Minimal Retentive Sets in Tournaments

– From Anywhere to TEQ –

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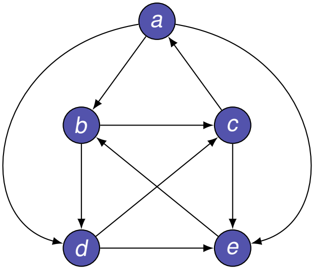
PREFERENCE AGGREGATION IN MULTIAGENT SYSTEMS

The Trouble with Tournaments

- Tournaments are oriented complete graphs
- Many applications: social choice theory, sports tournaments, game theory, argumentation theory, webpage and journal ranking, etc.
- **Question:** How to select the *winner(s)* of a tournament in the absence of transitivity?



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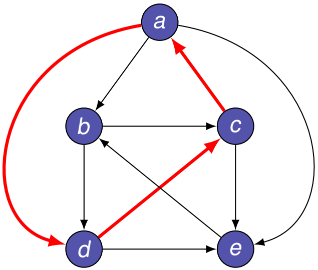
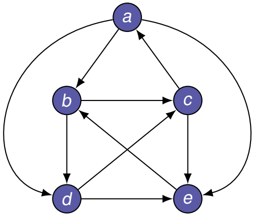


Overview

- Tournament solutions
- Retentiveness and Schwartz's *Tournament Equilibrium Set (TEQ)*
- Properties of minimal retentive sets
- 'Approximating' *TEQ*
- A new tournament solution

Tournament Solutions

- A *tournament* $T = (A, \succ)$ consists of:
  - a finite set A of *alternatives*
  - a complete and asymmetric relation $\succ$ on A



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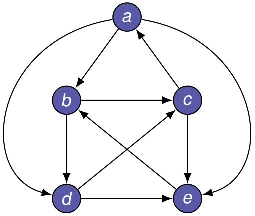

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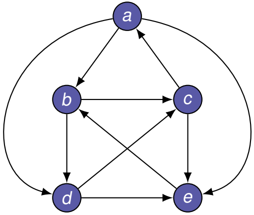

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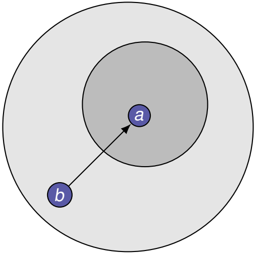
- Examples: Trivial Solution (TRIV), Top Cycle (TC), Uncovered Set, Slater Set, Copeland Set, Banks Set, Minimal Covering Set (MC), *Tournament Equilibrium Set (TEQ)*, ...

Basic Properties of Tournament Solutions

- Monotonicity (MON)
- Weak Superset Property (WSP)
- Strong Superset Property (SSP)
- Independence of Unchosen Alternatives (IUA)



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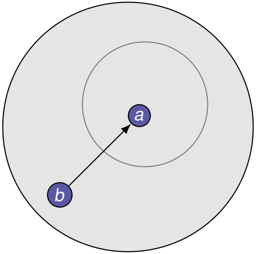

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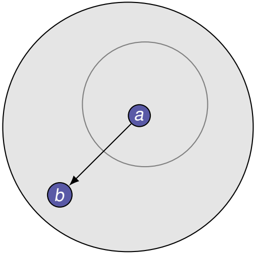

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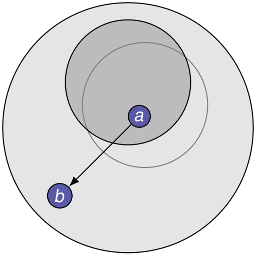

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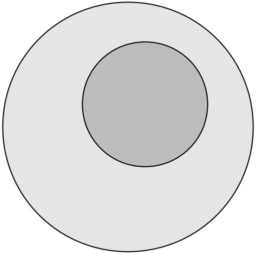

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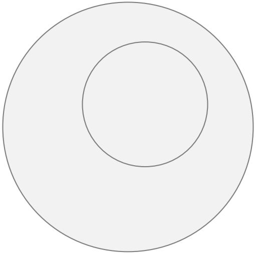

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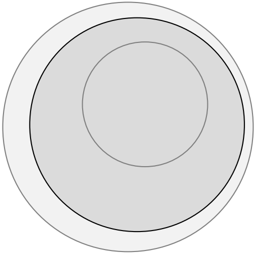

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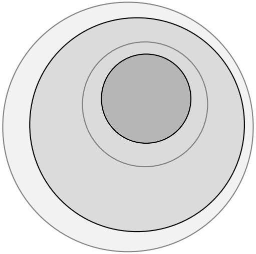

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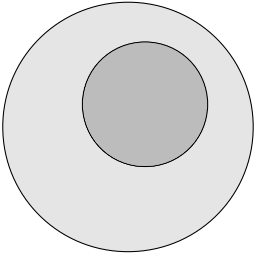

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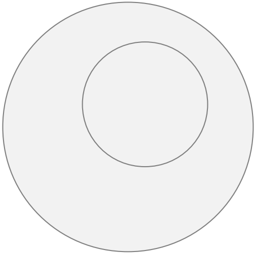

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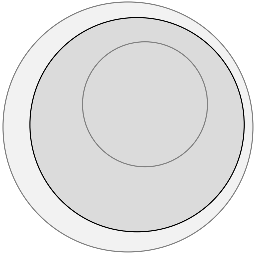

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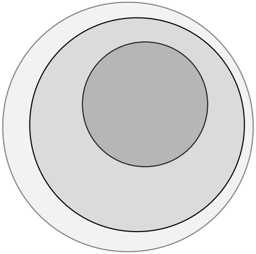

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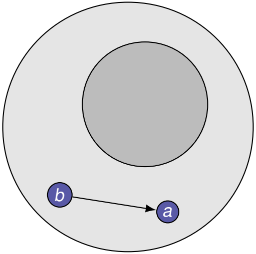

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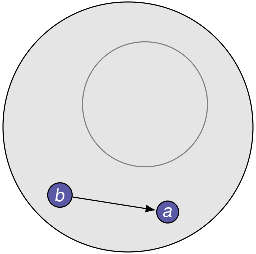

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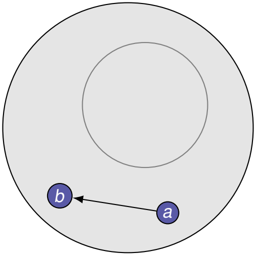

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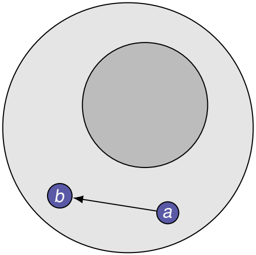

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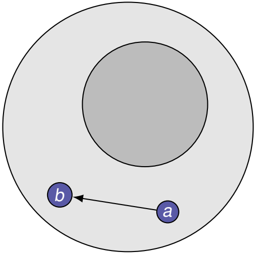

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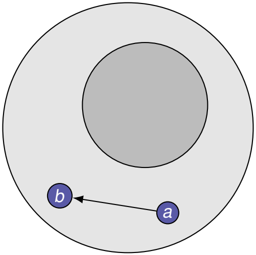

Note:

- SSP is equivalent to $\hat{\alpha}$ (see Felix's lecture)
- (SSP $\wedge$ MON) implies WSP and IUA

Examples

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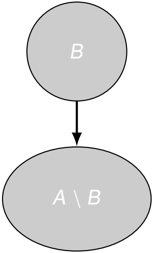

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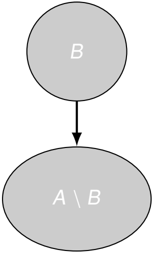

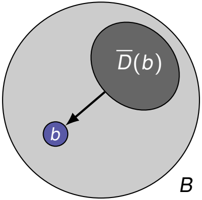

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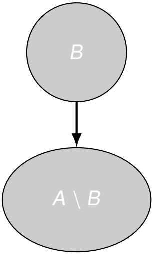
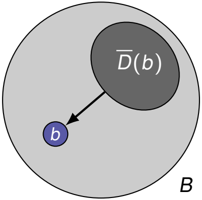
Both *TRIV* and *TC* satisfy all four basic properties





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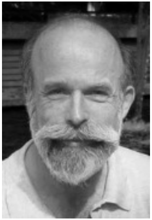

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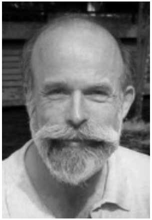

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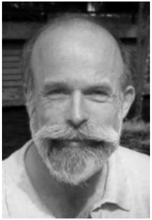

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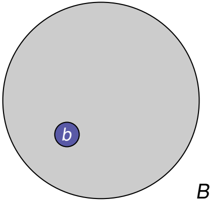

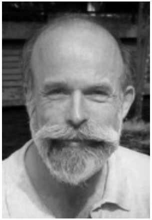

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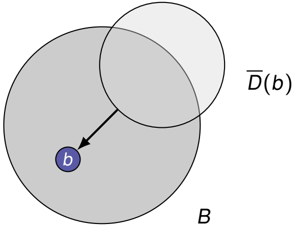

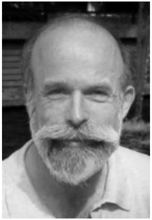

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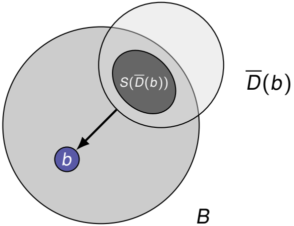

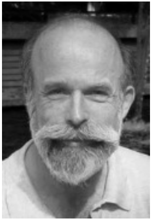

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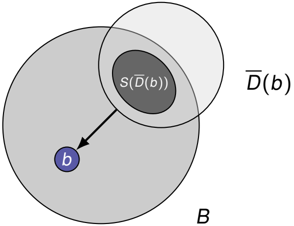

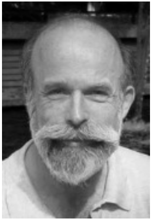

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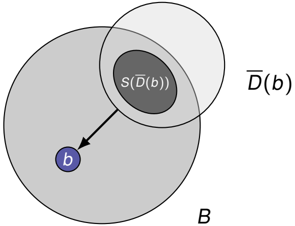

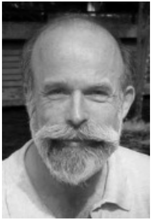

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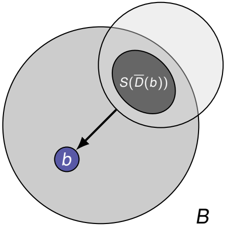

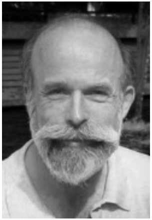

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Definition: $\mathring{S}$ returns the union of all minimal S-retentive sets

- Call $\mathring{S}$ *unique* if there always exists a unique minimal S-retentive set
- Minimal S-retentive sets exist for each tournament
- $\mathring{S}$ is unique if and only if there do not exist two disjoint S-retentive sets

Example

Proposition: $\mathring{TRIV} = TC$

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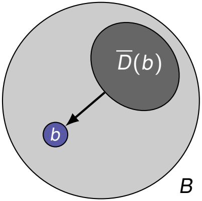
Proof: A set is *TRIV*-retentive if and only if it is dominating



$$TRIV(\overline{D}(b)) = \overline{D}(b)$$

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- well-defined because $|\overline{D}(a)| < |A|$ for each $a \in A$

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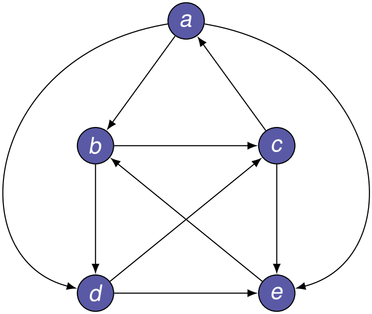

x	$\overline{D}(x)$
a	$\{c\}$
b	$\{a, e\}$
c	$\{b, d\}$
d	$\{a, b\}$
e	$\{a, c, d\}$

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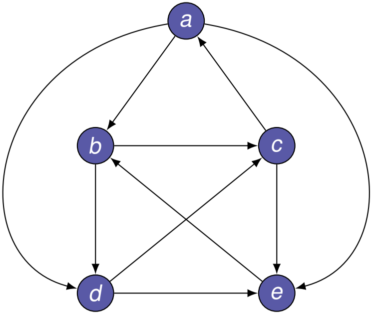

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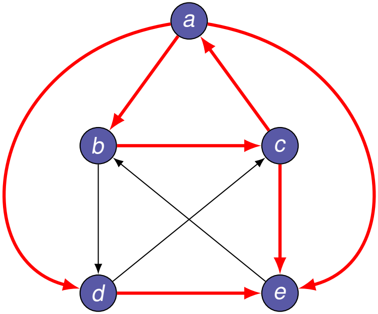

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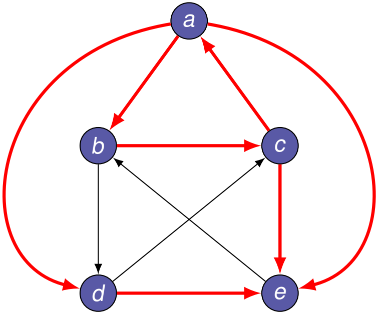

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b	$\{a, e\}$	$\{a\}$
c	$\{b, d\}$	$\{b\}$
d	$\{a, b\}$	$\{a\}$
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TEQ-retentive sets: $\{a, b, c, d, e\}$, $\{a, b, c, d\}$, $\{a, b, c\}$

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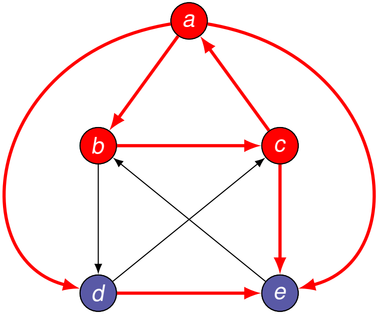

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$$TEQ(T) = \{a, b, c\}$$

The Tournament Equilibrium Set

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- well-defined because $|\overline{D}(a)| < |A|$ for each $a \in A$

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Theorem (Laffond et al., 1993, Houy, 2009): *TEQ* is unique if and only if *TEQ* satisfies any of MON, WSP, SSP, and IUA.

Inheritance of Basic Properties

Recall: $\mathring{S}$ returns the union of all minimal S-retentive sets

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Theorem: If S satisfies (MON $\wedge$ SSP), WSP, SSP, or IUA, so does $\mathring{S}$ *if $\mathring{S}$ is unique*.

Convergence

Define $S^{(0)} = S$ and $S^{(k+1)} = \mathring{S}^{(k)}$. Thus, we obtain sequences like:

$$TRIV, TC, \mathring{TC}, TC^{(2)}, TC^{(3)}, \ldots$$
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Definition: S *converges* to S' if for each T there is some $k_T \in \mathbb{N}$ such that

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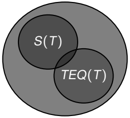
Proof: $S^{(n-1)}(T) = TEQ(T)$ for all tournaments T of order $\leq n$

Reaching the Limit

Theorem: If $S \neq TEQ$, then $S^{(k)} \neq TEQ$ for all $k \geq 0$.



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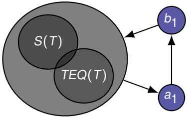

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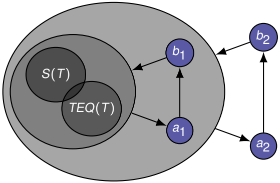

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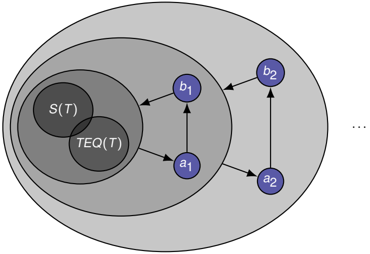

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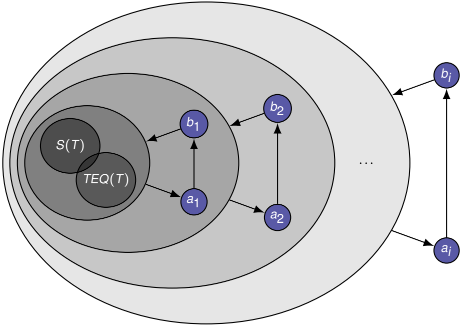

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As uniqueness of $TC^{(k)}$ implies uniqueness of $TC^{(k-1)}$, we have an infinite sequence of increasingly difficult *conjectures*.

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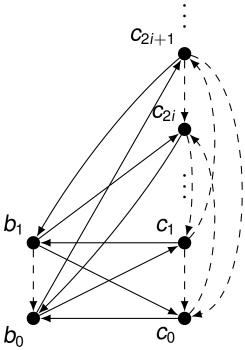

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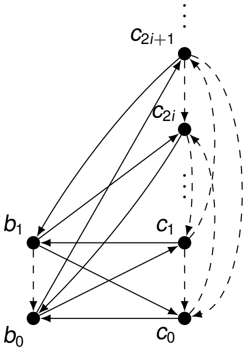
Theorem: $\mathring{TC}$ is unique.

Consequence:

- $\mathring{TC}$ satisfies MON, SSP, WSP, and IUA
- $\mathring{TC}$ lies between TC and TEQ
- $\mathring{TC}$ is efficiently computable



Conclusion

- Retentiveness as an operation on tournament solutions
- Inheritance of basic properties by minimal retentive sets
- Convergence and 'approximating' *TEQ*
- $\mathring{TC}$ first new concept in sequence with desirable properties
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Thank you!