Automated Theorem Proving for Impossibility Theorems Regarding Ranking Sets of Objects

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Two Goals

1. Formalize and automatically verify / prove the Kannai-Peleg Theorem ☑
2. Generalize and extend the developed framework for an automated and exhaustive theorem search for *Ranking Sets of Objects* (☑)

Outline

1. Ranking Sets of Objects
2. The Proof Technique
3. Application to the Kannai-Peleg Theorem
4. Generalization for an Automated Theorem Search
5. Questions and Discussion

P. Tang and F. Lin.
Computer-aided proofs of arrow's and other impossibility theorems.
Artificial Intelligence, 173(11):1041–1053, 2009.

Y. Kannai and B. Peleg.
A note on the extension of an order on a set to the power set.
Journal of Economic Theory, 32(1):172–175, 1984.

Setting and Notation for Ranking Sets of Objects

Question / concern: Given a an ordering of objects, is there a "compatible" ranking of all non-empty sets of objects?

Notation

- A finite set of *objects* (or *elements*) X (with cardinality $|X| = n$)
- A linear order $\dot{\geq}$ on X
  - reflexive, complete, transitive, antisymmetric
  - e.g., $x_1 \dot{>} x_2 \dot{>} x_3 \dot{>} \ldots \dot{>} x_n$
- The set $\mathcal{X}$ of all non-empty subsets of X (i.e., $\mathcal{X} := 2^X \setminus \{\emptyset\}$)
- A weak order $\succeq$ on $\mathcal{X}$
  - reflexive, complete, transitive
  - e.g., $A \succ B \sim C \succ D \ldots$

Example (A first easy "compatibility" requirement)

A weak order $\succeq$ on $\mathcal{X}$ satisfies the *principle of extension* if the following axiom holds:

$$(\texttt{EXT}) \quad x \dot{>} y \Rightarrow \{x\} \succ \{y\} \text{ for all } x, y \in X.$$

Some (Reasonable) Principles for a "Compatible" Weak Ordering

Causing an Impossibility: The Kannai-Peleg Theorem

Definition (The *Gärdenfors Principle* (also called *dominance*))

$$(\texttt{GF1}) \qquad ((\forall a \in A) x \mathrel{\dot{>}} a) \Rightarrow A \cup \{x\} \succ A \qquad \text{for all } x \in X \text{ and } A \in \mathcal{X},$$

$$(\texttt{GF2}) \qquad ((\forall a \in A) x \mathrel{\dot{<}} a) \Rightarrow A \cup \{x\} \prec A \qquad \text{for all } x \in X \text{ and } A \in \mathcal{X}.$$

Adding an element that is strictly better/worse ($\dot{>}$) than all the elements in a given set to that set produces a *strictly* better/worse set,

Definition (The principle of *independence*)

$$(\texttt{IND}) \qquad A \succ B \Rightarrow A \cup \{x\} \succeq B \cup \{x\} \qquad \text{for all } A, B \in \mathcal{X} \text{ and } x \in X \setminus (A \cup B).$$

If a set is strictly better than another one, then adding the same alternative to two sets does not reverse this strict order.

Theorem (Kannai, Peleg, 1984)

Let X be a linearly ordered set with $|X| \geq 6$. Then there exists no weak order $\succeq$ on $\mathcal{X}$ satisfying the Gärdenfors Principle (GF) *and independence* (IND).

Lin and Tang Use Induction to Prove Impossibility Theorems

Main idea?

1. Reduce to small base case using an inductive proof (manually)
2. Verify base case on a computer (SAT solver)

Successful?

- Four famous impossibility results (Arrow, Muller-Satterthwaite, Gibbard-Satterthwaite, Sen) verified by Lin and Tang
- Extension to *Ranking of Sets of Objects* and, specifically, the Kannai-Peleg Theorem

Inductive Approach also Successful for Kannai-Peleg Theorem

Lemma

If X is a linearly ordered set with $n+1$ elements ($n \in \mathbb{N}$) and there is a corresponding weak order $\succeq$ on $\mathcal{X}$ that satisfies the Gärdenfors Principle ($\mathtt{GF}$) and independence ($\mathtt{IND}$), then we can find another linearly ordered set Y with n elements only, as well as a corresponding weak order on $\mathcal{Y} := 2^Y \setminus \{\emptyset\}$ satisfying the same axioms ($\mathtt{GF}$) and ($\mathtt{IND}$).

Reading this contrapositively yields:

Corollary

If, for any linearly ordered set Y with n elements, there exists no weak order on $\mathcal{Y} = 2^Y \setminus \{\emptyset\}$ satisfying the Gärdenfors Principle ($\mathtt{GF}$) and independence ($\mathtt{IND}$), then also for any linearly ordered set X with $|X| = n+1$ there is no weak order on $\mathcal{X} = 2^X \setminus \{\emptyset\}$ that satisfies these axioms.

$\Longrightarrow$ Reduces the theorem to the base case with $n = 6$ elements, which is then checked on a computer.

- Straightforward check of all possible orderings would do
- But there are approximately $1.5254 \cdot 10^{97}$ such orderings

$\Longrightarrow$ Need some clever way of checking the base case

$\Longrightarrow$ Lin's and Tang's idea: propositional logic & SAT solver

SAT Solver zChaff Used in Our Implementation

- A SAT ($\cong$ satisfiability) solver is a program, which can check whether a formula φ in propositional logic has a satisfying assignment
- We used zChaff[1] which does this job for formulas in conjunctive normal form (CNF)
  - A propositional formula is in CNF if it is a conjunction of clauses, where a clause is a disjunction of literals (variables or their negations)
  - For instance $(p_1 \vee \neg p_2 \vee p_3) \wedge (\neg p_1 \vee p_3) \wedge (\neg p_2 \vee p_3 \vee \neg p_4)$ is in CNF
- $\mathbf{NP}$-complete problem, hence no nice upper bound on running time; but evolved and widely used heuristic algorithm

[1] SAT Research Group, Princeton University

Formalization of the Base Case (1/2)

Three steps

1. Identify underlying axioms
2. Formulate them in propositional logic (and transform the formulas to CNF)
3. Let SAT solver do the work

Lemma (base case)

Let X be a linearly ordered set with $|X| = 6$. Then there exists no weak order $\succeq$ on $\mathcal{X}$ satisfying the Gärdenfors Principle $(\mathtt{GF})$ *and independence* $(\mathtt{IND})$.

$\Longrightarrow$ Axioms: $(\mathtt{GF1})$, $(\mathtt{GF2})$, $(\mathtt{IND})$, $(\mathtt{LIN})$, $(\mathtt{WEAK})$

Problem: Axioms intuitively formulated in second-order logic $(\forall A \in \mathcal{X} \; \ldots)$

Solution: Make use of finiteness of instances

Formalization of the Base Case (2/2)

Lemma (base case)

Let X be a linearly ordered set with $|X| = 6$. Then there exists no weak order $\succeq$ on $\mathcal{X}$ satisfying the Gärdenfors Principle (GF) *and independence* (IND)*.*

- Propositional variables $l_{x,y}$ for all pairs $(x, y) \in X^2$ with intended meaning "x is ranked at least as high as y by the linear order $\dot{\geq}$" (or short: $x \mathrel{\dot{\geq}} y$)
- Propositional variables $w_{A,B}$ for all pairs $(A, B) \in \mathcal{X}^2$ with intended meaning "A is ranked at least as high as B by the weak order $\succeq$" (or short: $A \succeq B$)

Axiom of independence as example for conversion:

$$
\begin{array}{lcl}
(\texttt{IND}) & (\forall A, B \in \mathcal{X})(\forall x \in X \setminus (A \cup B)) & [A \succ B \rightarrow A \cup \{x\} \succeq B \cup \{x\}] \\
\equiv & \bigwedge_{A,B\in\mathcal{X}} \bigwedge_{\substack{x\in X \\ x\notin(A\cup B)}} & \left[(w_{A,B} \wedge \neg w_{B,A}) \rightarrow w_{A\cup\{x\},B\cup\{x\}}\right] \\
\equiv & \bigwedge_{A,B\in\mathcal{X}} \bigwedge_{\substack{x\in X \\ x\notin(A\cup B)}} & \left[\neg w_{A,B} \vee w_{B,A} \vee w_{A\cup\{x\},B\cup\{x\}}\right]
\end{array}
$$

Computer-aided instantiation of all axioms yields single, long formula (total: 4,005 variables, 252,681 clauses)

- But SAT solver returns result in about 5 seconds!
- Finishes the proof of the Kannai-Peleg Theorem

Automated and Exhaustive Theorem Search

Possible Because of General Inductive Step

Conjecture (General inductive step)

Formulas (or: axioms) of a certain logical form are preserved in substructures (with respect to *Ranking Sets of Objects*)

$\Longrightarrow$ Advantage: only base cases to check (can be done fast)

Results so far:

- 21 Axioms from literature, checked all subsets of axioms
  - Up to domain size 8: limit of SAT solver (2GB memory)
  - Approximately 16 million instances
- Found 173 (minimal) impossibilities (in about 6 hours running time)
  - Some trivial (e.g., *strict* or *extended* independence instead of independence)
  - Some new (e.g., correction of possibility & sizes 5, 7)
  - Reproved a few by hand (knowing what to do makes it easy)
- Conjectures about general possibilities / characterizations
  - Possibility for a large domain hints at general possibility
  - "Compatible" weak order can be extracted from satisfying assignment (output from SAT solver)

Conclusion and Discussion

- Mathematical framework for *Ranking Sets of Objects*
- General proof idea of using induction and computer-aided base case verification (developed by LIN and TANG)
- Formalization of Kannai-Peleg Theorem (about nonexistence of certain compatible orderings)
  - In propositional logic thanks to finiteness and computer-aided instantiation technique
  - Satisfiability checking using zChaff (fast: $\sim$ 5sec)
  - Automated proving of first-order formalization not successful so far

Framework can be used for:

- Formalization and automatic verification of known impossibility results
- Discovery of new (or variants of known) impossibility results
  - Started by LIN and TANG in their paper: relaxed some of Arrow's conditions
  - Exhaustive theorem search for *Ranking Sets of Objects*
  - Other axioms / areas?