Introduction to Voting and its Paradoxes

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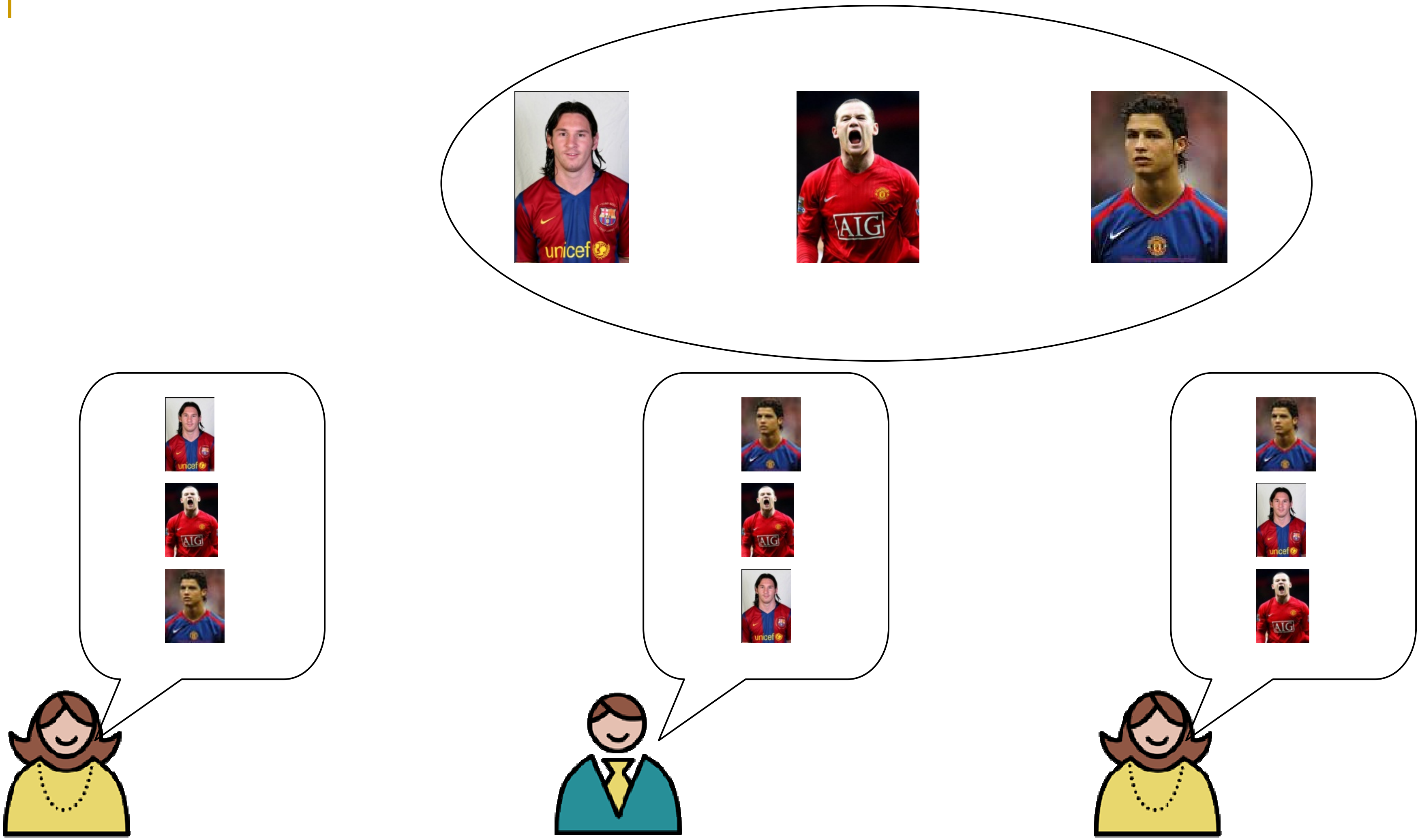
Introduction



So who is the best player?

Overview

- Introduction and Formal Framework
- Various Examples of Voting Rules
- Paradoxes in Voting Outcomes
- Condorcet Extensions
- Paradoxes and Properties of Voting Rules
- Conclusion and Literature

"Formal" Framework

- *X={a,b,c,...}* ... set of *n* alternatives/candidates
- *I* ... set of *m* individuals/voters
- Preference is a ranking of the alternatives

R_1
a
b
c

- Preference profile

R_1	R_2	R_3
a	b	c
b	c	b
c	a	a

- Social choice (or voting) rule (SCR) aggregates a preference profile into a social outcome
  - preference, set of alternatives, etc.

Introduction

- Collective decision making occurs often
  - Elections
  - Selecting committees
  - Choosing from job applicants
  - Experts choosing from a set of projects
  - Families deciding on holiday location, etc.

- There exist many different SCR
  - In what way do they differ?
    - Axiomatic approach
    - Outcome-based approach

Introduction

- The choice of the SCR is probably not much of a problem in *homogeneous* societies (groups).

- But what if the society (group) is *heterogeneous?* Especially there, a convincing social compromise seems compelling and therefore the SCR of importance.
  - So to see what differences can occur, might be of interest.

Introduction

How do we vote?

- Mostly by just marking ONE alternative.
- Does this really take into account a person's full preference?

Official Ballot
Election for the United States House of Representatives
District One

Voting Instructions
1. You only have ONE vote.
2. Place an X in the box next to the candidate for whom you wish to vote.
3. Your vote counts both for your candidate and your party.

Democratic	Republican	Reform	Green	Independent Candidate
Benjamin Pike	Fran Deutsch	Steven Wong	Tom Wartenberg	Robert Moll
Sam Rosen-Amy	Steve Grolnic	Deborah Gorlin	Juan Hernandez	
Megan Gentzler	Wendy Berg	Brad Crenshaw	Beata Panagopoules	
Ben Foster	Gerald Epstein	Daniel Czitrom	Alice Morey	
Colin Volz	Sarah McClurg	Meryl Fingrutd	Sarah Pringle	

a	a
b	e
c	d
d	c
e	b

Does not take into account quite a lot of information!

Introduction

Results - Great Britain and Northern Ireland

Party	No. of votes	Vote %	MEPs
Conservative	3,578,217	35.77%	36
Labour	2,803,821	28.03%	29
Lib Dem	1,266,549	12.66%	10
UKIP	696,057	6.96%	3
Green	625,378	6.25%	2
SNP	268,528	2.68%	2
Plaid Cymru	185,235	1.85%	2
PECP	138,097	1.38%	0
BNP	102,647	1.13%	0

That's the information we usually have after the election to determine the social outcome (seats in parliaments, committees, etc.).

Does the social outcome change a lot if we use more information or use the available information differently?

Historical Aspects

Voting theory as known today started during the French revolution

- Condorcet
- Borda

Simple Majority Rule (SMR)

- an alternative *a* is socially preferred to another alternative *b* if a majority prefers *a* to *b*

R_1	R_2	R_3
a	c	b
b	a	c
c	b	a

What is the social outcome for SMR with this profile?

$$a \succ b \succ c \succ a$$

Condorcet cycle

Example

Given a preference profile, does it make a difference what SCR we use?

$$X = \{a, b, c\},\ |I| = 3$$

R_1	R_2	R_3
a	a	a
b	b	b
c	c	c

unanimous profile

What is the social ranking/choice?

$$\begin{array}{c} a \\ b \\ c \end{array}$$

social ranking

Should not every reasonable rule provide that outcome?

(unanimity property)

Example

- What results do actual voting rules give?
  - **Plurality Rule**
    - vote for top-choice only and rank alternatives according to total number of votes
  - **Antiplurality Rule**
    - vote for all but bottom-choice

$$\begin{array}{ccc} R_1 & R_2 & R_3 \\ a & a & a \\ b & b & b \\ c & c & c \end{array}$$

PR	
a	$3p$
bc	$0p$

APR	
ab	$3p$
c	$0p$

> So problems do occur with 3 alternatives, 3 individuals and unanimous profiles already!

Voting Rules

R_1	R_2	R_3
a	a	a
b	b	b
c	c	c

- **Simple Majority Rule**

SMR
a
b
c

- **Borda Rule**

  assign *n-1* points to a top ranked, *n-2* points to second ranked, down to 0 points for a bottom ranked alternative.

  Rank alternatives according to total number of points.

$Borda$	
a	$6p$
b	$3p$
c	$0p$

Plurality Rule

PR has an interesting feature!

2	1	2	1
a	a	b	c
c	b	c	b
b	c	a	a

Plurality outcome is $a \succ b \succ c$

What if we all realized that we ranked from bottom to top. Is the PR outcome just the reversal?

NO! It remains exactly the same!

Example (Saari, 1995)

X = {Beer, Milk, Wine}, $|I|$ = 15

6	5	4
M	B	W
W	W	B
B	M	M

Plurality Rule: $M \succ B \succ W$

Antiplurality Rule: $W \succ B \succ M$

Majority Rule: $W \succ B \succ M$

Borda Rule: $W \succ B \succ M$

APR, MR and BR give the exact opposite of the PR outcome for the same profile!

… and the voters better not find out how the others voted when they use PR.

Example

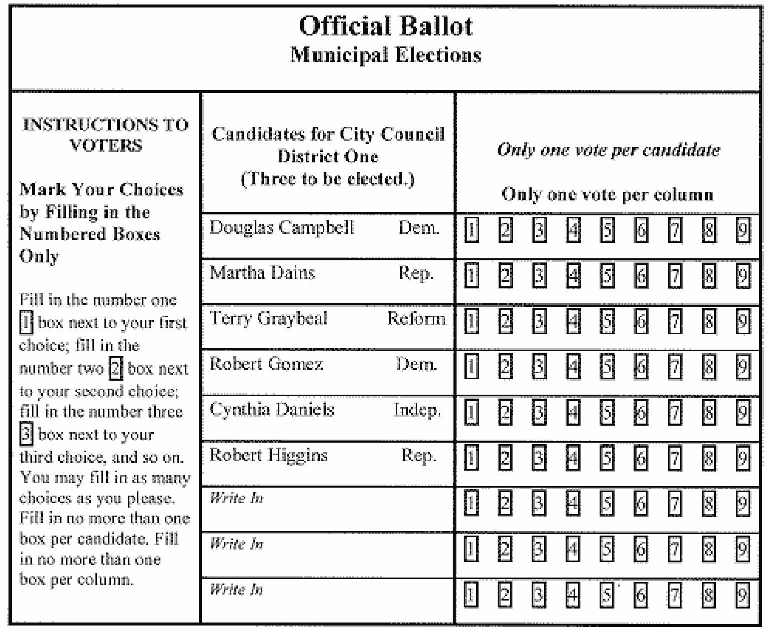
6	5	4
M	B	W
W	W	B
B	M	M

Plurality Runoff

- if no alternative has an absolute majority let the two alternatives with most votes run against each other
- first round: $M \succ B \succ W$ but no absolute majority, hence W is eliminated
- second round: $B \succ M$
- *Plurality runoff ranking:* $B \succ M \succ W$

  → different to plurality rule and Borda, etc.

Example



Official Ballot
Municipal Elections

INSTRUCTIONS TO VOTERS	Candidates for City Council District One (Three to be elected.)		Only one vote per candidate / Only one vote per column
Mark Your Choices by Filling in the Numbered Boxes Only	Douglas Campbell	Dem.	1 2 3 4 5 6 7 8 9
Fill in the number one 1 box next to your first choice; fill in the number two 2 box next to your second choice; fill in the number three 3 box next to your third choice, and so on.	Martha Dains	Rep.	1 2 3 4 5 6 7 8 9
You may fill in as many choices as you please.	Terry Graybeal	Reform	1 2 3 4 5 6 7 8 9
Fill in no more than one box per candidate.	Robert Gomez	Dem.	1 2 3 4 5 6 7 8 9
Fill in no more than one box per column.	Cynthia Daniels	Indep.	1 2 3 4 5 6 7 8 9
	Robert Higgins	Rep.	1 2 3 4 5 6 7 8 9
	Write In		1 2 3 4 5 6 7 8 9
	Write In		1 2 3 4 5 6 7 8 9
	Write In		1 2 3 4 5 6 7 8 9

6	5	4
M	B	W
W	W	B
B	M	M

Single transferable vote

- define a quota that has to be reached (e.g. 50%)
- first round: no alternative reaches quota with first rank votes
  - eliminate alternative with lowest number of first ranks
- second round: *B* reaches the quota as it gets 9 votes
- *STV ranking:* $B \succ M \succ W$
- also known as alternative vote or Hare's system
  - used e.g. in Australia, Ireland, etc.
  - however, in different versions

Another Example

X = {Beer, Milk, Wine}, $|\mathcal{I}|$ = 15

7	7	1
M	W	B
W	B	M
B	M	W

Majority Rule: $M \succ W$ (8:7); $W \succ B$ (14:1); $B \succ M$ (8:7)

→ Majority cycle!! There is no ***Condorcet winner***.

Alternative: sequential SMR

- vote on {M,W} first
- winner against B

What is the social preference?

$MR - seq$
B
M
W

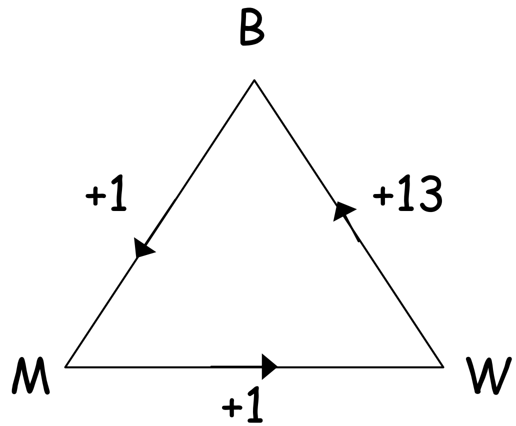
Starting with different pair leads to different outcome!

- controlling the agenda might be important

More Voting Rules

There exist many rules that break cycles
→ Condorcet extensions

7	7	1
M	W	B
W	B	M
B	M	W



Copeland rule

- rank the alternatives according to the difference between number of alternatives they win against (by a majority) and the number of alternatives they lose against.
- also of relevance in tournaments

More Voting Rules

7	7	1
M	W	B
W	B	M
B	M	W

Nanson rule

- Borda elimination procedure
- first round: B has lowest Borda score – eliminate
- second round: $M \succ W$
- *Nanson ranking:* $M \succ W \succ B$
- *Different to Borda ranking:* $W \succ M \succ B$
- why is this a Condorcet extension

Borda – Condorcet

There is a close relationship between majority margins and Borda score.

1	1	1
a	b	c
b	a	a
c	c	b

Majority margins: a $\succ$ b (2:1); b $\succ$ c (2:1); a $\succ$ c (2:1)

Borda scores: a (4); b (3); c (2)

As the sum of the majority margins equals the sum of the Borda scores, the average Borda score is

$$\frac{m\frac{(n-1)n}{2}}{n}$$

To be the Condorcet winner an alternative needs to have a majority over all (n-1) other alternatives. I.e. its score needs to be larger than

$$(n-1)\frac{m}{2}$$

which is more than the average and hence it cannot be ranked last.

Example Borda – Condorcet

Consider the following preference profile:

4	4	3	3	1
a	a	b	c	b
b	c	c	b	c
c	b	d	d	d
d	d	a	a	a

Using majority rule we get *a* as the Condorcet winner.

The Borda scores of the alternatives are as follows:

$Borda$	$scores$
a	24
b	30
c	29
d	7

Hence, the Condorcet winner is ranked next to last by the Borda rule.

Many other rules

Coombs rule

- similar to STV
  - eliminates alternative which is least preferred by the largest group of voters, i.e. with largest number of bottom ranks
  - does this until quota is reached

Maximin Rule

- rank the alternatives according to the minimal support they receive in pairwise comparisons, the higher the better.

Kemeny Rule

- choose the ranking which is closest to the individual rankings based on the total number of pairwise switches.

Others:

- Young
- Dodgson
- Black
- etc.

Example

4	5	5	1
a	c	e	e
b	d	a	b
c	b	d	c
d	e	b	d
e	a	c	a

	a	b	c	d	e
a		3	3	3	-7
b	-3		5	-5	3
c	-3	-5		5	3
d	-3	5	-5		3
e	7	-3	-3	-3	

- Coombs ranking is b ~ c ~ d $\succ$ e $\succ$ a
- Maximin ranking is e $\succ$ b ~ c ~ d $\succ$ a
- Kemeny ranking is a $\succ$ b $\succ$ c $\succ$ d $\succ$ e

Example

What if we allow to vote for a fixed number of candidates?

- vote for k candidates
- vote for 1
- vote for n - 1

2	2	2	3
a	a	c	d
b	d	b	b
c	c	d	c
d	b	a	a

- vote for 1
  - a
- vote for 2
  - b
- vote for 3
  - c
- Borda
  - d

Approval Voting

Another well known voting rule (see Brams and Fishburn) is approval voting (AV). Every voter votes for a subset of the set of alternatives, each alternative in the set getting one point. The alternatives are ranked according to the total number of votes they get.

→ “more” information needed than just preference rankings.

2	2	1	1	1	1
a	c	b	b	c	b
b	a	c	c	b	a
c	b	a	a	a	c

AV-outcome:

$a \succ b \succ a$

Actually, any outcome is possible with AV and certain approval sets given the above profile.

In contrast, the unique Borda ranking is $b \succ c \succ a$

Preliminary conclusions

- Same preference profile may lead to different outcomes depending on what voting rule used
  - differences based on outcomes
- How can we determine which voting rule we should use?
  - differences based on properties of voting rules
  - two properties whose violation give rise to interesting paradoxes are
    - monotonicity
      - additional support for a candidate should not be harmful for it
    - consistency
      - if the electorate is partitioned into several groups and an alternative is among the winners in all groups, then it should also be among the winners if the voting rule is applied on the whole electorate.

Paradoxes

Additional support paradox: is a violation of the *monotonicity property*, i.e. if "x" wins under profile u, then "x" should also win under any profile u' in which every voter ranks "x" at least as high as in profile u.

34	35	31
a	b	c
c	c	b
b	a	a

Using *plurality runoff*, "b" wins.

What if 4 of the 34 voters state the preference bac instead, increase "b"s support?

30	4	35	31
a	b	b	c
c	a	c	b
b	c	a	a

Now "c" wins, although "b" has received additional support.

Non-monotonicity is a feature of many voting rules that work sequentially, Nanson, STV, Coombs.

Paradoxes

No-show paradox: part of the voters may be better off by not voting than by voting according to their preferences.

- In a similar spirit as before as there is a change in voters' behavior.

26	47	2	25
a	b	b	c
b	c	c	a
c	a	a	b

Using plurality runoff, "a" wins.

Had the 47 voters not voted, the outcome would have been "c" and hence preferred by the abstaining voters.

Moulin (1988): If |X|>3, all procedures that choose the Condorcet winner – if one exists – are vulnerable to the no-show paradox.

Paradoxes

Violation of consistency by majoritarian rules

- Let |X|=3 and |I|=75 partitioned into two groups

17	8	5
a	b	c
b	c	a
c	a	b

a is Condorcet winner

14	16	15
a	b	c
c	a	b
b	c	a

Condorcet cycle

Looking at the whole electorate, *b* is the Condorcet winner!

- this is a violation of consistency for all Condorcet extensions that consider *a,b,c* indifferent in the second group
- e.g. Copeland rule
- but also for maximin rule, Plurality runoff, Nanson, etc.

Various other paradoxes

Anscombe paradox: is a compound majority paradox, i.e. it deals with the way in which issues are voted upon.

Example: 5 voters, 3 issues, binary choices (Y,N)

voter	issue 1	issue 2	issue 3
voter 1	N	Y	Y
voter 2	Y	N	Y
voter 3	Y	Y	N
voter 4	N	N	N
voter 5	N	N	N
outcome	N	N	N

A majority of the voters can be on the loosing side on a majority of the issues

Various other paradoxes

Ostrogorski's paradox: is also a compound majority paradox

Example: 5 voters, 3 issues, binary choices (Y,N)

voter	issue 1	issue 2	issue 3	party supported
voter 1	N	Y	Y	Y
voter 2	Y	N	Y	Y
voter 3	Y	Y	N	Y
voter 4	N	N	N	N
voter 5	N	N	N	N

Shows that a party (Y) may win a two party contest, but still the loser (N) might share the views of a majority of the voters on every single issue.

Similar structure of problems comes up in the theory of **judgment aggregation**!

Conclusions

- There exist many different reasonable voting rules.
- Almost for any pair of social choice rules there exist preference profiles for which those rules lead to different outcomes.
- Comparison of voting rules via satisfied or violated properties.
- Paradoxes related to monotonicity and consistency aspects.

But in general it should be clear that a voting outcome is not so much depending on the individuals preferences but probably more so on the voting rule chosen!

Literature

Some interesting literature on this topic:

- Nurmi, H. (1999): Voting Paradoxes and How to Deal with Them. Springer, Berlin.
- Nurmi, H. (2002): Voting Procedures under Uncertainty. Springer, Berlin.
- Riker, W.H. (1982): Liberalism Against Populism. W.H. Freeman and Company.
- Saari, D.G. (1995): Basic Geometry of Voting. Springer, Berlin.