Borda, Condorcet, and Pareto optimality in ordinal group activity selection

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Abstract

We consider the situation in which group activities need to be organized for a set of agents. Each agent can take part in at most one activity, and the agents' preferences depend both on the activity and the number of participants in that activity. In particular, the preferences are given by means of strict orders over such pairs "(activity, group size)", including the possibility "do nothing". Our goal will be to assign agents to activities on basis of their preferences, the minimum requirement being that no agent prefers doing nothing, i.e., not taking part in any activity at all. We aim at establishing such an assignment by two approaches. On the one hand, taking a voting-theoretical perspective, we apply Borda scores and the Condorcet criterion to our setting. On the other hand, we target at a Pareto optimal assignment. We analyse the computational complexity involved in finding such desired assignments, with focus on two natural special cases of the agents' preferences.

1 Introduction

In many situations activities need to be organized for a set of agents, with the agents having preferences over the activities. However, often the preferences of the agents do not depend solely on the activity itself, but also on the number of participants in the activity (see also [17]). We consider such a scenario and assume that each agent can be assigned to at most one activity. E.g., consider a company which would like to provide free sports classes in order to achieve a high employee satisfaction [26], or the organizer of a social or business event (such as a workshop), who wants to arrange social activities for the free afternoon [17]. In the former case, for cost reasons the company might allow each employee to take part in at most one activity; in the latter case, since the activities take place at the same time, each agent can take part in at most one activity. Now, often the agents have preferences not only over the available activities, but also over the number of attendees of the activity. For example, one would be willing to take a sauna with up to 5 attendees, but does not wish to take part if the sauna is more crowded. On the other hand, for activities connected with costs that need to be shared by the attendees, an agent might only take part if a high number of attendees joins, while the desired numbers of attendees may be different for each agent. These examples already indicate two natural special cases we will consider in this paper: vaguely speaking, in the case of decreasing preferences the agents want the number of other agents joining an activity to be as low as possible; in the case of increasing preferences, the agents would like as many agents as possible to join the same activity.

Thus, in this work we consider a setting in which the agents have preferences over pairs of the form "(activity, group size)". In these pairs, we include the possibility "do nothing" to which we refer as the void activity $a_\emptyset$. Note that $a_\emptyset$ allows the agents to express which pairs "(activity, group size)" they are not really happy with by ranking them below $a_\emptyset$ (they would rather do nothing than join the respective activity with the corresponding total number of attendees). Throughout this paper, we assume the agents' preferences to be strict orders over such pairs (including $a_\emptyset$).

The goal, of course, would be a "good" assignment of agents to activities. As a minimum requirement, the assignment should be *individually rational*, i.e., no agent should be forced to take part in an activity with a total number of attendees such that she would prefer doing nothing. Taking into account the special cases of increasing and decreasing preferences, we follow two different approaches to find a "good" assignment, and provide computational complexity results for each of them. First, by applying Borda scores and the Condorcet criterion respectively, we derive a group decision from the agents' rankings (Sections 3 and 4). Second, we aim at finding a Pareto optimal outcome (Section 5).

Related work.

In [17], the general *group activity selection problem (GASP)* is introduced, where the agents' preferences are weak orders over the pairs "(activity, group size)". The authors analyse computational complexity aspects of the special case of a-GASP, where the agent's preferences are not strict orders but trichotomous; i.e., each agent partitions the set of pairs "(activity,group size)" into the following three clusters: pairs approved by an agent (i.e., pairs that are preferred to the void activity), the void activity itself, and pairs that are disapproved by the agents (i.e, pairs the void activity is preferred to). In that framework, the focus is laid on maximum individually rational assignments (i.e., an individually rational assignment with the maximum number of agents assigned to a non-void activity) and stable assignments, with different notions of stability such as Nash and core stability. In their computational study, several special cases including increasing and decreasing preferences are considered.

In our work, we focus on another obvious variant of the group activity selection problem, in which the agents' preferences are strict orders. This variant, the *group activity selection problem with ordinal preferences* (o-GASP), has been introduced in [15]. As done here, in [15] two approaches towards desirable outcomes have been taken, each of which has been analysed with respect to the computational complexity involved: On the one hand, a voting-theoretical perspective led to the use of k-approval scores; on the other hand, game-theoretical concepts such as Nash and core stability have been considered. In particular, Darmann [15] shows that finding an individually rational assignment that maximizes k-approval scores is, in general, computationally hard. On the positive side, when all agents have increasing preferences an individually rational assignment maximizing k-approval scores can be found in polynomial time for any fixed $k \in \mathbb{N}$, while for decreasing preferences the problem remains solvable in polynomial time for $k \leq 3$ and becomes intractable for $k \geq 4$ (see [15]). In addition, Darmann [15] gives a number of complexity results for deciding whether a stable assignment exists, for different notions of stability such as Nash and core stability.

We add to both approaches of [15] by applying Borda scores and the Condorcet criterion from voting theory (see [11] for a survey) on the one hand, and by considering Pareto optimality on the other.

Positional scores from voting theory have been considered outside of their original framework for instance in fair division problems (see, e.g., [7] and [16]) or in the context of optimization problems such as the traveling salesman problem [24]. In Section 3, we provide computational complexity results for finding an individually rational assignment maximizing Borda scores.

In addition, we consider the well-known Condorcet-criterion [12] in our framework: desired is an outcome that beats each other outcome (or, in its weaker variant, is not beaten by any other outcome) in a simple majority comparison. In Section 4 we analyse the computational complexity involved in finding an outcome that satisfies such a criterion. Originally, the Condorcet criterion has been used for single winner elections (see, e.g., [11]), but has also been generalized to electing sets of agents (committees) in several ways (see [20], [21], [23]); in that respect, also the computational complexity of finding such an outcome has been considered (see [19] and [13]). In addition, the computational complexity involved in finding a Condorcet solution has been investigated outside of its classical framework, for instance in the context of matchings (see, e.g., [1], [9], and [22]), rankings [27], or spanning trees [14].

The model presented in this paper is closely related to that of *hedonic games* [6, 10], and in particular *anonymous hedonic games* [6]. In the latter setting, the goal is to find a "good" partition of the set of agents into several groups, while the agents have preferences over the size of the possible group they are part of. In contrast, in our setting the agents' preferences depend on the group size and the specific activity they would like to join.

In general (i.e., non-anonymous) hedonic games, the agents' preferences are over the possible groups (i.e., the composition of the group rather than only its size) they are part of. By introducing dummy agents for the activities and suitable preferences, the general group activity selection problem – and hence our model – can be embedded in the general hedonic game framework (see [17] for a detailed description of that representation). As already pointed out in [17], note that our setting has useful structural properties that distinguish it from a hedonic game though – e.g., it allows for a succinct representation of agents' preferences. In addition, in our model two natural special cases are inherent that admit efficient algorithms for finding good outcomes.
Both anonymous and non-anonymous hedonic games have been studied by Ballester [5] from a computational viewpoint. In particular, Ballester [5] shows that – for a variety of stability concepts such as core or Nash stability – deciding whether there is a stable outcome is NP-complete for both anonymous and non-anonymous hedonic games. In addition, the computational complexity of stable partitions has also been considered in the context of additively separable hedonic games, e.g., in the works of [18], [25], and [3]; the latter also consider the concept of Pareto optimality.

Aziz et al. [4] focus on the computational complexity involved in finding Pareto optimal solutions in hedonic games and some of its variants including anonymous hedonic games and roommate games. Aziz et al. [4] show that it is NP-hard to find a Pareto optimal solution both in anonymous and non-anonymous hedonic games, and provide an algorithm which determines a Pareto optimal solution for some of its variants, including roommate games, in polynomial time. These results, however, do not immediately carry over to our setting. In particular, while o-GASP can be embedded in the general (non-anonymous) hedonic game framework, for o-GASP we provide an algorithm that computes a Pareto optimal assignment in polynomial time (see Section 5).

2 Formal framework

The setting we consider in this paper is as follows (see also [15]).

Definition 1 *An instance* (N, A, P) *of the* Group activity selection problem with ordinal preferences *(*o-GASP*) is given as follows.* N *is a set of agents with* $n = |N|$*; unless stated otherwise, the agents are denoted by* $N = \{1, \dots, n\}$. $A = A^* \cup \{a_\emptyset\}$ *is a set of activities, where* $A^* = \{a_1, \dots, a_m\}$*;* $X = (A^* \times \{1, \dots, n\}) \cup \{a_\emptyset\}$ *is the set of alternatives. Finally, the profile* $P = \langle V_1, \dots, V_n \rangle$ *consists of* n *votes, one for each agent. For agent* i*, a* vote V_i *(also denoted by* $\succ_i$*) is a strict order over* X*; the set* $S_i \subseteq X$ *such that for each* $x \in S_i$ *we have* $x \succ_i a_\emptyset$*, is the* induced approval vote *of agent* i*. We say that agent* i *approves of the alternatives in* S_i*.*

Definition 2 *Given an instance* (N, A, P) *of* o-GASP*, we say that agent* i *has* increasing preferences *if for each* $a \in A^*$, $(a, k) \succ_i (a, k-1)$ *holds for each* $k \in \{2, \dots, n\}$.
An agent i *has* decreasing preferences *if for each* $a \in A^*$, $(a, k-1) \succ_i (a, k)$ *holds for each* $k \in \{2, \dots, n\}$.

An instance (N, A, P) *of* o-GASP *has increasing (decreasing) preferences, if each agent* $i \in N$ *has increasing (decreasing) preferences.*

Definition 3 *Given an instance* (N, A, P) *of* o-GASP*, a mapping* $\pi : N \to A$ *is called* assignment*. The set* $\pi^a := \{i \in N | \pi(i) = a\}$ *denotes the set of agents assigned to* $a \in A$*. The set* $\pi_i := \{j \in N | \pi(j) = \pi(i)\}$ *denotes the set of agents assigned to the same activity as agent* $i \in N$*.*

An assignment is said to be individually rational *if for every* $a \in A^*$ *and every agent* $i \in \pi^a$ *it holds that* $(a, |\pi^a|) \succ_i a_\emptyset$*.*

As in [17, 15], we consider individual rationality as a minimum requirement an assignment must satisfy: If an assignment is not individually rational, then there is an agent that would rather join the void activity (i.e., do nothing) than taking part in the assigned activity (and hence this agent wants

to deviate from that activity).
Note that the assignment π defined by $\pi(i) = a_\emptyset$ for all $i \in N$ is always individually rational; therewith, an individually rational assignment always exists. Let $\#(\pi) = |\{i \in N \mid \pi(i) \neq a_\emptyset\}|$ denote the number of agents assigned to a non-void activity. Finally, π is *maximum individually rational* if π is individually rational and $\#(\pi) \geq \#(\pi')$ for every individually rational assignment π'.

In this paper, we consider individually rational assignments only. In particular, in any instance (N, A, P) of o-GASP we will restrict the attention to the part of the profile which excludes alternatives ranked below $a_\emptyset$. In addition, throughout the paper we assume that $S_i \neq \emptyset$ holds for each agent i, since otherwise in any individually rational assignment i can only participate in the void activity.

2.1 Borda scores, the Condorcet criterion, and Pareto optimality

In an instance of o-GASP, a scoring function f maps an assignment to a non-negative real number by means of $f(\pi) := \sum_{i \in N} f_i(\pi(i), |\pi_i|)$ with $f_i : X \to \mathbb{R}_0^+$. In *Borda scores*, for $i \in N$ we have $f_i(x) = |\{x' \in X : x \succ_i x'\}|$.

Our first task will be to find an individually rational assignment that maximizes Borda scores (Section 3).

In our second approach, we apply the Condorcet criterion to o-GASP. In that context, desirable would be an individually rational assignment which, in a simple majority comparison, beats each other individually rational assignment. Another goal would be the simple majority comparison-winner among all maximum individually rational assignments.

Definition 4 *Given an instance (N, A, P) of* o-GASP*, we say that agent i* prefers *assignment π over assignment π' (denoted by $\pi \rhd_i \pi'$), if $(\pi(i), |\pi_i|) \succ_i (\pi'(i), |\pi'_i|)$ holds.*
An assignment π is IR-Condorcet*, if π is individually rational and for all individually rational assignments $\pi' \neq \pi$ we have $|\{i \in N : \pi \rhd_i \pi'\}| > |\{i \in N : \pi' \rhd_i \pi\}|$.*
π is MIR-Condorcet*, if π is maximum individually rational and for all maximum individually rational assignments $\pi' \neq \pi$ we have $|\{i \in N : \pi \rhd_i \pi'\}| > |\{i \in N : \pi' \rhd_i \pi\}|$.*

Finally, we will consider *Pareto optimality*. A Pareto optimal assignment is an individually rational assignment that cannot be "globally improved upon" in the sense that there is no other assignment under which at least one agent is better off while no agent is worse off.

Definition 5 *Given an instance (N, A, P) of* o-GASP*, an individually rational assignment $\pi : N \to A$ is* Pareto optimal *if there is no assignment π' such that there is no $i \in N$ with $(\pi'(i), |\pi'_i|) \prec_i (\pi(i), |\pi_i|)$ and for at least one $i \in N$ $(\pi'(i), |\pi'_i|) \succ_i (\pi(i), |\pi_i|)$ holds.*

We begin our study with the task of finding an individually rational assignment of maximum Borda score in Section 3 (we also refer to the respective decision problem as MAX BORDA). In Section 4, we focus on the decision problem whether an IR-Condorcet resp. MIR-Condorcet assignment exists (we refer to the respective problem as IR-CONDORCET-EXISTENCE and MIR-CONDORCET-EXISTENCE). In Section 5 we consider the problem of finding a Pareto optimal assignment (we refer to this problem as PARETO).

An overview of the computational complexity results achieved in this paper is given in Table 1.

3 Maximizing Borda scores

Unfortunately, as it turns out the problem of finding an individually rational assignment of maximum Borda score is computationally intractable for both special cases of increasing and decreasing instances already. Due to space constraints, the proof of the following theorem is omitted.

	general pref.	INC	DEC
MAX BORDA	NP-complete		
IR-CONDORCET-EXISTENCE	coNP-hard	coNP-hard	in P
MIR-CONDORCET-EXISTENCE	coNP-hard	coNP-hard	?
PARETO	in P		

Table 1: Overview of complexity results for the considered problems, with respect to general, increasing (INC), and decreasing (DEC) preferences

Theorem 1 *Given $b \in \mathbb{N}$, it is* NP*-complete to decide whether* o-GASP *admits an individually rational assignment π with Borda score of at least b, even when all agents have increasing preferences.*

Theorem 2 *Given $b \in \mathbb{N}$, it is* NP*-complete to decide whether* o-GASP *admits an individually rational assignment π with Borda score of at least b, even when all agents have decreasing preferences.*

Proof: We provide a reduction from 3-SAT. In an instance $\mathcal{I} = (X, C)$ of 3-SAT we are given a set X of variables and a set C of 3-clauses over X; we ask if there is a truth assignment for X such that each clause of C is satisfied. It is known that 3-SAT remains NP-complete even if each literal occurs exactly twice [8]. Let $\mathcal{I}$ be such an instance with $X = \{x_1, \ldots, x_n\}$ and $C = \{C_1, \ldots, C_m\}$. For some $z_{j_1}, z_{j_2}, z_{j_3} \in \{x_i, \bar{x}_i | x_i \in X\}$, we can rewrite C_j as $C_j = (z_{j_1} \vee z_{j_2} \vee z_{j_3})$, $1 \leq j \leq m$. We construct an instance (N, A, P) of o-GASP with decreasing preferences as follows. Recall that we look at individually rational assignments; hence, in the specification of profile P we only need to consider alternatives ranked above $a_\emptyset$. Let

$$\ell := 5(m + n)$$

and $A^* = \{d_1, \ldots, d_\ell\} \cup \{x_i, \bar{x}_i | x_i \in X\}$. In addition, let $N = \{1 \leq j \leq m+n\} \cup \{D_1, \ldots, D_\ell\}$ and

- for $1 \leq h \leq \ell$, let $V_{D_h} = (d_h, 1) \succ (d_h, 2) \succ \ldots \succ (d_h, \ell + m + n) \succ a_\emptyset$
- for $1 \leq j \leq m$, let $V_j = (z_{j_1}, 1) \succ (z_{j_1}, 2) \succ (z_{j_2}, 1) \succ (z_{j_2}, 2) \succ (z_{j_3}, 1) \succ (z_{j_3}, 2) \succ (d_1, 1) \succ (d_2, 1) \succ \ldots \succ (d_\ell, 1) \succ a_\emptyset$
- for $1 \leq i \leq n$, let $V_{m+i} = (x_i, 1) \succ (\bar{x}_i, 1) \succ (d_1, 1) \succ (d_2, 1) \succ \ldots \succ (d_\ell, 1) \succ a_\emptyset$

Let M denote the Borda score of the top-ranked alternative, i.e., $M = (\ell + 2n)(\ell + m + n)$.

Claim. $\mathcal{I}$ is a "yes"-instance of 3-SAT if and only if (N, A, P) admits an individually rational assignment π with Borda score $f(\pi) \geq M \cdot \ell + (M - 5)(m + n)$.

Proof of claim. "$\Rightarrow$": Let τ be a satisfying truth assignment. Let $\pi(D_h) = d_h$, for each $1 \leq h \leq \ell$. For $1 \leq i \leq n$, if $\tau(x_i) = \mathsf{true}$, then set $\pi(m + i) = \bar{x}_i$, otherwise set $\pi(m + i) = x_i$. For $1 \leq j \leq m$, if j is not yet assigned to an activity, let π assign j (arbitrarily) to one of the activities $z_{j_1}, z_{j_2}, z_{j_3}$ for which the literal of the same label is set true under τ.
It is not hard to see that π is well-defined and individually rational. Clearly, $f(\pi) \geq M \cdot \ell + (M - 5)m + (M - 1)n > M \cdot \ell + (M - 5)(m + n)$ holds.

"$\Leftarrow$": Let π be an individually rational assignment with $f(\pi) \geq M \cdot \ell + (M - 5)(m + n)$.
First, we show that for each $1 \leq h \leq \ell$, $\pi(D_h) = d_h$ must hold. Assume the opposite, i.e., for some h, $\pi(D_h) \neq d_h$. By individual rationality of π, $\pi(D_h) = a_\emptyset$ follows. Note that $a_\emptyset$ yields a Borda score of $M - (\ell + m + n)$ for agent D_h. As a consequence,

$$\begin{aligned} f(\pi) &\leq [M - (\ell + m + n)] + M(\ell - 1 + m + n) \\ &= M(\ell + m + n) - (\ell + m + n) \\ &= M(\ell + m + n) - 6(m + n) \\ &< M(\ell + m + n) - 5(m + n) \end{aligned}$$

This is a contradiction with our assumption.
Assume that one of the remaining agents is assigned to the void activity $a_\emptyset$. Then,

$$\begin{array}{rcl} f(\pi) & \leq & [M-(\ell+2)]+M(\ell-1+m+n) \\ & = & M(\ell+m+n)-(\ell+2) \\ & = & M(\ell+m+n)-(5(m+n)+2) \\ & < & M(\ell+m+n)-5(m+n) \end{array}$$

which again contradicts our assumption.

Thus, each agent is assigned to a non-void activity. Because for each $1 \leq h \leq \ell$ we have (i) $\pi(D_h) = d_h$ and (ii) $(d_h, k) \notin S_j$ for any $k \neq 1$ and $j \in \{1, \ldots, m+n\}$, it follows that the following hold:

- for each $1 \leq i \leq n$, $\pi(m+i) \in \{x_i, \bar{x}_i\}$
- for each $1 \leq j \leq m$, $\pi(j) \in \{z_{j_1}, z_{j_2}, z_{j_3}\}$

Since π is an individually rational assignment, $\pi(m+i) = x_i$ (resp. $\pi(m+i) = \bar{x}_i$) implies that $m+i$ is the only agent assigned to x_i (resp. $\bar{x}_i$), and in particular none of the agents $1 \leq j \leq m$ is assigned to x_i (resp. $\bar{x}_i$.) On the other hand, each agent j, $1 \leq j \leq m$, is assigned to an activity for which the literal of the same label is contained in the clause C_j. Thus, the truth assignment τ defined by $\tau(x_i) = \mathsf{true}$ if and only if $\pi(m+i) = \bar{x}_i$, is well-defined and satisfies each of the clauses in C. ■

4 Determining Condorcet assignments

In our second approach, we focus on IR-Condorcet and MIR-Condorcet assignments. Before turning to our computational complexity study, we point out that in general (even if both of them exist) these assignments do not coincide. In particular, there are instances in which (i) there is an IR-Condorcet assignment but no MIR-Condorcet assignment (see Example 1), (ii) there is an MIR-Condorcet assignment but no IR-Condorcet assignment (see Example 2), and (iii) there is both an IR-Condorcet and an MIR-Condorcet assignment, but they do not coincide (see Example 3).

Example 1 *Let* $N = \{1, 2, 3\}$ *and* $A^* = \{a, b, c\}$. *Let*

- $V_1 : (a, 2) \succ_1 (a, 3) \succ_1 (b, 3) \succ_1 (c, 3) \succ_1 a_\emptyset$
- $V_2 : (a, 2) \succ_2 (c, 3) \succ_2 (a, 3) \succ_2 (b, 3) \succ_2 a_\emptyset$
- $V_3 : (b, 3) \succ_3 (c, 3) \succ_3 (a, 3) \succ_3 a_\emptyset$

The assignment π *with* $\pi(1) = \pi(2) = a$ *and* $\pi(3) = a_\emptyset$ *is an IR-Condorcet assignment. There are three maximum individually rational assignments* π', τ, σ*:* $\pi'(i) = a$*,* $\tau(i) = b$ *and* $\sigma(i) = c$ *for all* $i \in N$*. However, an MIR-Condorcet assignment does not exist, because the agents* $2, 3$ *prefer* σ *over* π'*, agents* $1, 3$ *prefer* τ *over* σ*, and the agents* $1, 2$ *prefer* π' *over* τ*.*

Example 2 *Let* $N = \{1, 2, 3\}$ *and* $A^* = \{a, b, c\}$. *Let*

- $V_1 : (a, 2) \succ_1 (b, 2) \succ_1 (c, 2) \succ_1 (a, 3) \succ_1 a_\emptyset$
- $V_2 : (c, 2) \succ_2 (a, 2) \succ_2 (b, 2) \succ_2 (a, 3) \succ_2 a_\emptyset$
- $V_3 : (b, 2) \succ_3 (c, 2) \succ_3 (a, 2) \succ_3 (a, 3) \succ_3 a_\emptyset$

The assignment π with $\pi(i) = a$ for all $i \in N$ is the unique maximum individually rational assignment, and hence an MIR-Condorcet assignment.
Clearly, the assignment that assigns each agent to $a_\emptyset$ is no IR-Condorcet assignment. The individually rational assignment $\pi'(2) = \pi'(3) = c$ and $\pi'(1) = a_\emptyset$ is preferred by the agents $2, 3$ over each individually rational assignment that assigns exactly two agents to a. In addition, the assignment $\tau(1) = \tau(3) = b$ and $\tau(2) = a_\emptyset$ is preferred by the agents $1, 3$ over any assignment that assigns exactly two agents to c. Finally, the assignment $\sigma(1) = \tau(2) = a$ and $\tau(3) = a_\emptyset$ is preferred by the agents $1, 2$ over each assignment that assigns exactly two agents to b. Therefore, an IR-Condorcet assignment does not exist.

Example 3 *Let $N = \{1, 2, 3\}$ and $A^* = \{a\}$. Let $V_i : (a, 2) \succ_i (a, 3) \succ_i a_\emptyset$ for $i \in \{1, 2\}$ and $V_3 : (a, 3) \succ_3 a_\emptyset$. Then the assignment π with $\pi(1) = \pi(2) = a$ and $\pi(3) = a_\emptyset$ is an IR-Condorcet assignment, while the MIR-Condorcet assignment is given by $\pi'(1) = \pi'(2) = \pi'(3) = a$.*

As it turns out, even for increasing preferences it is computationally hard to decide whether an IR-Condorcet assignment exists (Theorem 3). On the positive side, we can show that the problem becomes easy in the case of decreasing preferences (Theorem 4).

Theorem 3 *It is* coNP*-hard to decide whether* o-GASP *admits an IR-Condorcet assignment, even when all agents have increasing preferences.*

Proof: We provide a reduction from $\overline{\text{X3C}}$, the complementary problem to EXACT COVER BY 3-SETS (X3C). In an instance $\langle X, \mathcal{Y} \rangle$ (with $X = \{1, \ldots, 3q\}$ and $\mathcal{Y} = \{Y_1, \ldots, Y_p\}$) of $\overline{\text{X3C}}$ we ask if it is true that X can *not* be covered by exactly q sets from $\mathcal{Y}$. Clearly, this problem is coNP-complete. Again, we can restrict to instances in which each element of X appears in exactly three sets of $\mathcal{Y}$; recall that in such a case $p = 3q$ holds. For each $i \in X$ let $Y_{i_1}, Y_{i_2}, Y_{i_3}$ with $i_1 < i_2 < i_3$ denote the sets that contain i.
Given such an instance $\langle X, \mathcal{Y} \rangle$ of $\overline{\text{X3C}}$, we construct an instance $\mathcal{I}$ of o-GASP as follows. Let $n := 3q$. Now, we set $N := \{i' | 1 \leq i' \leq 2n-1\}$ and $A^* := \{y_i | 1 \leq i \leq n\} \cup \{b, c\}$. The agents preferences are as follows, with $1 \leq i \leq n-1$ and $n+1 \leq j \leq 2n-1$:

i	n	j
$(b, 2n-1)$	$(y_{n_1}, 2n-1)$	$(c, 2n-1)$
$(b, 2n-2)$	$(y_{n_1}, 2n-2)$	$a_\emptyset$
$\vdots$	$\vdots$	
$(b, n-1)$	$(y_{n_1}, 3)$	
$(y_{i_1}, 2n-1)$	$(y_{n_2}, 2n-1)$	
$(y_{i_1}, 2n-2)$	$(y_{n_2}, 2n-2)$	
$\vdots$	$\vdots$	
$(y_{i_1}, 3)$	$(y_{n_2}, 3)$	
$(y_{i_2}, 2n-1)$	$(y_{n3}, 2n-1)$	
$(y_{i_2}, 2n-2)$	$(y_{n_3}, 2n-2)$	
$\vdots$	$\vdots$	
$(y_{i_2}, 3)$	$(y_{n_3}, 3)$	
$(y_{i_3}, 2n-1)$	$(c, 2n-1)$	
$(y_{i_3}, 2n-2)$	$a_\emptyset$	
$\vdots$		
$(y_{i_3}, 3)$		
$(c, 2n-1)$		
$a_\emptyset$		

We show that $\langle X, \mathcal{Y} \rangle$ is a "yes"-instance of $\overline{\text{X3C}}$ if and only if $\mathcal{I}$ admits an IR-Condorcet assignment.

Let the assignment π be defined by $\pi(i') = c$ for all $i' \in N$.

Assume $\langle X, \mathcal{Y} \rangle$ is a "yes"-instance of $\overline{\text{X3C}}$, i.e., there is no exact cover for X. Then π is an IR-Condorcet assignment: Any assignment $\tilde{\pi} \neq \pi$ assigns at most $n-1$ agents of $\{1, \dots, n\}$ to a non-void activity, and all the agents of $\{n+1, \dots, 2n-1\}$ to $a_\emptyset$; thus, the number of agents who prefer over π over $\tilde{\pi}$ is at least n, while the number of agents who prefer $\tilde{\pi}$ over π is at most $n-1$.

Assume $\langle X, \mathcal{Y} \rangle$ is a "no"-instance of $\overline{\text{X3C}}$, i.e., there is an exact cover for X. Hence, there is a set of indices $I \subseteq \{1, \dots, p\}$ such that $|I| = q$ and $\cup_{h \in I} Y_h = X$. Consider the assignment $\hat{\pi}$ with

- for $i \in \{1, \dots, n\}$, let $\hat{\pi}(i) = y_j$ with $j \in \{i_1, i_2, i_3\} \cap I$
- for $i > n$, let $\hat{\pi}(i) = a_\emptyset$

$\hat{\pi}$ is preferred over π by n agents, and hence π is not IR-Condorcet. On the other hand, π' with $\pi'(i) = b$ for $i \in \{1, \dots, n-1\}$ and $\pi'(i) = a_\emptyset$ for $i \geq n$ is preferred over any assignment $\tilde{\pi} \notin \{\pi, \pi'\}$ by a majority of agents. However, π is preferred over π' by n agents. Therewith, an IR-Condorcet assignment does not exist. ■

Theorem 4 *Given an instance of* o-GASP *with decreasing preferences, it can be decided in polynomial time whether an IR-Condorcet assignment exists. In the case of existence, such an assignment can be determined in polynomial time.*

Proof: Let π be an IR-Condorcet assignment. By decreasing preferences, for each agent i a pair $(a_i, 1)$ is top-ranked. Assume there is an $a \in A^*$ with $|\pi^a| \geq 2$. Let $i \in \pi^a$. Let π' be defined by $\pi'(j) = \pi(j)$ for $j \notin \pi^a$, $\pi'(i) = a_\emptyset$ and $\pi'(j) = a$ for $j \in \pi^a \setminus \{i\}$. By decreasing preferences, $|\pi^a| - 1 \geq 1$ agents prefer π' over π, while only agent 1 prefers π over π'. This contradicts with π being an IR-Condorcet assignment.
Thus, $|\pi^a| \leq 1$ holds for each $a \in A^*$. Let, for $i \in N$ the pair $(a_i, 1)$ denote i's top-ranked alternative. If, for some agent i, $\pi(i) = a$ for some $a \in A \setminus \{a_i\}$ or $\pi(i) = a_\emptyset$, then i would prefer the assignment $\tilde{\pi}$ defined by $\tilde{\pi}(j) = \pi(j)$ for $j \notin \pi^{a_i}$, $\tilde{\pi}(j) = a_\emptyset$ for $j \in \pi^{a_i}$ and $\tilde{\pi}(i) = a_i$. Agent i prefers $\tilde{\pi}$ over π, while only the agents in π^{a_i} prefer π over $\tilde{\pi}$. By $|\pi^a| \leq 1$, this contradicts with the choice of π.
Thus, π is an IR-Condorcet assignment if and only if each agent i is the only agent assigned to a_i. The existence of such an assignment (and its determination if it exists) can be performed in polynomial time. ■

Considering maximum individually rational assignments only, in the case of increasing preferences it is also computationally hard to decide whether an MIR-Condorcet assignment exists (the proof is omitted due to space constraints).

Theorem 5 *It is* coNP-*hard to decide whether* o-GASP *admits an MIR-Condorcet assignment, even when all agents have increasing preferences.*

Unfortunately, we do not have a hardness or easiness result regarding the existence of an MIR-Condorcet assignment in the case of decreasing preferences. In addition, note that the hardness results for the case of increasing preferences also hold for the weak variants (if in Definition 4 the strict inequalities are replaced by weak inequalities) of IR-Condorcet and MIR-Condorcet. However, for these variants, the computational complexity in the case of decreasing preferences is also open.

5 Pareto optimality

Finally, we consider the problem of finding a Pareto optimal assignment. Clearly, in an instance of o-GASP a Pareto optimal assignment always exists. Consider an agent i with top-choice (a, k). Assume it is possible to assign i and $k-1$ other agents to a while respecting individual rationality. Since agent i's top-choice is (a, k) it follows that there is a Pareto optimal assignment π with $\pi(i) = a$ and $|\pi^a| = k$. Thus, for determining a Pareto optimal assignment π we can already fix $\pi(i) = a$ and $|\pi^a| = k$.

Now, the basic algorithmic idea is as follows. Consider an individually rational assignment in which (i) for some agents the activities they are assigned to and (ii) for some activities the number of agents assigned to the activity have already been fixed. Pareto-improve the assignment, i.e., find an assignment that respects (i) and (ii) and is better for at least one agent while making no agent worse off. In fact, applying this idea we can find a Pareto optimal assignment in an instance (N, A, P) of o-GASP in polynomial time by means of procedure *Pareto* and its subroutines *test* and *improve* described below.

Procedure *Pareto*

Let $r_\ell(i)$ denote agent i's ℓ-th ranked alternative. We assume that the profile P contains only alternatives (a, k) which are approved by at least k agents. The idea of procedure *Pareto* (algorithm 1) is as follows. Start with the void assignment, i.e., with the assignment $\pi(i) = a_\emptyset$ for all $i \in N$.

Pick an agent, say agent 1. For her top-ranked alternative (a, k), assign agent 1 to a together with $k - 1$ arbitrarily chosen other agents approving of (a, k). Clearly, for agent 1 there can be no assignment she prefers to the actual assignment. I.e., there is a Pareto optimal assignment π that assigns 1 to a such that $|\pi^a| = k$. Hence, we add 1 to the (initially empty) set N' of agents for which the finally assigned activity is determined already and also fix the number of agents assigned to a to be k in the finally determined assignment. Thus, if $|\{h \in N : (a, k) \in S_h\}| = k$, then we can add all these agents in π^a to N' and proceed with the reduced instance in which a and π^a are removed from the original instance.

However, some of the agents of $\pi^a \setminus \{1\}$ might be better off with an alternative different from (a, k). Hence, if $|\{h \in N : (a, k) \in S_h\}| > k$, then it might be possible to assign some agent of $\pi^a \setminus \{1\}$ to an alternative she prefers to (a, k) without making any agent worse off. So, for each of these agents j, we verify, for alternatives (b, x) with $(b, x) \succ_j (a, k)$, if there is an assignment π' such that a total of x agents (including j) is assigned to b without making any agent worse off (note that in this first step, from the possible choices we must exclude $(b, x) = (a, \ell)$ for any choice of ℓ since it is already fixed that exactly k agents including agent 1 must be assigned to a). In general, in procedure *Pareto* the set S denotes the set of agents j for which we check if such an assignment π' exists.

For a given agent j, procedure *test* checks if there is such an assignment π', for each alternative (b, x) which j prefers to $(\pi(j), |\pi_j|)$ (the alternatives are considered in increasing order of the rank in $\succ_j$). If a respective assignment exists or all alternatives are exhausted, j is added to N'; in the former case, π is replaced by π' and the set S of agents is updated accordingly (now, some agents might want to "deviate" from (b, x)).

Finally, if such a Pareto-improvement is not possible anymore, we remove all agents of N' and all activities to which an agent of N' is assigned from the instance, and iterate *Pareto* with the resulting reduced instance.

Algorithm 1 Procedure for deriving an Pareto optimal assignment in instance $\mathcal{I}$

procedure *Pareto*($\mathcal{I}$)
$\pi(h) := a_\emptyset$ for all $h \in N$, $N' := \emptyset$, $i := 0$, $j := 0$, $\ell := 1$, $S := \emptyset$, $R := \emptyset$
$i := \min N \setminus N'$
if $r_1(i) = a_\emptyset$ **then**
  $\pi(i) = a_\emptyset$
  $N' := N' \cup \{i\}$
else
  for i's top-ranked alternative (a, k) assign i and $k-1$ other agents to a
  $R := \{h \in N : (a, k) \in S_h\}$
  if $|R| = k$ **then**
    $N' := N' \cup R$
  else
    $N' := N' \cup \{i\}$
    $S := \pi^a \setminus \{i\}$
    while $\cup_{a \in A^*} \pi^a \neq N'$ **do**
      while $S \neq \emptyset$ **do**
        $j := \min S$
        procedure *test*($\mathcal{I}, N', j, \pi$)
        $S := S \setminus \{j\}$
        $N' := N' \cup \{j\}$
$N := N \setminus N'$, $A^* := A^* \setminus \{a \in A^* : \pi^a \neq \emptyset\}$,
P is the reduced profile consisting of agents in N and alternatives with $a \in A^*$ only, where only alternatives (a, k) are considered which are approved by at least k agents in N
$\mathcal{I} := (N, A, P)$
if $N \neq \emptyset$ **then**
  Pareto($\mathcal{I}$)

Algorithm 2 Procedure *test* used in procedure *Pareto*

procedure *test*($\mathcal{I}, N', j, \pi$)
$\ell := 1$
while $\ell < mn + 1$ and $r_\ell(j) \succ_j (\pi(j), |\pi_j|)$ **do**
  Let (b, x) denote $r_\ell(j)$.
  if $\pi^b = \emptyset$ or $(b, x) \in \{(\pi(h), |\pi_h|) \mid h \in N'\}$ **then**
    $T := \{g \in N : (b, x) \succ_g (\pi(g), |\pi_g|)\} \cup \pi^b$
    if $|T| \geq x$ **then**
      $\pi' :=$*Improve*($\mathcal{I}, N', j, (b, x), \pi$)
      if $\pi'(j) = b$ **then**
        $\ell := mn + 1$
        $\pi := \pi'$
        $S := S \cup \pi^b$
      else
        $\ell := \ell + 1$

Procedures *test* and *improve*

Given instance $\mathcal{I}$, agent j, assignment π and agent set N', the goal of procedure *test* is an individually rational assignment π' which assigns j to the best-ranked among the alternatives (b, x) that j prefers to $(\pi(j), |\pi_j|)$ such that for each agent of N' the assignment remains unchanged, i.e., for all $i \in N'$ $(\pi'(i), |\pi'_i|) = (\pi(i), |\pi_i|)$ holds. Procedure *test* considers the alternatives (b, x) in increasing order of their rank in $\succ_j$, checks if such an assignment exists and terminates if such an assignment is found or all alternatives (b, x) which j prefers to $(\pi(j), |\pi_j|)$ are exhausted. Note that agent j cannot be assigned to (b, x) if the total number of agents assigned to b is already fixed to a

number different from x (line 5 of procedure *test*).
Summing up, for considered alternative (b, x) assignment π' must satisfy the following properties:

$$\begin{array}{l} 1. \text{ for all } i \in N' \text{ we have } (\pi'(i), |\pi'_i|) = (\pi(i), |\pi_i|) \\ 2. \ (\pi'(j), |\pi'_j|) = (b, x) \\ 3. \text{ for all } h \in N\text{: } (\pi'(h), |\pi'_h|) = (\pi(h), |\pi_h|) \text{ or } (\pi'(h), |\pi'_h|) \succ_h (\pi(h), |\pi_h|) \end{array} \tag{1}$$

In order to check whether such an assignment exists for a given alternative (b, x), *test* executes the subroutine *improve* described below.

Procedure *improve* works as follows. Given as input an instance $\mathcal{I} = (N, A, P)$, a subset $N' \subset N$, dedicated agent j and alternative (b, x), and assignment π, *improve* checks if there is an assignment π' such that (1) is satisfied.
This is done by solving a feasible flow problem with lower and upper bounds on the edge capacities in the following directed graph $G = (V, E)$, with $V := \{s, t, b\} \cup N \cup \{a|\pi^a \neq \emptyset, a \in A^*\}$. The edge set E is constructed as follows:

- introduce the edges (s, j) and (j, b) of lower and upper capacity bound 1
- for each $i \in N \setminus \{j\}$,
  - introduce the edges (s, i)
    - of upper capacity bound 1 if $\pi(i) = a_\emptyset$
    - of lower and upper capacity bound 1 if $\pi(i) \neq a_\emptyset$.
  - for each $a \in A^*$ with $\pi^a \neq \emptyset$, introduce
    - the edge (i, a) of lower and upper capacity bound 1 if $\pi(i) = a$ and $i \in N'$
    - the edge (i, a) of upper capacity bound 1 if $\pi(i) = a$ and $i \notin N'$
    - the edge (i, a) of upper capacity bound 1 if $(a, |\pi^a|) \succ_i (\pi_i, |\pi_i|)$
  - introduce the edge (i, b) of upper capacity bound 1 if $(b, x) \succ_i (\pi_i, |\pi_i|)$
- for each $a \in A^*$ with $\pi^a \neq \emptyset$ introduce edge (a, t) of lower and upper capacity bound $|\pi^a|$.
- introduce the edge (b, t) of lower and upper capacity bound x.

Unless otherwise specified above, the lower capacity bound of an edge is 0. An example of the graph is given in Figure 1, where l, u denote the lower and upper edge capacity bounds. In Figure 1, agents 1 and j are as described above; for agent i we have $\pi(i) = b$ and $(d, |\pi^d|) \succ_i (b, x)$; agent n is currently not assigned to any activity but approves of $(d, |\pi^d|)$ and $(r, |\pi^r|)$.

Let f denote a feasible integer flow in the above graph (if such a flow exists). Now, *improve*$(\mathcal{I}, N', j, (b, x), \pi)$ outputs

- π if there is no feasible flow
- the assignment π' defined by, for $i \in N$ and $a \in A^*$, $\pi'(i) = a$ if and only f sends flow along the edge (i, a), and $\pi'(i) = a_\emptyset$ if there is no flow sent through vertex i.

Lemma 1 *In polynomial time, procedure* improve$(\mathcal{I}, N', j, (b, x), \pi)$ *outputs either an assignment* π' *that satisfies the conditions stated in (1) or* π *in the case that such an assignment* π' *does not exist.*

Proof: The polynomial running time of *improve* follows from the facts that the size of graph G is polynomial in n and m and an integer feasible flow can be computed in polynomial time (see, e.g., [2]).

Assume the procedure outputs an assignment $\pi' \neq \pi$. By construction of graph G, conditions 1. and 2. of (1) are satisfied by π'. In order to verify that 3. is satisfied as well, it is sufficient to see that an agent i with $\pi(i) \neq a_\emptyset$ by construction cannot be assigned to a worse-ranked non-void alternative; in addition, by the fact that the lower and upper capacity of (s, i) is 1, i cannot be

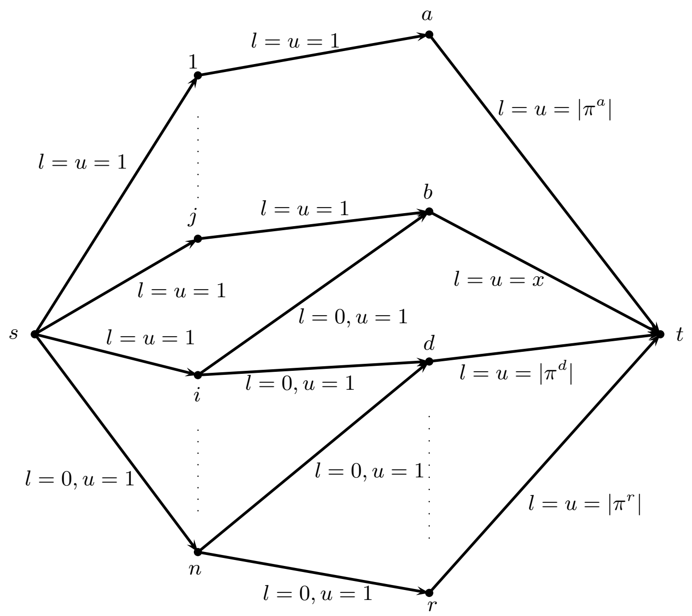

Figure 1: Graph G used in procedure *improve*.

assigned to the void activity either. Thus, for i with $\pi(i) \neq a_\emptyset$ either $\pi'(i) = b$ or $\pi'(i) = \pi(i)$ holds. The first case, by construction implies $(\pi'_i, |\pi'_i|) = (b, x) \succ_i (\pi_i, |\pi_i|)$; in the second case, we have $(\pi'(i), |\pi'_i|) = (\pi(i), |\pi_i|)$.

On the other hand, it is not hard to see that in the case that an assignment satisfying (1) exists, procedure *improve* finds one. ■

As a consequence, in polynomial time procedure *test*$(\mathcal{I}, N', j, \pi)$ checks if there is an assignment π' such that (1) holds for some $(b, x) \succ_j (\pi(j), |\pi_j|)$; in case of existence, π is updated to coincide with an assignment π' that assigns j to her best-ranked such alternative. Given the fact that for all $i \in N'$, the pair $(\pi(i), |\pi(i)|)$ is already the respective alternative of the finally computed assignment, *test*$(\mathcal{I}, N', j, \pi)$ hence determines the activity (and corresponding group size) to which agent j is finally assigned; therewith, after executing *test*$(\mathcal{I}, N', j, \pi)$ agent j is added to the set N'.

Procedure *Pareto* executes procedure *test* at most once for each agent, and the latter evokes *improve* for a given agent at most once for each alternative (b, x), hence procedure *improve* is executed at most mn^2 times. Thus, with Lemma 1 we can conclude the following theorem.

Theorem 6 *In* o-GASP*, a Pareto optimal assignment can be found in polynomial time.*

6 Conclusion

We have analysed the computational complexity involved in finding solutions to the problem of assigning agents to activities on basis on their preferences over pairs made up of an activity and the size of the group of agents participating in that activity. Taking two different perspectives, solution concepts are provided for this problem. With respect to the two natural special cases of decreasing and increasing preferences, computational complexity results are given for the task of finding such a solution. For future research, a finer placement of IR-CONDORCET-EXISTENCE and MIR-CONDORCET-EXISTENCE in the polynomial hierarchy, and the computational complexity of the latter problem in the case of decreasing problems would be of interest.

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