomatic sense, Bloc is closer to Plurality than SNTV. This is quite interesting since one's first idea of generalizing Plurality would likely be to think of SNTV. Yet, Bloc is certainly not the only committee scoring rule that satisfies our criterion (e.g., the Perfectionist rule satisfies it as well).

In the next theorem we establish which committee scoring rules satisfy the fixed majority criterion. (Note that in this theorem we only consider the case where $m \geq 2k$, i.e., where it is possible to form two disjoint committees.[5])

Theorem 2 (Faliszewski et al. [13]). *Let $f = (f_{m,k})_{2k \leq m}$ be a family of committee scoring functions with the corresponding family $(g_{m,k})_{2k \leq m}$ of counting functions. Then, $\mathcal{R}_f$ satisfies the fixed-majority criterion if and only if for every $k, m \in \mathbb{N}$, $2k \leq m$ (i) $g_{m,k}$ is not constant, and (ii) for each pair of nonnegative integers k_1, k_2 with $k_1 + k_2 \leq k$, it holds that: $g_{m,k}(k) - g_{m,k}(k - k_2) \geq g_{m,k}(k_1 + k_2) - g_{m,k}(k_1)$.*

Condition (ii) in Theorem 2 is a relaxation of the convexity property for function $g_{m,k}$. Indeed, for most practical considerations, the next corollary is more useful and appealing.

Definition 6. Let $g_{m,k}$ be a counting function for some top-k-counting function $f_{m,k} \colon [m]_k \to \mathbb{N}$. We say that $g_{m,k}$ is *convex* if for each k' such that $2 \leq k' \leq k$, it holds that $g_{m,k}(k') - g_{m,k}(k' - 1) \geq g_{m,k}(k' - 1) - g_{m,k}(k' - 2)$. On the other hand, we say that g is *concave* if for each k' with $2 \leq k' \leq k$ it holds that $g_{m,k}(k') - g_{m,k}(k' - 1) \leq g_{m,k}(k' - 1) - g_{m,k}(k' - 2)$.

Corollary 3 (Faliszewski et al. [13]). *Let $f = (f_{m,k})_{2k \leq m}$ be a family of top-k-counting committee scoring functions with the corresponding family $(g_{m,k})_{2k \leq m}$ of counting functions. The following hold:*

(1) if $g_{m,k}$ are convex, then $\mathcal{R}_f$ satisfies the fixed-majority criterion, and

(2) if $g_{m,k}$ are concave but not linear (that is, $\mathcal{R}_f$ is not Bloc) then $\mathcal{R}_f$ fails the fixed-majority criterion.

As examples, notice that the counting function for the Bloc rule is linear (and, thus, both convex and concave), and the counting function for the Perfectionist rule is convex, so these two rules satisfy the fixed-majority criterion. On the other hand, the counting function for α_k-CC is concave and, so, this rule fails the criterion.

4.3 Variants of Noncrossing Monotonicity

Elkind et al. [9] introduced two monotonicity notions for multiwinner rules: candidate monotonicity and noncrossing monotonicity. In the former one, we require that if, in some vote, we shift forward a candidate from a winning committee, then this candidate still belongs to some winning committee after the shift (possibly a different winning committee). Elkind et al. [9] show that all committee scoring rules have this property.

For noncrossing monotonicity, we require that if we shift forward a member of a winning committee W (without him or her passing another member of W) then W is still winning.

[5] This assumption is not a major drawback. Indeed, if $m < 2k$ then perhaps one should organize an election for who should *not* be on the committee, and not the other way round.

Definition 7 (Elkind et al. [9])**.** A multiwinner rule $\mathcal{R}$ is said to be noncrossing-monotone if, for each election $E = (C, V)$ and each $k \in [|C|]$, the following holds: if $c \in W$ for some $W \in \mathcal{R}(E, k)$, then, for each E' obtained from E by shifting c forward by one position in some vote without passing other members of W, we have that $W \in \mathcal{R}(E', k)$.

Elkind et al. [9] have shown that weakly separable rules are noncrossing-monotone, and we will now show that these are the only such committee scoring rules (in particular, the following theorem provides a tool for showing that a rule is not weakly separable).

Theorem 3 (Faliszewski et al. [12])**.** *Let $\mathcal{R}_f$ be a committee scoring rule defined through a family $f = (f_{m,k})_{k \leq m}$ of scoring functions $f_{m,k} \colon [m]_k \to \mathbb{N}$. $\mathcal{R}_f$ is noncrossing-monotone if and only if $\mathcal{R}_f$ is weakly separable.*

Based on the idea of noncrossing monotonicity, we can define many other similar notions. Consider a multiwinner rule $\mathcal{R}$, committee size k, and let $M = \{m_1, \ldots, m_t\}$ be a subset of $[k]$, where $t \leq k$ and $m_1 < m_2 < \cdots < m_t$. Consider a vote v (over a candidate set C, $|C| \geq k$) and a committee $W = \{w_1, \ldots, w_k\}$. Let $(i_1, \ldots, i_k)$ be the position of W in v, so that for each ℓ, $i_\ell = \mathrm{pos}_v(w_\ell)$. By an M-shift of W in v, we mean shifting by one position forward w_{m_1}, then w_{m_2}, and so on, until w_{m_t}. We say that a given M-shift is legal with respect to W if executing it is possible (i.e., w_{m_1} is not ranked first) and the shifted members of W do not overtake other members of W.

Definition 8 (Faliszewski et al. [12])**.** Let $\mathcal{M} = (\mathcal{M}_k)_{k \geq 1}$ be a sequence of sets, where each $\mathcal{M}_k$ is a family of subsets of $[k]$. We say that a multiwinner rule $\mathcal{R}$ is $\mathcal{M}$-monotone if for each election $E = (C, V)$ and each $k \in [|C|]$, the following holds: For each $W \in \mathcal{R}(E, k)$, each $M \in \mathcal{M}_k$, and each E' obtained from E by applying a legal M-shift (with respect to W), we have that $W \in \mathcal{R}(E', k)$.

The notion of $\mathcal{M}$-monotonicity is quite powerful. We can use it, e.g., to define the standard noncrossing monotonicity and several other interesting notions:

1. Consider a sequence $\mathcal{M}^{\mathrm{nc}} = (\mathcal{M}^{\mathrm{nc}}_k)_{k \geq 1}$, where each $\mathcal{M}^{\mathrm{nc}}_k$ consists of all the singleton subsets of $[k]$. Then $\mathcal{M}^{\mathrm{nc}}$-monotonicity is simply noncrossing monotonicity.

2. Consider a sequence $\mathcal{M}^{\mathrm{pre}} = (\mathcal{M}^{\mathrm{pre}}_k)_{k \geq 1}$, where each $\mathcal{M}^{\mathrm{pre}}_k$ is of the form $\{\{1\}, \{1, 2\}, \{1, 2, 3\}, \ldots, \{1, \ldots, k\}\}$. We refer to $\mathcal{M}^{\mathrm{pre}}$-monotonicity as prefix monotonicity.

3. Consider a sequence $\mathcal{M}^{\mathrm{top}} = (\mathcal{M}^{\mathrm{top}}_k)_{k \geq 1}$, where each $\mathcal{M}^{\mathrm{top}}_k$ is equal to $\{\{1\}\}$. We refer to $\mathcal{M}^{\mathrm{top}}$-monotonicity as top-member monotonicity.

Intuitively, if a rule satisfies the prefix monotonicity criterion, then shifting forward some top members of a winning committee, within a given vote, never prevents this committee from winning. Top-member monotonicity is a refinement of prefix monotonicity, where we are restricted to shifting only the top-ranked member of a winning committee.

Naturally, all noncrossing-monotone rules (i.e., all weakly separable rules) satisfy all types of $\mathcal{M}$-monotonicity. Thus it is impossible to use $\mathcal{M}$-monotonicity notions to characterize classes of rules that do not contain weakly separable ones. Yet, it is possible to show that some such classes do satisfy specific types of $\mathcal{M}$-monotonicity. For example, top-member monotonicity is satisfied by all representation-focused rules.

Theorem 4. *Every representation-focused rule is top-member monotone.*

Moreover, only decomposable rules can be prefix-monotone (and mostly, though not only, those based on convex functions; see also the explanations below Theorem 6).

Theorem 5 (Faliszewski et al. [12]). *Let $\mathcal{R}_f$ be a committee scoring rule. If $\mathcal{R}_f$ is prefix-monotone then $\mathcal{R}_f$ is decomposable.*

Theorem 6 (Faliszewski et al. [12]). *Let $\mathcal{R}_f$ be a decomposable committee scoring rule defined through a family of scoring functions:*

$$f_{m,k}(i_1,\dots,i_k) = \gamma^{(1)}_{m,k}(i_1) + \gamma^{(2)}_{m,k}(i_2) + \cdots + \gamma^{(k)}_{m,k}(i_k),$$

where $\gamma = (\gamma^{(t)}_{m,k})_{t \le k \le m}$ is a family of single-winner scoring functions. A sufficient condition for $\mathcal{R}_f$ to be prefix-monotone is that for each m and each $k \in [m]$ we have that:

(i) for each $i \in [k]$ and each $p, p' \in [m-1]$, $p < p'$, it holds that $\gamma^{(i)}(p) - \gamma^{(i)}(p+1) \ge \gamma^{(i)}(p') - \gamma^{(i)}(p'+1)$, and

(ii) for each $i, j \in [k]$, $j > i$, and each $p \in [m]$, $j \le p < m-(k-i)$, it holds that $\gamma^{(i)}(p) - \gamma^{(i)}(p+1) \ge \gamma^{(j)}(p) - \gamma^{(j)}(p+1)$.

Intuitively, condition (i) says that functions in the γ family are convex, and condition (ii) says that, for each m and k, if $i < j$ then $\gamma^{(i)}_{m,k}$ decreases at least as rapidly as $\gamma^{(j)}_{m,k}$.

Example 6. α_k-CC is a decomposable rule (since it is top-k-counting) that is not prefix-monotone. To see this, consider $k = 2$ and an election that includes the vote $v\colon a \succ b \succ c \succ d$. Let us say that in the whole election the winning committees are $W = \{b, c\}$ and $W' = \{c, d\}$. If α_k-CC were prefix-monotone, then shifting b and c by one position forward in v (to obtain $b \succ c \succ a \succ d$) should keep W winning. However, doing so does not change the score of W and increases the score of W', so W no longer wins. In this case, it happens because the k-approval single-winner scoring functions α_k are not convex.

5 Computational Remarks

Many committee scoring rules are NP-hard to compute. Procaccia et al. [22], Lu and Boutilier [18], and Betzler et al. [5] show hardness of winner determination for various representation-focused rules, Skowron et al. [23] and Aziz et al. [3] show strong hardness results for large families of OWA-based rules, and Faliszewski et al. [13] do the same for many top-k-counting rules. (On the other hand, weakly separable rules are polynomial-time computable.) Fortunately, many of these papers also provide means to go around their hardness results (e.g., considering parameterized algorithms or approximation algorithms). Here we present a few selected results regarding decomposable and top-k-counting rules.

First, we show a $(1 - \frac{1}{e})$-approximation algorithm (based on optimizing submodular functions [21]) for a class of decomposable rules that satisfy a certain fairly natural condition (in particular, it applies to OWA-based rules based on non-increasing OWA operators, and to top-k-counting rules based on concave counting functions).

Theorem 7 (Faliszewski et al. [12]). *Let $\mathcal{R}_f$ be a decomposable committee scoring rule defined through a family of scoring functions $f_{m,k}(i_1,\dots,i_k) = \gamma^{(1)}_{m,k}(i_1) + \gamma^{(2)}_{m,k}(i_2) + \cdots + \gamma^{(k)}_{m,k}(i_k)$, where $\gamma = (\gamma^{(t)}_{m,k})_{t \le k \le m}$ is a family of polynomial-time computable single-winner scoring functions. If for each m, each $k \in [m]$, each $i \in [k-1]$ and each $p \in [m]$, it holds that $\gamma_{i-1}(p) \ge \gamma_i(p)$, then there is a polynomial-time algorithm that, given an election $E = (C, V)$ and a committee size k, outputs a committee W such that $\mathrm{score}_E(W) \ge (1 - \frac{1}{e}) \max_{S \subseteq C, |W| = k} \mathrm{score}_E(S)$.*

For special cases of committee scoring rules (and, as in the case below, special restriction on committee size) it is possible to provide far stronger results. Below we show an example of a polynomial-time approximation scheme (PTAS).

Theorem 8 (Faliszewski et al. [12]). *Let $\mathcal{R}_f$ be a top-k-counting committee scoring rule, where the family $f = (f_{m,k})_{k \leq m}$ is defined through a family of counting functions $(g_{m,k})_{k \leq m}$ that are: (a) polynomial-time computable and (b) constant for arguments greater than some given value ℓ. If $m = o(k^2)$, then there is a PTAS for computing the score of a winning committee under $\mathcal{R}_f$.*

It is also possible to derive FPT algorithms for committee scoring rules. Not surprisingly, there is an FPT algorithm, parameterized by the number of candidates, for all committee scoring rules (the exact formulation of the result is new to this manuscript).

Proposition 8. *Let $\mathcal{R}_f$ be a committee scoring rule, where all the functions in the family $f = (f_{m,k})_{k \leq m}$ are computable in* FPT*-time with respect to m. There is an algorithm that, given a committee size k and an election E, computes a winning committee from $\mathcal{R}_f(E, k)$ in* FPT*-time with respect to the number m of candidates.*

It is also very tempting to consider other parametrizations, such as, e.g., parameterization by the number of voters. In this case, we have a much more restricted result.

Theorem 9 (Faliszewski et al. [12]). *Let $\mathcal{R}_f$ be a top-k-counting committee scoring rule, where the family $f = (f_{m,k})_{k \leq m}$ is defined through a family of concave counting functions $(g_{m,k})_{k \leq m}$. There is an algorithm that, given a committee size k and an election E, computes a winning committee from $\mathcal{R}_f(E, k)$ in* FPT*-time with respect to the number n of voters.*

6 Summary

We have provided an axiomatic and computational study of committee scoring rules, focusing mostly on the hierarchy formed by its subclasses and on the properties of monotonicity and fixed-majority consistency.

There are several natural directions for future research. First, it seems that our *semantic* understanding of committee scoring rules is lagging behind our *syntactic* understanding of them. We have considered various subclasses of committee scoring rules by restricting the allowed scoring functions defining these rules, but we only had limited success in capturing the same classes axiomatically. So far, we were able to characterize weakly separable rules and fixed-majority consistent top-k-counting rules. Ideally, for *each* of our subclasses of committee scoring rules we would like to have both a syntactic characterization (by providing appropriate restrictions on the committee scoring functions), and a semantic one (by characterizing it axiomatically).

We are also interested in efficient algorithms for (approximately) computing committee scoring rules. For example, it would be interesting to identify for which classes of committee scoring rules there is a PTAS, for which there is a constant-ratio polynomial-time approximation algorithm, and for which computing approximate solutions is hard. Similarly, it would be interesting to know for which rules and for which parameters there are FPT algorithms.

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