Consensus Formation via Preference Updating

COST-ADT Doctoral School on Computational Social Choice in Estoril

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Conformism and achieving Consensus

- What happens in a conformist society?

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- What happens in a dynamic setting of aggregation where people compromise (or conform) to achieve consensus?

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- What happens in a conformist society?
- What happens in a dynamic setting of aggregation where people compromise (or conform) to achieve consensus?
- A society which changes their opinions towards the representative agent (i.e., towards the outcome of the elections).

Conformism and achieving Consensus

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Conformism and achieving Consensus

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- Were the elections conducted again after individuals get "closer" to the initial outcome, would the consensus still be the same representative agent?

Notation

- N is set of individuals.
- A is the set of alternatives.
- $\mathcal{L}(A)$ is the set of all possible linear orders over A.
- $\mathcal{L}(A)^N$ is the set of all possible profiles.
- $p \in \mathcal{L}(A)^N$ is a generic profile of linear orders of agents in N.
- A social welfare function/correspondence $\alpha : \mathcal{L}(A)^N \to 2^{\mathcal{L}(A)}$ assigns a nonempty set of linear orderings to each profile $p \in \mathcal{L}(A)^N$.

Kemeny Distance

- Given $R, R' \in \mathcal{L}(A)$, $\delta(R, R') = \frac{|R \backslash R'| \cup |R' \backslash R|}{2}$ is the distance between R and R'.

Kemeny Distance and Updating

- Assume $R = \begin{matrix} a \\ b \\ c \end{matrix}$,

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- So $\delta(R, R') = 1$.

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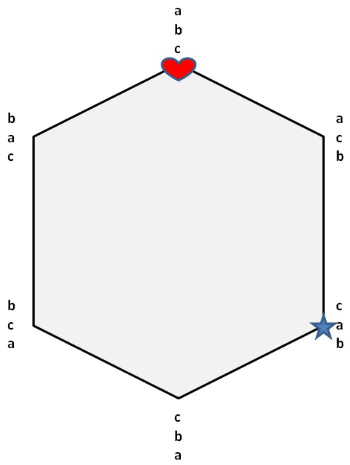
- Make one swap of adjacent alternatives a and b,
- $R' = \begin{matrix} b \\ a \\ c \end{matrix}$,
- So $\delta(R, R') = 1$.

- The maximum distance between rankings in $\mathcal{L}(A)$ is $\left(\frac{|A|.|A-1|}{2}\right)$

  (i.e., between $\begin{matrix} a \\ b \\ c \end{matrix}$ and $\begin{matrix} c \\ b \\ a \end{matrix}$ the distance is 3).

Kemeny Distance and the Lattice

with three alternatives



Kemeny Distance and the Lattice

with four alternatives

R_1	R_2	R_3	R_4	R_5	R_6	R_7	R_8	R_9	R_{10}	R_{11}	R_{12}
a	a	b	a	a	b	b	c	a	a	d	b
b	b	a	c	d	a	c	a	c	d	a	d
c	d	c	b	b	d	a	b	d	c	b	a
d	c	d	d	c	c	d	d	b	b	c	c

R_{13}	R_{14}	R_{15}	R_{16}	R_{17}	R_{18}	R_{19}	R_{20}	R_{21}	R_{22}	R_{23}	R_{24}
b	c	c	d	d	b	c	c	d	d	c	d
c	b	a	a	b	d	b	d	c	b	d	c
d	a	d	c	a	c	d	a	a	c	b	b
a	d	b	b	c	a	a	b	b	a	a	a

Kemeny Distance and the Lattice

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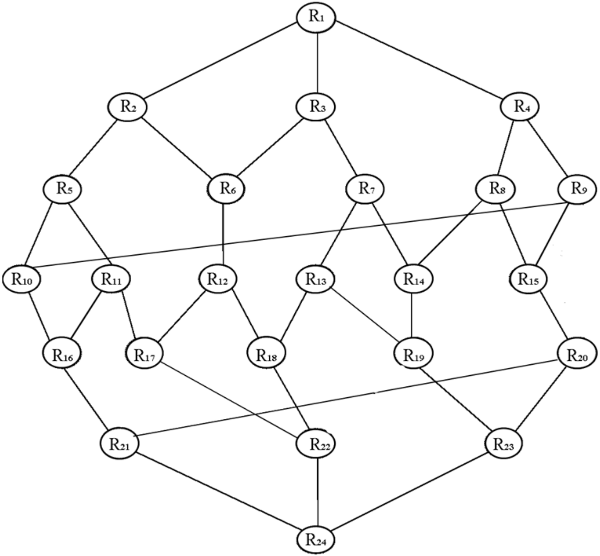
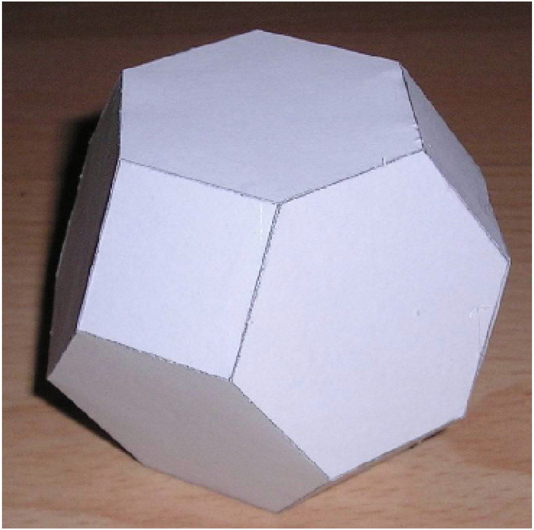


Kemeny Distance and the Lattice

with four alternatives (Truncated Octahedron)



Types of Updating

- Extreme Updating

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- General Updating

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$$p = \begin{bmatrix} a & c & b \\ b & a & c \\ c & b & a \end{bmatrix}$$ and

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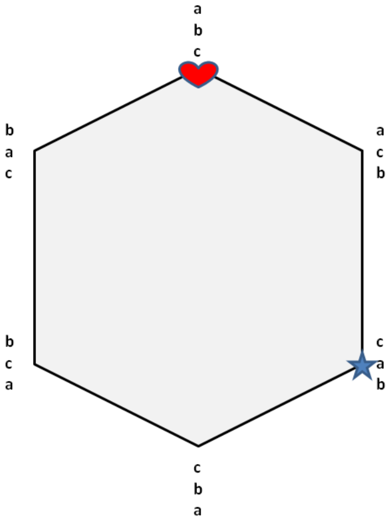
identical to the outcome.

- Then the updated profile
is $q = \begin{bmatrix} a & \mathbf{a} & b \\ b & \mathbf{b} & c \\ c & \mathbf{c} & a \end{bmatrix}$.

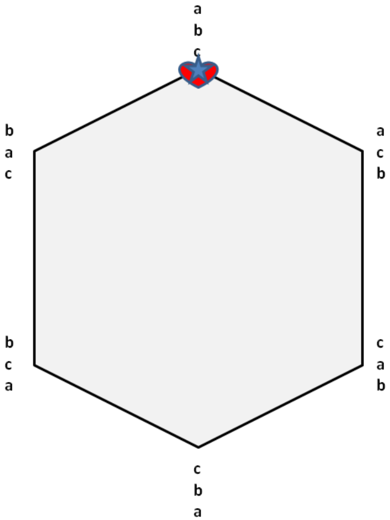
Illustrations

Extreme Updating



Illustrations

Extreme Updating



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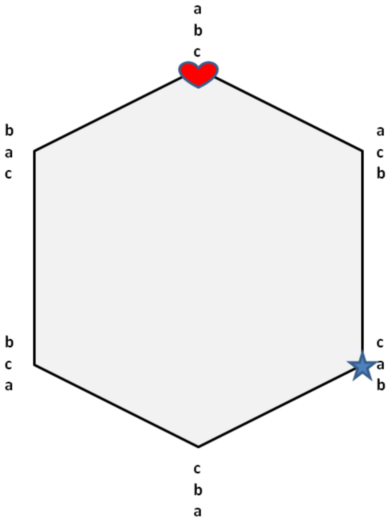
Illustrations
Short-path Updating



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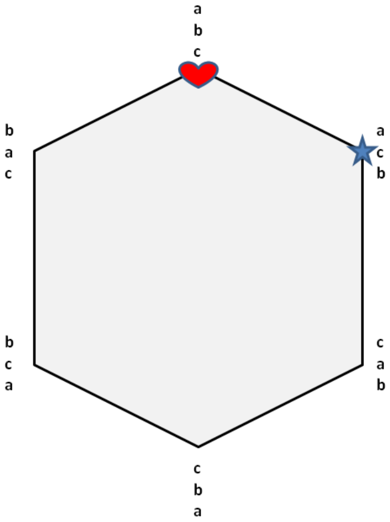

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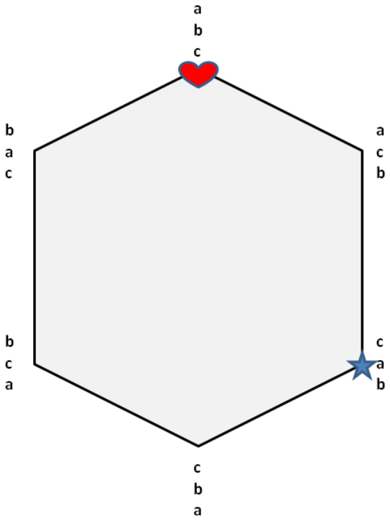
Illustrations

General Updating



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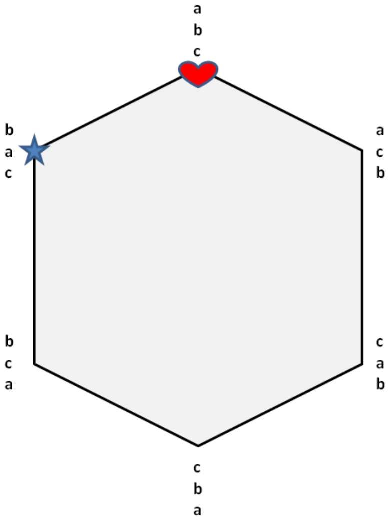

Types of Updating

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- Note that short-path updating is a special case of general updating.

Kemeny-Young Method and Updating

- Given any $p \in \mathcal{L}(A)^N$, R is a *Kemeny ranking* if and only if for all $R' \in \mathcal{L}(A)$, $\sum_{i \in N} \delta(p(i), R) \leq \sum_{i \in N} \delta(p(i), R')$.

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- The Kemeny Young method chooses all Kemeny rankings of a profile.
- The method chooses the rankings, whose sum of distances from each agent is minimum.
- The top alternative in a Kemeny ranking is called *Kemeny winner* and the bottom alternative is called *Kemeny loser.*

Kemeny-Young Method and Updating

- What happens when people's opinion gets even closer to the outcome?

Kemeny-Young Method and Updating

- What happens when people's opinion gets even closer to the outcome?
- Is the initial outcome still elected as a Kemeny ranking?

Kemeny-Young Method and Updating

- Consider the example below where $|N| = 7$ and $A = \{a_1, a_2, a_3, a_4\}$. Let the profile p be as follows:

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a_1	a_1	a_2	a_3	a_4	a_4		a_1
a_2	a_3	a_4	a_4	a_2	a_3	$\rightarrow$	a_2
a_3	a_2	a_3	a_2	a_1	a_1		a_3
a_4	a_4	a_1	a_1	a_3	a_2		a_4

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a_1	a_1	a_2	a_3	a_4	a_4		a_1
a_2	a_3	a_4	a_4	a_2	a_3	$\rightarrow$	a_2
a_3	a_2	a_3	a_2	a_1	a_1		a_3
a_4	a_4	a_1	a_1	a_3	a_2		a_4

- Agent who has ranking R_4, updates and we have

Kemeny-Young Method and Updating

$v(R'_1) = 2$	$v(R'_2) = 1$	$v(R'_3) = 1$	$v(R'_4) = 1$	$v(R'_5) = 1$	$v(R'_6) = 1$		R'_k
a_1	a_1	a_2	a_1	a_4	a_4		a_1
a_2	a_3	a_4	a_2	a_2	a_3	$\rightarrow$	a_2
a_3	a_2	a_3	a_4	a_1	a_1		a_4
a_4	a_4	a_1	a_3	a_3	a_2		a_3

Kemeny-Young Method and Updating

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a_2	a_3	a_4	a_2	a_2	a_3	$\rightarrow$	a_2
a_3	a_2	a_3	a_4	a_1	a_1		a_4
a_4	a_4	a_1	a_3	a_3	a_2		a_3
- Agents who have ranking R'_1, update and we have

Kemeny-Young Method and Updating

$v(R''_1) = 2$	$v(R''_2) = 1$	$v(R''_3) = 1$	$v(R''_4) = 1$	$v(R''_5) = 1$	$v(R''_6) = 1$		R''_k
a_1	a_1	a_2	a_1	a_4	a_4		a_1
a_4	a_2	a_4	a_2	a_2	a_3	$\rightarrow$	a_4
a_2	a_3	a_3	a_4	a_1	a_1		a_2
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Kemeny-Young Method and Updating

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a_1	a_1	a_2	a_1	a_4	a_4		a_1
a_4	a_2	a_4	a_2	a_2	a_3	$\rightarrow$	a_4
a_2	a_3	a_3	a_4	a_1	a_1		a_2
a_3	a_4	a_1	a_3	a_3	a_2		a_3

- Agent who has ranking R''_5, updates and we have

Kemeny-Young Method and Updating

$v(\tilde{R}_1) = 2$	$v(\tilde{R}_2) = 1$	$v(\tilde{R}_3) = 1$	$v(\tilde{R}_4) = 1$	$v(\tilde{R}_5) = 1$	$v(\tilde{R}_6) = 1$		$\tilde{R}_k$
a_1	a_1	a_2	a_1	a_4	a_4		a_4
a_4	a_2	a_4	a_2	a_1	a_3	$\rightarrow$	a_1
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a_4	a_2	a_4	a_2	a_1	a_3	$\longrightarrow$	a_1
a_2	a_3	a_3	a_4	a_2	a_1		a_2
a_3	a_4	a_1	a_3	a_3	a_2		a_3
- Note that initial Kemeny-loser in profile p is now the Kemeny winner.

Update proofness

- On the class of general updating, the Kemeny-Young method fails to preserve the outcome.

Update proofness

- On the class of general updating, the Kemeny-Young method fails to preserve the outcome.
- We analyse which rules can preserve the outcome, under which type of updating.

Update proofness (A new monotonicity concept)

- **Extreme-update proofness:** A rule φ is *extreme update proof* if for all R in $\varphi(p)$ and all preference profiles q we have that $R \in \varphi(q)$ whenever

$$p(i) = q(i) \text{ or } q(i) = R \text{ for all } i \text{ in } N.$$

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- **Short-path update proofness:** A rule φ is *short path update proof* if for all R in $\varphi(p)$ and all preference profiles q we have that $R \in \varphi(q)$ whenever

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Update proofness (A new monotonicity concept)

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$$p(i) \cap R \subseteq q(i) \subseteq p(i) \cup R \text{ for all } i \text{ in } N.$$
- **General update proofness**: A rule φ is *general update proof* if for all R in $\varphi(p)$ and all preference profiles q we have that $R \in \varphi(q)$ whenever
$$\delta\left(q(i), R\right) \leq \delta\left(p(i), R\right) \text{ for all } i \text{ in } N.$$

Scoring Rules and Extreme Updating

Lemma

For any number of agents and any number of alternatives, Scoring rules are not extreme-update proof.

- Hence, scoring rules are also not *short-path update proof.*

Pairwise Methods and Extreme Updating

- Pairwise methods

Pairwise Methods and Extreme Updating

- Pairwise methods
- Convex images property

Pairwise Methods and Extreme Updating

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Pairwise Methods and Extreme Updating

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Pairwise Methods and Extreme Updating

Lemma

Among Pairwise Condorcet methods that satisfy neutrality and convex images property, no extreme-update proof rule exists.

Pairwise Methods and Extreme Updating

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Among Pairwise Condorcet methods that satisfy neutrality and convex images property, no extreme-update proof rule exists.

- The proof of the lemma covers all cases except for the cases $n = 2$ and $n = 4$.

Kemeny-Young Method and Short-path Updating

Lemma

The Kemeny-Young method is short-path update proof.

- Hence, the method is also *extreme update proof.*

Kemeny-Young Method and "Strong" Short-path Updating

Lemma

Let p and q be profiles in L^N. Let $R \in \varphi_K(p)$. For all $i \in N$ let $R \cap p(i) \subseteq q(i)$. Then $\varphi_K(q) \subseteq \varphi_K(p)$. (i.e. when q is a short-path update of p towards R.)

Characterization of Kemeny Young Method

- Pareto Optimality

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- Pareto Optimality
- Consistency

Characterization of Kemeny Young Method

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Characterization of Kemeny Young Method

- Pareto Optimality
- Consistency
- Neutrality
- Short-path update proofness (Pairwise Monotonicity)

Characterization of Kemeny Young Method

Theorem

A rule is Pareto optimal, Consistent, Neutral and Monotone if and only if it is the Kemeny-Young Method

Conclusion

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- Even if conformism is extreme (in extreme update sense), many rules fail to keep the representative agent unchanged.

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- So conformism may lead to changes in the society's representative agent.
- Even if conformism is extreme (in extreme update sense), many rules fail to keep the representative agent unchanged.
- Things can get very unpredictable as seen in the example in the beginning, where the worst alternative eventually becomes a best alternative as the society changes.

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- Which rules are short-path update proof?

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- How about other metric-distances?

Conclusion

- Which rules are short-path update proof?
- So far only Kemeny-Young method.
- How about other metric-distances?
- How about other lattice structures on preferences?

Thanks

Thank you!

Future

Anonymity, neutrality, Pareto-optimality, convexity, cancellation and monotonicity are not consistent.

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Anonymity, neutrality, Pareto-optimality, convexity, consistency and monotonicity are not consistent.

Future

Anonymity, neutrality, Pareto-optimality, convexity, replication invariance and strong monotonicity if and only if Oligarchical Pareto correspondence.