# Non Myopic Collaborators (nearly) Get Their Way

Yair Zick, Yoram Bachrach, Ian Kash and Peter Key

**Abstract**

We consider revenue division problems in iterative settings. In our model, a group of players has some initial resources, used in order to generate revenue. At every time-step, the revenue shares received at time $t$ are player resources at time $t+1$, and the game is repeated. The key issue here is that the way resources are shared has a dramatic effect on long-term social welfare, so in order to maximize individual long-term revenue one must consider the welfare of others, a behavior not captured by other models of cooperation among economic agents. Our work focuses on homogeneous production functions. We identify conditions that ensure that what players find best for themselves is "nearly" what is best for the group. We apply our results to some families of utility functions, and discuss their implication in these domains.

## 1 Introduction

We consider a setting where players repeatedly engage or compete with each other over time and need to balance individual welfare and group or social welfare; this is necessary because the future is conditional on aggregate welfare. We adopt a model where a finite set of players periodically decide to divide up some resources amongst themselves, and where the resources available to all of them at the next allocation period depend upon this current division; in other words, there are complementarities between the individuals or between their associated resources. If we express players' preferences using utility functions, then the future utility of an individual at time $t$ depends upon the aggregate utility of all the players up to time $t$.

This type of interplay between individual desires and global utility is common in settings where budgets need to be iteratively shared. Consider, for example, a large software company with several departments. Each department works on a specific product and is allocated a budget at the end of some time period (say, biannually). Each department uses its budget to produce a product (a tablet, a gaming console, an operating system etc.), which is pooled at the end of the "fiscal round" and redistributed. Here, money plays a dual role: when assigned to the divisions, money is a resource, who use it in order to generate money-as-revenue. We assume that there are complementarities between divisions, i.e., profits are increased by investing in all divisions; this may be due to the fact that different products appeal to different markets —it is better to participate in several markets, rather than focusing solely on OS development— or because some divisions, such as IT and HR, provide services to other divisions. Divisions are faced with two conflicting agendas. On the one hand, each division wants to maximize the revenue share that it receives (a division with higher revenue can increase employee salaries, hire more employees to decrease workload on others, invest in better equipment and software, etc.); on the other hand, no division wants to receive the entire share of the revenue, as this may hurt future profits and result in lower future revenue for the division. If one division receives a disproportionately large share of the profits, total company revenue may dramatically decrease, which, in turn, hurts future revenue. In this setting, what is the socially optimal way of dividing revenue? What is the revenue sharing scheme that is best for an individual division? Is there a way to divide profits such that individual divisions are happy and social welfare is high?

We consider the following revenue sharing problem: given a valuation function $v : \mathbb{R}^n_+ \to \mathbb{R}$ and an initial endowments vector $\mathbf{w}(0) \in \mathbb{R}^n$, find a sequence of resource allocations that maximize social welfare on the one hand, and on the other hand, are such that players do not want to increase their share. We begin by showing a simple, yet unfortunate result: any socially optimal sequence of revenue divisions is not individually optimal. However, we show that when $v$ is homogeneous of degree $k \geq 1$, individual incentives align with group incentives in the limit. In more detail, assuming that players care about their future utility for a sufficiently long time horizon, individually optimal payoff divisions tend to be socially optimal payoff divisions. On the other hand, if $v$ is homogeneous of degree $k < 1$, players remain selfish even in the limit. We apply our results to CES, Cobb-Douglas and Leontief production functions to obtain sequences of optimal and individually optimal contracts, and we discuss the implication of our results to network flow games.

Intuitively, since long-run optimal divisions for players are aligned with the social optimum, in any "reasonable" strategic setting we would expect players to be able to reach the socially optimal outcome in equilibrium if they are sufficiently patient. We give two examples of strategic settings where this is the case. In the first, players must directly negotiate about how to divide production revenue. In the second, players have private information about their part of the production function that they report to a mechanism that then decides on the contract.

We conclude with a discussion of related work and future directions.

# 2 Preliminaries

The key assumption in our work is that while players are interested in receiving a fair share of the profits at every time step, they are also interested in increasing their long-term profits. Hence it would not be in the best interest of any player to demand all the profits, as leaving others without a reasonable share could actually result in lower future profits.

Formally, a group of players $N = \{1, \ldots, n\}$ has an *initial resource vector* $\mathbf{w}(0) = (w_1(0), \ldots, w_n(0)) \in \mathbb{R}^n_+$, there is some *production function* $v : \mathbb{R}^n_+ \to \mathbb{R}_+$, which, for every $\mathbf{x} \in \mathbb{R}^n_+$ , determines the production level $v(\mathbf{x}) \in \mathbb{R}_+$. Here and throughout the paper, boldface letters refer to vectors; the set $\mathbb{N}$ is the set of all positive integers (excluding 0), and the set $\mathbb{R}_+$ is the set of all non-negative real numbers (including 0).

Time is divided into discrete epochs indexed by $t$. At time $t = 1$, the total payoff (production value) is $v(\mathbf{w}(0))$, which we denote as $V_1$; players select an allocation vector $\mathbf{x}_1$ from the $n-1$ dimensional simplex, $\mathbf{x}_1 \in \Delta_n$, where $\Delta_n = \{\mathbf{x} \in \mathbb{R}^n_+ \mid \sum_{i=1}^n x_i = 1\}$, and each player $i \in N$ receives an individual share of $w_i(1) = x_{i,1}V_1$.

At time 2, the aggregate production is $V_2 = v(\mathbf{w}(1))$, players select $\mathbf{x}_2 \in \Delta_n$, and the process repeats.

A *contract* $\chi$ is a sequence $(\mathbf{x}_t)_{t=1}^{\infty}$ of vectors in $\Delta_n$. At every time step $t$, player $i$ receives a share of $w_i(t) = x_{i,t}V_t$, where $V_t = v(\mathbf{w}(t-1))$. The revenue that a contract $\chi$ generates at time $t$ is denoted $V_t(\chi)$, and is computed recursively as described above. Observe that the value $V_t(\chi)$ depends only on the first $t-1$ payoff divisions; we sometimes write $V_t(\mathbf{x}_1, \ldots, \mathbf{x}_{t-1})$ when this property needs to be emphasized.

We say that a contract is *stationary* if for all $t$ we have $\mathbf{x}_t = \mathbf{x}$ for some fixed $\mathbf{x} \in \Delta_n$. A stationary contract is then identified with $\mathbf{x}$, rather than the constant sequence $(\mathbf{x})_{t=1}^{\infty}$.

Given a contract $\chi = (\mathbf{x}_t)_{t=1}^{\infty}$, the *total welfare at time $T$* is simply $\sum_{t=1}^{T} V_t(\chi)$, and is denoted $\mathrm{sw}_T(\chi)$. We say that a contract $\chi^*$ is *pointwise optimal at time $T$* if $\chi^*$ maximizes $\mathrm{sw}_T(\chi)$ over the space of all possible contracts. We say that $\chi^*$ is *universally optimal* if $\chi^*$ is optimal at time $T$ for all $T$.

We assume that players are not interested in the global social welfare provided by a contract, but rather in their own revenue. Given a player $i \in N$, the benefit that player $i$ receives from the contract $\chi = (\mathbf{x}_t)_{t=1}^{\infty}$ at time $t$ is given by $x_{i,t}V_t(\chi)$; we also write $U_{i,T}(\chi) = \sum_{t=1}^{T} x_{i,t}V_t(\chi)$, to be player $i$'s *utility* (or *individual revenue*) at time $T$. We say that $\chi^*$ is *individually optimal for player $i$ at time $T$* if $\chi^*$ maximizes $U_{i,T}$ over the space of all possible contracts. Observe that if $v$ is continuous, then for all $T$ there exists some contract $\chi^*$ that is individually optimal for $i$ at time $T$.

*Remark* 2.1. By taking the share that player $i$ receives as his utility, we implicitly assume that player $i$ wants to receive a higher share of the profits at every time step. However, we also assume that the players invest all of their revenue at time $t$ in time $t+1$; that is, no part of the players' revenue goes to generate "happiness". This assumption is without loss of generality: our model can easily accommodate a scenario where a *fixed* part of player resources goes to their welfare. More explicitly, we can add a parameter $c_i$ for every $i \in N$, where $c_i \in [0,1]$ is the share of the profits that $i$ keeps for himself; $c_i$ can represent labor costs, server costs or employee benefits; all of our results can be modified accordingly.

# 3 Optimal Contracts: First Observations

Finding a socially optimal contract at time $T$ amounts to solving the following problem:

$$\begin{aligned} \max \quad & \textstyle\sum_{t=1}^{T} v(\mathbf{w}_t) \quad \text{over} \quad (\mathbf{w}_t)_{t=1}^{T} \\ \text{s.t.} \quad & \textstyle\sum_{i=1}^{n} w_{i,1} = V_0 \\ & \textstyle\sum_{i=1}^{n} w_{i,t} = v(\mathbf{w}_{t-1}) \qquad \forall t, 2 \le t \le T \end{aligned} \tag{1}$$

We first show that if $v$ is monotone increasing[1], one can find an optimal solution to Equation (1) using a greedy procedure. Given a non-negative constant $C$, let us write $\Delta_n(C) = \{\mathbf{w} \in \mathbb{R}_+^n \mid \sum_{i=1}^{n} w_i = C\}$; in particular, $\Delta_n$ equals $\Delta_n(1)$.

**Proposition 3.1.** *Let $opt(V)$ be the value of Equation* (1) *when $V_0 = V$; if $v$ is monotone increasing, then $opt(V)$ is monotone increasing in $V$.*

As an immediate corollary of Proposition 3.1 (proof omitted), we have that in order to find an optimal contract at time $T$, one must find a point $\mathbf{w}_1 \in \Delta_n(V_1)$ that maximizes $v$; then, a point $\mathbf{w}_2 \in \text{argmax}_{\mathbf{w}\in\Delta_n(V_2)} v(\mathbf{w})$ and so on. In other words, greedily maximizing the revenue at every time step will result in a universally optimal contract. From a computational point of view, finding universally optimal contracts is relatively straightforward when $v$ is monotone, and that finding a point that maximizes $v$ over $\Delta_n(V)$ can be done in polynomial time for any $V$ (assuming that it is possible to find a maximum of $v$ over $\Delta_n$ in polynomial time).

**Corollary 3.2.** *Let $\tau(n)$ be the time required to find a maximum of $v$ over $\Delta_n(V)$; if $v$ is monotone increasing, then finding an optimal contract at time $T$ is in $\mathcal{O}(\tau(n)T)$*

When $v$ is homogeneous, finding an optimal contract is even easier. Recall that a function $v : \mathbb{R}_+^n \to \mathbb{R}_+$ is called *homogeneous of degree $k$*, or $k$-homogeneous, if for all $\alpha \ge 0$ and all $\mathbf{w} \in \mathbb{R}_+^n$, $v(\alpha\mathbf{w}) = \alpha^k v(\mathbf{w})$. We now make an important observation (proof omitted): if $v$ is homogeneous of degree $k$, then there exists a stationary socially optimal contract.

**Proposition 3.3.** *If $v$ is monotone and homogeneous of degree $k$, then there is a universally optimal stationary contract $\mathbf{x}^* \in \Delta_n$. Also, $\mathbf{x}^*$ is the maximum of $v$ over $\Delta_n$.*

If $v$ is homogeneous, we get a closed formula for utility under a contract $\chi = (\mathbf{x}_t)_{t=1}^{\infty}$.

**Proposition 3.4.** *If $v$ is homogeneous of degree $k$, then the utility at time $T$ under the contract $\chi = (\mathbf{x}_t)_{t=1}^{\infty}$ is $V_T(\chi) = V_1^{k^{T-1}} \prod_{t=1}^{T-1} v(\mathbf{x}_t)^{k^{T-1-t}}$.*

*Proof.* We use induction on $T$. For $T = 1$ we have $V_1(\chi) = V_1$ agreeing with the above formula. Assuming the claim holds for $T$, we show it holds for $T+1$:

$$\begin{aligned} V_{T+1}(\chi) &= v(V_T(\chi)\mathbf{x}_T) = V_T(\chi)^k v(\mathbf{x}_T) = \left( V_1^{k^{T-1}} \prod_{t=1}^{T-1} v(\mathbf{x}_t)^{k^{T-t-1}} \right)^k v(\mathbf{x}_T) \\ &= \left( V_1^{k^T} \prod_{t=1}^{T-1} v(\mathbf{x}_t)^{k^{T-t}} \right) v(\mathbf{x}_T) = V_1^{k^T} \prod_{t=1}^{T} v(\mathbf{x}_t)^{k^{T-t}}. \end{aligned}$$

[1] If $w_i \le w'_i$ for all $i$ then $v(\mathbf{w}) \le v(\mathbf{w}')$

□

For stationary contracts, we obtain an even simpler formula.

**Corollary 3.5.** *Given a stationary contract* $\mathbf{x}$*, if* $v$ *is homogeneous of degree* $k$*, then* $V_T(\mathbf{x}) = V_1^{k^{T-1}} v(\mathbf{x})^{\sum_{t=0}^{T-2} k^t}$.

Using Proposition 3.4, we obtain the following formula for the individual utility of player $i$ at time $T$. $U_{i,T}(\chi) = \sum_{t=1}^{T} x_{i,t} V_1^{k^{t-1}} \prod_{h=1}^{t-1} v(\mathbf{x}_h)^{k^{t-1-h}}$, and for a stationary contract, $U_{i,T}(\mathbf{x}) = x_i \left( \sum_{t=1}^{T} V_1^{k^{t-1}} v(\mathbf{x})^{\sum_{h=0}^{t-2} k^h} \right)$.

*Remark* 3.6 (Discounted Future Welfare). Discounting future returns can also be modelled in our framework, but are omitted due to space constraints. Briefly, for a large enough discount factor, our results still hold.

# 4 Individually Optimal Contracts For Homogeneous Production Functions

The results in Section 3 show that the class of homogeneous production functions has a highly desirable property: namely, it is possible to use the same revenue division in all rounds and derive a universally optimal contract.

Hence, when $v$ is homogeneous, universally optimal contracts are easy to characterize and find: simply find a maximum of $v$ over $\Delta_n$. Identifying individually optimal contracts is a more complex task. Using the closed formulas for individual player utility, we now turn to show the main result of this paper: under certain assumptions, the socially optimal contract is "nearly" individually optimal. Our results are asymptotic in nature, showing that individually optimal contracts at time $T$ converge to the socially optimal contract as $T$ grows. Taking a large value of $T$ implies that collaborative players are far-sighted: only when players care about their long term utility do their goals match those of the group.

In order to have a robust model of cooperation and individual incentives, we wish to capture some notion of complementarity among players. If players are actually better off without allocating resources to some subset of players, then there is little a-priori incentive to collaborating with them. This notion is captured via the idea of *mutual dependency*.

**Definition 4.1.** A function $v : \mathbb{R}^n \to \mathbb{R}$ satisfies mutual dependency if for all $\mathbf{x} \in \Delta_n$ and all $i \in N$, if $x_i = 0$, then $v(\mathbf{x}) = 0$.

Mutual dependency is an important aspect of negotiating optimal contracts; if mutual dependency does not hold, then there are some players who need not contribute to the group effort, and are somewhat expendable. In other words, suppose that $\mathbf{x}^*$ is an optimal stationary contract for $v$ and that $x_i^* = 0$; then the set $N \setminus \{i\}$ can argue that player $i$ should receive no share of the profits, as there exist optimal contracts that do not require any of his resources (in terms of a company, $i$ would be a department that is completely redundant). In what follows we assume that the function $v$ satisfies mutual dependency, and that there are some points in $\Delta_n$ for which $v$ assumes strictly positive values. The following example shows the importance of the mutual dependency property.

**Example 4.2.** Consider a setting where for any $\mathbf{x} \in \mathbb{R}_+^n$ we have that $v(\mathbf{x}) = f(\sum_{i=1}^n x_i)$, where $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a strictly monotone function. In that case, the value of $v$ at time $t$ is simply $f^t(W)$, where $W = \sum_{i \in N} w_i(0)$, and $f^t$ is the composition of $f$ with itself $t$ times. In this setting, the only thing determining the worth of a player is the amount of money he brings to the table at time $t$. Mutual dependency does not hold here. In this setting, any choice $\mathbf{x} \in \Delta_n$ is universally socially optimal.

We begin with a basic question regarding properties of optimal versus individually optimal stationary contracts. Given an optimal (or pointwise optimal) contract $\mathbf{x}^* \in \Delta_n$, is it possible that $\mathbf{x}^*$ is also individually optimal for some $i \in N$? As Lemma 4.3 shows, if $v$ is continuously differentiable, this is not possible.

**Lemma 4.3.** *Suppose that $v$ is continuously differentiable, and define $F : \mathbb{R}^n \to \mathbb{R}$ with $F \in \{V_t, \mathrm{sw}_t\}$ (where $t$ is a fixed time period); Given some $\mathbf{x}^*$ that is a global maximum of $F$ in the interior of $\Delta_n$, then $\mathbf{x}^*$ does not maximize individual utility for any $i \in N$ at time $t$.*

*Proof.* First, observe that if $v$ is continuously differentiable, then $F$ is as well. Moreover, since $\mathbf{x}^*$ is in the interior of $\Delta_n$, $x_i^* < 1$ for all $i \in N$.

Let $j \in N$ be a player for whom $x_j^* > 0$, where $i \neq j$. We show the claim holds by showing that 0 is not a maximum of the function $g_{ij}(x) = (x_i^* + x)F(\mathbf{y}(x))$, where $\mathbf{y}(x)$ equals $\mathbf{x}^*$ on all coordinates but $i, j$, and $y_i(x) = x_i^* + x, y_j(x) = x_j^* - x$.

We note that if $\mathbf{x}^* \in \Delta_n$ is a global maximum of $F$ in the interior of $\Delta_n$, then $\frac{\partial F}{\partial x_i}(\mathbf{x}^*) = \frac{\partial F}{\partial x_j}(\mathbf{x}^*)$ for all $i, j \in N$. Taking the derivative of $g_{ij}$, we get $\frac{\partial g_{ij}}{\partial x}(x) = F(\mathbf{y}(x)) + (x_i^* + x)(\frac{\partial F}{\partial x_i}(\mathbf{y}(x)) - \frac{\partial F}{\partial x_i}(\mathbf{y}(x)))$; Thus $\frac{\partial g_{ij}}{\partial x}(0) = F(\mathbf{x}^*) > 0$. □

Lemma 4.3 implies that there exist stationary contracts that are better for individuals than optimal stationary contracts. The proof of Lemma 4.3 is simple, but we wish to stress its importance. For an individual player, this lemma presents a rather unfortunate state of affairs: any contract that is socially optimal is *necessarily* individually sub-optimal. This means that social welfare and individual gains are always at odds. Can we, under certain conditions, mitigate this effect? In what follows, we show when this is indeed the case.

Before we proceed, let us recall the following property of homogeneous functions.

**Proposition 4.4.** *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a $k$-homogeneous, continuously differentiable function, then the point $\mathbf{x}^*$ is a critical point of $f$ in the interior $\Delta_n$ if and only if $\frac{\partial f}{\partial x_i}(\mathbf{x}^*) = \frac{\partial f}{\partial x_j}(\mathbf{x}^*)$ for all $i, j \in N$; in that case, $\frac{\partial f}{\partial x_i}(\mathbf{x}^*) = kf(\mathbf{x}^*)$ for all $i \in N$.*

Using the formulas for individual utility derived from Proposition 3.4 and Corollary 3.5, we now show that partial derivatives with respect to player $j$ of the individually optimal contract for player $i$ are equal for all $j \neq i$ and all time. To prove this result we require that the optimization problem faced by player $i$ is Strong Lagrangian, which implies that taking the maximum of the associated Lagrangian is indeed the maximum of the original optimization. Since we are working with functions and optimizing over a closed convex set, a sufficient condition is that $v$ is concave.

**Lemma 4.5.** *Assume $v$ is monotone, homogeneous of degree $k$ and that the constrained optimization problem for player $i$ is strong Lagrangean. Let $\chi_T^* = (\mathbf{x}_t^*)_{t=1}^{\infty}$ be an individually optimal contract for player $i$ at time $T$, then for any $q < T$ and any $j, j' \in N$ such that $j, j' \neq i$, $\frac{\partial v}{\partial x_{j,q}}(\mathbf{x}_q^*) = \frac{\partial v}{\partial x_{j',q}}(\mathbf{x}_q^*)$; moreover, $\frac{\partial v}{\partial x_{i,q}}(\mathbf{x}_q^*) = \frac{\partial v}{\partial x_{j,q}}(\mathbf{x}_q^*) - \frac{v(\mathbf{x}_q^*)k^q V_q(\chi_T^*)}{\sum_{t=q+1}^{T} k^{t-1} x_{i,t} V_t(\chi_T^*)}$.*

Lemma 4.5 (proof omitted) implies that if $\chi_T^* = (\mathbf{x}_t^*)_{t=1}^{\infty}$ is an individually optimal contract for $i \in N$ at time $T$, every point $\mathbf{x}_q^*$ is "nearly" a critical point of $v$: the partial derivatives of $v$ at $\mathbf{x}_q^*$ are all equal, except that of player $i$, which differs by an amount that is $O(k^{-(T-q)})$ for $k > 1$, and $O(1/(T-q))$ when $k = 1$. Hence, by fixing a time $q$ and taking the horizon $T$ to infinity, we can ensure that $\mathbf{x}_q^*$ *approaches* a critical point of $v$ over $\Delta_n$.

First, let us observe a simple property of individually optimal contracts (proof omitted).

**Lemma 4.6.** *Suppose that $v$ is $k$ homogeneous. Let $\chi_T^* = (\mathbf{x}_t^*)_{t=1}^{\infty}$ be an individually optimal contract for player $i$ at time $T$, and let $\mathbf{x}^*$ be an optimal stationary contract, then $x_{i,t}^* \geq x_i^*$ for all $t$.*

Since Lemma 4.6 holds for any individually optimal contract, we get as a corollary that there is some constant $c > 0$ such that $x^*_{i,t} > c$ for all individually optimal contracts and for all $t \in \mathbb{N}$.

**Theorem 4.7.** *Suppose that we are given a sequence of contracts $(\chi^*_T)_{T=1}^{\infty}$ such that*

1. *$\chi^*_T$ is individually optimal for player $i$ at time $T$ for all $T$.*
2. *Player $i$ receives a weakly increasing share at every round; i.e., for every $\chi^*_T = (\mathbf{x}_t(T))_{t=1}^{\infty}$, $x_{i,t}(T)V_t(\chi^*_T) \le x_{i,t+1}(T)V_{t+1}(\chi^*_T)$.*

*If $v$ is continuously differentiable and homogeneous of degree $k \ge 1$ , for any $q \in \mathbb{N}$, if $(\mathbf{x}_q(T))_{T=1}^{\infty}$ converges, then $\lim_{T\to\infty} \mathbf{x}_q(T)$ is a critical point of $v$ over $\Delta_n$, where the convergence speed is linear for $k > 1$, and sublinear for $k = 1$.*

*Proof.* By Lemma 4.5, $\frac{\partial v}{\partial x_{j,q}}(\mathbf{x}_q(T)) = \frac{\partial v}{\partial x_{j',q}}(\mathbf{x}_q(T))$ for all $j, j' \ne i$. It remains to show that $\lim_{T\to\infty} \frac{\partial v}{\partial x_{i,q}}(\mathbf{x}_q(T)) - \frac{\partial v}{\partial x_{j,q}}(\mathbf{x}_q(T)) = 0$, which under our assumptions is equivalent to showing that $\lim_{T\to\infty} \frac{V_q(\chi^*_T)k^q v(\mathbf{x}_q(T))}{\sum_{t=q+1}^{T} x_{i,t}(T)k^{t-1}V_t(\chi^*_T)} = 0$. Let $\mathbf{x}^* \in \operatorname{argmax}_{\mathbf{x}\in\Delta_n} v(\mathbf{x})$ be some optimal stationary contract; recall from Lemma 4.6, $x_{i,q}(T) \ge x^*_i$, and $x^*_i > 0$ from mutual dependency. Thus:

$$\frac{V_q(\chi^*_T)k^q v(\mathbf{x}_q(T))}{\sum_{t=q+1}^{T} x_{i,t}(T)k^{t-1}V_t(\chi^*_T)} \le \frac{V_q(\chi^*_T)k^q v(\mathbf{x}_q(T))}{\sum_{t=q+1}^{T} x_{i,q}(T)k^{t-1}V_q(\chi^*_T)}$$
$$= \left(\frac{k^q v(\mathbf{x}_q(T))}{x_{i,q}(T)}\right)\frac{1}{\sum_{t=q+1}^{T} k^{t-1}} \le \frac{v(\mathbf{x}^*)}{x^*_i}\frac{1}{\sum_{t=0}^{T-q} k^t}.$$

Since $v$ is homogeneous of degree $k \ge 1$, we have that $\sum_{t=0}^{T-q} k^t$ goes to infinity is $T$ grows, thus $\lim_{T\to\infty} \frac{\partial v}{\partial x_{i,q}}(\mathbf{x}_q(T)) - \frac{\partial v}{\partial x_{j,q}}(\mathbf{x}_q(T)) = 0$. According to Proposition 4.4, this implies that the limit of the sequence $(\mathbf{x}_q(T))_{T=1}^{\infty}$ is a critical point of $v$ over $\Delta_n$. □

As an immediate corollary of Theorem 4.7, if $v$ has a *unique* critical point in the interior of $\Delta_n$ (e.g., if $v$ is a strictly concave function), then we can obtain a stronger claim.

**Corollary 4.8.** *Let $(\chi^*_T = (\mathbf{x}_t(T))_{t=1}^{\infty})_{T=1}^{\infty}$ be a sequence satisfying the conditions stated in Theorem 4.7; suppose that $v$ is continuously differentiable, concave and homogeneous of degree $k \ge 1$; then for any $q$, $\lim_{T\to\infty} \mathbf{x}_q(T)$ exists and is an optimal contract for $v$.*

Corollary 4.8 can be interpreted as follows. Suppose that in a collaborative setting, things are going well: the production function $v$ exhibits increasing returns to scale (it is homogeneous of degree $k \ge 1$), and every player $i \in N$ can guarantee an increasing share of the profits for an appropriate choice of contracts. If player $i$ is far-sighted enough —that is, he would consider his overall utility for a large enough time period $T$— then it is in his best interest to choose contracts that are nearly globally optimal, at least at time periods $q$ that are sufficiently distant from the horizon $T$. In other words, when players are not myopic and the production function indicates favorable future conditions, *what is best for the group is also best for the individual.* We stress that $q$ must be sufficiently far from $T$; otherwise, the claims trivially do not hold. For example, at the last time step, $T$, it is clearly optimal for a player to choose an allocation that gives him $100\%$ of the revenue.

We mention that the convergence result shown in Theorem 4.7 only requires that $\lim_{T\to\infty} \frac{V_q(\chi^*_T)}{\sum_{t=q+1}^{T} k^{t-1}V_t(\chi^*_T)} = 0$. This can occur even if the conditions stated in Theorem 4.7 do not hold. For example, if $k$ is large enough, it may compensate for any decrease in the value of $V_t(\chi^*_T)$. In general, a high degree of homogeneity would result in faster convergence of $(\mathbf{x}_q(T))$ to a critical point of $v$ over $\Delta_n$. However, the degree of homogeneity is not the only factor in play here; even if $k < 1$ we may have convergence, if $\lim_{T\to\infty} \sum_{t=q+1}^{t} k^{t-1}V_t(\chi^*_T) = \infty$. However, if $k < 1$, this requires that $V_t(\chi^*_T)$ exhibits an extremely fast growth rate (at least exponential) in $t$.

## 4.1 Individual Regret Under Stationary Contracts

Suppose that instead of letting players choose any contract, players may only choose stationary contracts. In other words, rather than choosing a different allocation at every time step, players choose a single allocation at time $t = 1$, and stick to this allocation for all future time steps. This may happen if changing the allocation at every round is costly, or if computing a maximum of $v$ over $\Delta_n(V)$ at every step is computationally expensive.

If we assume that players are only allowed to reason about stationary contracts, then they would be far more agreeable to choosing what is optimal for the group.

We say that a contract $\mathbf{x}$ has *non-vanishing welfare* if $\lim_{T\to\infty} \mathrm{sw}_T(\mathbf{x}) = \infty$; that is, the sum $\sum_{t=1}^{T} V_t(\mathbf{x})$ does not converge. This condition ensures that the expected future revenue of players expect does not decay to zero with time. We now show the following claim (proof omitted)

**Theorem 4.9.** *Suppose that $v : \mathbb{R}^n \to \mathbb{R}$ is $k$-homogeneous and continuously differentiable, such that an optimal contract for $v$ satisfies the non-vanishing welfare condition, and for each player $i$ the individual optimization problem is strong Lagrangian. Given some $i \in N$, let $(\mathbf{x}^*_T)_{T=1}^{\infty}$ be a sequence of individually pointwise optimal stationary contracts for $i$, such that $\lim_{T\to\infty} \mathbf{x}^*_T = \mathbf{x}^*$, and there exists some point $\mathbf{y}^*$ such that $\mathbf{x}^* > \mathbf{y}^*$, and $\mathbf{y}^*$ satisfies the non-vanishing welfare condition. Then $\mathbf{x}^*$ is a critical point of $v$ over $\Delta_n$ if and only if $k \geq 1$.*

Again, assuming that $v$ has a unique maximum, we obtain the following corollary (proof omitted).

**Corollary 4.10.** *Under the same assumptions as in Theorem 4.9 with $v$ $k$-homogeneous with $k \geq 1$, if $v$ has a unique maximum over $\Delta_n$, then any sequence $(\mathbf{x}^*_T)_{T=1}^{\infty}$ of individually optimal stationary contracts is convergent, and its limit is a group optimal contract.*

We conclude that if the scope of players' reasoning about alternative outcomes is limited to stationary contracts, then we can obtain a much stronger result than that given in Theorem 4.7. Namely, a contract that maximizes individual welfare (over the space of *stationary* contracts) converges to a global maximium of $v$.

Corollary 4.10 is not true if we drop the unique maximum assumption, as the following simple example shows.

**Example 4.11.** Suppose that the function $v$ is homogeneous of degree $k \geq 1$, is differentiable over $\Delta_n$ and satisfies mutual dependency. Let $\mathbf{x}^*, \mathbf{y}^* \in \Delta_n$ be two different global maxima of $v$ in $\Delta_n$. If $\mathbf{x}^* \neq \mathbf{y}^*$, then in particular there exist $i, j \in \Delta_n$ such that $x^*_i > x^*_j$ and $y^*_i < y^*_j$. Under $\mathbf{x}^*$, player $j$ will do much better, and similarly, under $\mathbf{y}^*$, player $i$ will do much better. In this case, our results imply that for every $i \in N$, there is a socially optimal contract that is "nearly" individually optimal for $i$.

## 4.2 Non-Differentiable Utility Functions

The results given so far in this Section 4 heavily rely on $v$ being continuously differentiable; we can, in fact, prove similar claims for functions that are not differentiable, but are uniform limits of continuously differentiable functions. Such functions are not mere pathologies but arise in some of the applications we describe later, in Section 5. The main result of this section is Theorem 4.12, the proof is omitted due to space constraints.

**Theorem 4.12.** *Suppose that $(v_j)_{j=1}^{\infty}$ is a sequence of functions that are continuously differentiable, strictly concave and homogeneous of degree $k \geq 1$; such that $v_j$ converge uniformly to $v$. Moreover, suppose that $v$ has a unique global maximum over $\Delta_n$. Given a universally optimal stationary contract $\mathbf{x}^* \in \Delta_n$ for $v$, then for any fixed $q$, and any sequence of contracts $(\chi^*_T)_{T=1}^{\infty}$ that are individually optimal for $i$ at time $T$, we have that if $\lim_{T\to\infty} \mathbf{x}_q(T)^*$ exists, then it is $\mathbf{x}^*$.*

Theorem 4.7 states that when $v$ is differentiable, strictly concave and homogeneous of degree $k \geq 1$, individual incentives coincide with social welfare in the limit. Theorem 4.12 allows us to drop the differentiability requirement, if we know that $v$ is the uniform limit of a sequence of differentiable functions that satisfy Theorem 4.7. We mention that a result analogous to Theorem 4.12 can be shown for Theorem 4.9. That is, if $v$ is the uniform limit of differentiable functions for which Theorem 4.9 can be applied, then an analogue of Theorem 4.12 can be shown.

# 5 Applications

In this section we analyze certain classes of functions to which our results apply. In first set of examples, we explore common production functions used in the economic literature, where $a_i$ are a set of positive weights whose sum is $a = \sum_{i=1}^{n} a_i$.

## 5.1 CES Production Functions

We first look at the *CES production functions* (Constant Elasticity of Substitution). Let $v_r(\mathbf{x}) = c \cdot \left(\frac{1}{n}\sum_{i=1}^{n}\left(\frac{x_i}{a_i}\right)^r\right)^{\frac{1}{r}}$, where $c$ and $a_1, \ldots, a_n$ are positive constants, and $r \neq 0$. We note that $v_r$ is homogeneous of degree 1 and differentiable for all $r$. When $r > 1$ $v_r$ is convex, when $r = 1$ it is linear, and when $r < 1$ it is concave. We can in fact show that the optimal contract (for any $r < 1$) is $x_i^* = \frac{\beta_{i,r}}{\sum_{j=1}^{n}\beta_{j,r}}$, where $\beta_{i,r} = a_i^{\frac{r}{r-1}}$; this is a unique critical point of $v_r$ in the interior of $\Delta_n$, but when $r > 1$, it is a minimum of $v_r$. Applying our results, we obtain that if $r < 1$, then any sequence of individually optimal contracts necessarily converges to the universally optimal contract described above. In other words, for all strictly concave CES production functions, individual incentives eventually align with social welfare.

It is also known that $\lim_{r\to 0} v_r(\mathbf{x})$ is the Cobb-Douglas production function (Section 5.2), and $\lim_{r\to -\infty} v_r(\mathbf{x})$ is the Leontief production function (Section 5.3).

## 5.2 Cobb-Douglas Production Functions

A *Cobb-Douglas production function* is a function of the form $v_c(\mathbf{x}) = c\prod_{i=1}^{n} x_i^{a_i}$. Note that the Cobb-Douglas production function is $a$-homogeneous: $v_c(\lambda\mathbf{x}) = c\prod_{i=1}^{n}(\lambda x_i)^{a_i} = \lambda^a v_c(\mathbf{x})$. It is well-known in the economic literature that the maximum of $v_c$ over $\Delta_n$ is unique, and equals $(\frac{a_1}{a}, \ldots, \frac{a_n}{a})$, with the shares directly proportional to the weights $a_i$. This is also example where the individual utility functions have the same form as the production function, namely $U_i$ is the same functional form with parameters $a_1, ..., a_{i-1}, a_i + 1, a_{i+1} ... a_n$.

Since $\mathbf{x}^* = (\frac{a_1}{a}, \ldots, \frac{a_n}{a})$ is a universally optimal stationary contract, and Cobb-Douglas functions are differentiable and homogeneous of degree $a$, according to Theorem 4.9, we know that for $a \geq 1$, both Theorems 4.7 and 4.9 apply.

## 5.3 Leontief Functions

Consider the function $v_\ell(\mathbf{x}) = c\min_{i\in N}\{\frac{x_i}{a_i}\}$, known as a *Leontief* production function, where $c$ and $(a_i)_{i\in N}$ are all strictly positive constants. Now $v_\ell$ is 1-homogeneous, which implies, by Lemma 3.5 and Corollary 3.3 that the global maximum of $v_\ell$ is a universally optimal contract. Since $v_\ell$ is not differentiable, Theorem 4.9 does not apply here. However, observe that $\lim_{r\to-\infty} v_r(\mathbf{x}) = v_\ell(\mathbf{x})$. If we show that $v_\ell$ has a unique maximum over $\Delta_n$, then we can apply Theorem 4.12 and show that the unique optimal contract has no regret for any player in the limit. Let $\mathbf{x}^*$ be in $\mathrm{argmax} v_\ell$; it is clear that $\frac{x_i^*}{a_i} = \frac{x_j^*}{a_j}$ for all $i, j \in N$; combining this fact with $\sum_{i=1}^{n} x_i^* = 1$, implies that $x_i^* = \frac{a_i}{a}$. Since this optimum is unique, we know that $\mathbf{x}^*$ is nearly individually optimal for all players, according to Theorem 4.12.

### 5.4 Network Flow Games

Suppose we are given a directed, weighted graph $\Gamma = \langle V, E\rangle$, with a source-terminal node pair $s, t \in V$, where the edge set $E = \{e_1, \dots, e_n\}$ and the weight of the edge $e \in E$ is a positive integer $w_e$. Then we define the maximum flow game $v_\Gamma : \mathbb{R}^n \to \mathbb{R}$ to be the maximum flow from $s$ to $t$ that can be achieved on $\Gamma$ multiplied by some constant value $c$. Here we assume that $E = \{e_1, \dots, e_n\}$. The constant $c$ can be thought of as the per unit value of the commodity that is passed through the network; in other words, the edges are players, and $v_\Gamma(\mathbf{x})$ is the maximum flow through $\Gamma$, given that $e$ has a capacity of $c_e x_e$. The edge $e$ uses the amount of money it has, $x_e$, to purchase capacity, which is multiplied by the factor $c_e$. Given $\mathbf{x} \in \mathbb{R}^n_+$, we write $\Gamma(\mathbf{x})$ to be the graph $\Gamma$ with capacities $c_e x_e$ instead of $c_e$; thus $\Gamma(\mathbf{1}^n) = \Gamma$, and we indeed assume that $\mathbf{w}(0) = \mathbf{1}^n$. This means that $v_\Gamma(\mathbf{x})$ equals the maximum flow through $\Gamma(\mathbf{x})$. This is a straightforward generalization of the classic network flow cooperative game [10].

The first observation we make is that $v_\Gamma$ is homogeneous of degree 1: changing all edge capacities by a factor of $\lambda$ results in a change of $\lambda$ to the maximum flow as well, hence $v_\Gamma$ is homogeneous of degree 1. In consequence, finding the optimal contract takes polynomial time.

We contrast our payoff division with the canonical core-stable payoff division, which pays only edges that are in the minimum cut (for details, see [8]). Paying only the edges in the minimum cut in our setting is clearly not optimal: only the edges in the minimum cut survive the first iteration, and the graph cannot pass any further flow (unless $\Gamma$ is a degenerate graph where there are no paths of length more than 1 from $s$ to $t$).

Let $\mathcal{C}(\Gamma)$ be the set of $(s, t)$ cuts of $\Gamma$. Given a cut $C \in \mathcal{C}(\Gamma)$, we can write $C$ as a vector $\mathbf{w}_C$ in $\mathbb{R}^n$, with $\mathbf{w}_C(e) = w_e$ for all $e \in C$, and $\mathbf{w}_C(e) = 0$ otherwise. We let $M_\Gamma$ be an $|\mathcal{C}(\Gamma)| \times n$ matrix whose rows are the vectors $\mathbf{w}_C$. This means that $v_\Gamma(\mathbf{x})$ can be rewritten as $v_\Gamma(\mathbf{x}) = \min_{C \in \mathcal{C}(\Gamma)} \mathbf{w}_C \cdot \mathbf{x}$. Now, given a vector in $\mathbb{R}^{|\mathcal{C}|}$, we let $f_r(\mathbf{x})$ be a CES production function, much like in Section 5.3, i.e. $f_r(\mathbf{x}) = c\left(\sum_{j=1}^{|\mathcal{C}|} \frac{1}{|\mathcal{C}|} x_i^r\right)^{\frac{1}{r}}$ (note that here we assume that $a_1, \dots, a_n = 1$). We write $v_{\Gamma,r} : \mathbb{R}^n \to \mathbb{R}$ to be $v_{\Gamma,r}(\mathbf{x}) = f_r(M_\Gamma \mathbf{x})$, and noting that $\lim_{r \to -\infty} v_{\Gamma,r}(\mathbf{x}) = v_\Gamma(\mathbf{x})$ with uniform convergence. We note that all $v_{\Gamma,r}$ are homogeneous of degree 1, differentiable in the interior of $\Delta_n$, and when $r < 1$, are strictly concave. Therefore, if $v_\Gamma$ has a unique global maximum that is in the interior of $\Delta_n$, Theorem 4.12 holds for all $v_{\Gamma,r}$, which means that we can apply the results of Section 4 to network flow games to show that stationary contracts are asymptotically individually optimal.

## 6 Strategic Behavior

In Section 4, we described the individual entities in our setting as players; however, this is a slight abuse of the term. It is usually the case that players are strategic: that is, they are allowed to choose actions and affect the outcome in order to benefit themselves. In the model described so far, players could evaluate outcomes based on the utility that they grant them, but they could not take action to affect them. It was simply assumed that a central authority chooses the socially optimal outcome, and, given that this was the choice, our analysis shows that individual entities are happy with this choice in the limit.

The reasons we have done so are twofold. First, our basic model applies to a wide variety of situations, and the details of the strategic interaction may vary between them. Defining equilibria for a particular setting can also be somewhat delicate, as players' utility at each time-step tends to infinity. Second, in some sense, our results show that under certain assumptions the strategic model is irrelevant. That is, we describe conditions that ensure that all players desire to reach the same outcome (up to an $\varepsilon$); thus, any "reasonable" setup will allow them to do so. In the remainder of this section, we examine two particular settings that illustrate this intuition. In the first, players collectively "vote" on how to divide the surplus. In the second, players have private information about their part of the production function that they can strategically submit to a mechanism.

## 6.1 The Division Game

One simple way our model could be turned into a game is to allow each player to propose a division of resources for each round and then divide the resources according to the (weighted) average of the proposals. This allows players to be greedy and try and take more resources for themselves, but our results show that sufficiently farsighted players will not choose to do so.

More formally, we assume that players "vote" on a contract in the following manner: each player $i \in N$ proposes a contract $\chi_i \in \Delta(n)^T$, and the contract chosen is a weighted average of players' choices. That is, we are given constant, non-negative weights $\alpha_1, \dots, \alpha_n$, such that $\sum_{i=1}^n \alpha_i = 1$. Given $\Omega = (\chi_1, \dots, \chi_n)$, the contract that is chosen is $\chi = \sum_{i=1}^n \alpha_i \chi_i$. Each player has a utility function $U_i(\Omega)$.

Recall that a strategy profile $\Omega$ is an *$\varepsilon$-Nash equilibrium* if for all $i \in N$, and for any contract $\bar{\chi}$, $U_i(\Omega) \geq U_i(\Omega_{-i}, \bar{\chi}) - \varepsilon$. Here $(\Omega_{-i}, \bar{\chi})$ denotes the strategy profile where player $i$ proposes the contract $\bar{\chi}$ and all other players propose the same contracts as under $\Omega$. In words, a profile of contracts is an $\varepsilon$-Nash equilibrium if no player can gain more than $\varepsilon$ by proposing a different contract.

Under this model, we can immediately apply the results in Section 4 to obtain the following theorem.

**Theorem 6.1.** *Assume that:*

*(a) The production function $v$ is differentiable, homogeneous of degree $k \geq 1$, strictly concave and its maximum over $\Delta_n$ is $\mathbf{x}^*$.*

*(b) There exists a point $\mathbf{y}^* \in \mathbb{R}^n_+$ such that the non-vanishing welfare condition holds for $\mathbf{y}^*$, and $\mathbf{x}^* > \mathbf{y}^*$.*

*(c) Players may only choose stationary contracts.*

*Then, for any $\varepsilon > 0$ there exists some $T_0 \in \mathbb{N}$ such that for all $T > T_0$, if player utility is given by $U_{i,T}$, then the strategy profile $(\mathbf{x}^*)_{i=1}^n$ is an $\varepsilon$-Nash equilibrium.*

*Proof.* Suppose that all players propose the socially optimal contract; then for any player $i \in N$, the best that he can do by changing his proposal is an individually optimal stationary contract at time $T$. According to Corollary 4.10, under the conditions stated in the theorem, for any $\varepsilon > 0$ there is some $T_0$ such that for all $T > T_0$ $U_i(\mathbf{x}^*)$ is within a factor of $\varepsilon$ from what is individually optimal for $i$. □

When players can only make a single decision about a contract at the start of negotiation, and are limited to choosing a single payoff division that will apply to all subsequent rounds, choosing what is socially optimal is an $\varepsilon$-Nash equilibrium for players who consider their long-term utility for a long enough time-horizon. Assuming that players may only propose a single payoff division that will apply to all future rounds is not completely unreasonable, if one assumes that negotiations take place only in the first round: a dynamic contract is a complex object, comprising of a sequence of vectors in $\Delta_n$, and would thus require a lot of reasoning on the side of players.

## 6.2 Truthful Contracts

We now turn to a setting with private information. For concreteness, let us assume that $v$ is a CES production function with $r < 1$ and player $i \in N$ has private information $a_i$. A natural choice of a mechanism (without money) that takes the report from each player and selects a contract is the mechanism $\mathcal{M}$ that chooses the unique optimal stationary contract (see Section 5.1). Therefore, when reporting their private information, players are essentially choosing among a set of static contracts. This immediately implies the following theorem.

**Theorem 6.2.** *For any $\varepsilon > 0$, there exists a time $T_0$ such that for all $T > T_0$, $\mathcal{M}$ is $\varepsilon$-truthful.*

*Proof.* The claim is an immediate corollary of Theorem 4.9. □

In order for Theorem 6.2 to hold, it is necessary that the homogeneity and concavity of $v$ do not depend on the parameters that are reported. Because $v$ is a CES production function with $r < 1$, the concavity of $v$ does not depend on the coefficients that the players report. Thus, if all players believe that social welfare is adequately represented by such a function and want to maximize their long-term revenue, they will truthfully report their parameters. However, in the case of Cobb-Douglas production functions (see Section 5.2), a player is only incentivized to truthfully report his coefficient if the sum of the coefficients is greater or equal to 1.

# 7 Related Work

There are several theoretical models that aim to capture a notion of collaboration among rational players, mostly from the realm of cooperative game theory and social choice. When monetary incentives are in play, the general setup is as follows: players form the coalition $N$ (or alternatively, partition into disjoint coalitions, with the goal of maximizing total revenue, see [1]), divide revenue, and the game ends. In the terms of our paper, cooperative game theory focuses on a single step of our iterative process, with no repeated interactions, and no far-sightedness on the part of the players. Other models of dynamics in collaborative sharing do exist (see [7] for a literature review, or [4] for a more recent take on dynamics in payoff allocation), but these models do not take into consideration resource allocation dynamics: in our setting, the payoff to players in previous rounds can affect production volume in future rounds. There are several papers that study dynamics of coalition formation among players (see, e.g. [11, 12]; or a literature review in [2]) however, revenue division and its effects are not studied. The main research effort in this front is computing the maximum of $v$ given player resources, and doing so in an iterative manner. Thus, no revenue sharing issues are studied.

Our main result is that the socially optimal outcome is the one that is most desirable to all players in the limit. This idea is similar to the concept of regret minimization. Regret minimization is a well-known benchmark in learning [5, 13] (see [9, Chapter 4] for an overview). While we use a similar benchmark, our analysis and optimization objectives are quite different from that of the regret minimization literature.

Our work is also similar to the problem of *portfolio selection* [3]: given a set of $n$ stocks and a sequence of changes to the stock values (given by vectors in $\mathbb{R}^n$), find an optimal investment portfolio, i.e. an investment strategy that maximizes total wealth. There is a significant body of work analyzing the performance of algorithms for portfolio selection, and its connection to regret has been previously studied (see, e.g. [6] and the citations within). One can think of our setting as one where the individual stocks have incentives and would like more money invested in them.

## 8 Conclusions and Future Work

In this paper, we explore a new model of long-term cooperation. As highlighted in Section 7, the key novel point in our work is the fact that payoffs at time $t$ affect revenue at time $t + 1$. This approach can be naturally applied to several settings, essentially any strategic setting that can be expressed as a function $v : \mathbb{R}^n \to \mathbb{R}$. This includes, for example, several models from cooperative games such as weighted matching games, market exchange games, weighted voting games, weighted graph games and others. Of course, not all these settings have the homogeneity property that we heavily relied on, but we hope that when restricted to classes of games, structural properties of the game can be utilized in order to obtain some understanding of the behavior of individuals in these settings. We expect that in settings where homogeneity does not hold, strategic behavior would be much more pronounced. As we mention in Section 6, under our assumptions, all players want the same outcome; that is, if they are not myopic, they would be (nearly) fully collaborative.

Alternative versions of our results can be also shown for the case where mutual dependency does not hold. We mainly require mutual dependency to ensure that all maxima (for players as a whole and as individuals) occur in the interior of $\Delta_n$. Weaker versions of our results hold in the case where maxima occur on the boundary of $\Delta_n$. Briefly, if there are global maxima on the boundary of $\Delta_n$, then in the limit, there are some players who experience no regret at those points. However, in this setting there may be a difference between points that minimize total regret and points that maximize utility. Studying the relation between the two would be an interesting direction for future research.

Finally, a natural extension of our model would be the introduction of some measure of uncertainty regarding the value of $v$. Adding noise to the model would be an interesting extension not only due to it being more realistic, but also due to the fact that notions such as mutual dependency may only hold in expectation; in this case, some players may be paid at a round even if they are not contributive whatsoever, since they may be useful in future iterations.

## References


[1] R.J. Aumann and J.H. Drèze. Cooperative games with coalition structures. *International Journal of Game Theory*, 3:217–237, 1974.

[2] G. Chalkiadakis, E. Elkind, and M. Wooldridge. *Computational Aspects of Cooperative Game Theory*. Morgan and Claypool, 2011.

[3] T.M. Cover. Universal portfolios. *Mathematical Finance*, 1(1):1–29, 1991.

[4] E. Elkind, D. Pasechnick, and Y. Zick. Dynamic weighted voting games. In *AAMAS'13*, pages 515–522, 2013.

[5] E. Even-dar, Y. Mansour, and U. Nadav. On the convergence of regret minimization dynamics in concave games. In *Proceedings of the 41st annual ACM symposium on Theory of computing*, STOC '09, pages 523–532, 2009.

[6] E. Gofer and Y. Mansour. Pricing exotic derivatives using regret minimization. In *Algorithmic Game Theory*, pages 266–277. Springer, 2011.

[7] E. Lehrer and M. Scarsini. On the core of dynamic cooperative games. *CoRR*, abs/1203.2832, 2012.

[8] M. Maschler, E. Solan, and S. Zamir. *Game Theory*. Cambridge Unversity Press, 2013.

[9] N. Nisan, T. Roughgarden, É. Tardos, and V.V. Vazirani, editors. *Algorithmic Game Theory*. Cambridge University Press, 2007.

[10] B. Peleg and P. Sudhölter. *Introduction to the Theory of Cooperative Games*, volume 34 of *Theory and Decision Library. Series C: Game Theory, Mathematical Programming and Operations Research*. Springer, Berlin, second edition, 2007.

[11] Tuomas Sandholm, Kate Larson, Martin Andersson, Onn Shehory, and Fernando Tohmé. Coalition structure generation with worst case guarantees. *Artificial Intelligence*, 111(1):209–238, 1999.

[12] O. Shehory and S. Kraus. Formation of overlapping coalitions for precedence-ordered task-execution among autonomous agents. In *ICMAS'96*, pages 330–337, 1996.

[13] M. Zinkevich. Online convex programming and generalized infinitesimal gradient ascent. In *Proceedings of the Twentieth International Conference on Machine Learning*, ICML '03, 2003.