Majority voting on restricted domains: a summary

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Abstract. In judgment aggregation, unlike preference aggregation, not much is known about domain restrictions that guarantee consistent majority outcomes. We introduce several conditions on individual judgments sufficient for consistent majority judgments. Some are based on global orders of propositions or individuals, others on local orders, still others not on orders at all. Some generalize classic social-choice-theoretic domain conditions, others have no counterpart. Our most general condition generalizes Sen's triplewise value-restriction, itself the most general classic condition. Taken together, our results suggest that majority inconsistencies can be avoided in practice, provided that disagreements are appropriately structured. This rehabilitates majority voting as a potential way to reach collective judgments.

1 Introduction

In the theory of preference aggregation, it is well known that majority voting on pairs of alternatives may generate inconsistent (i.e., cyclical) majority preferences even when all individuals' preferences are consistent (i.e., acyclical). The most famous example is Condorcet's paradox. Here one individual prefers x to y to z, a second y to z to x, and a third z to x to y, and thus there are majorities for x against y, for y against z, and for z against x, a 'cycle'. But it is equally well known that if individual preferences fall into a suitably restricted domain, majority cycles can be avoided (for an excellent overview, see Gaertner [16]). The most famous domain restriction with this effect is Black's single-peakedness [1]. A profile of individual preferences is single-peaked if the alternatives can be ordered from 'left' to 'right' such that each individual has a most preferred alternative with decreasing preference for other alternatives as we move away from it in either direction. Inada [18] showed that another condition called single-cavedness and interpretable as the mirror image of single-peakedness also suffices for avoiding majority cycles: a profile is single-caved if, for some 'left'-'right' order of the alternatives, each individual has a least preferred alternative with increasing preference for other alternatives as we move away from it in either direction. Sen [37] introduced a very general domain restriction, called triplewise value-restriction, that garantees acyclical majority preferences and is implied by Black's, Inada's and other conditions; it therefore unifies several domain-restriction conditions, yet has a technical flavour without straightforward interpretation.

The wealth of domain-restriction conditions for avoiding majority cycles was supplemented by another family of conditions based not on 'left'-'right' orders of the alternatives, but on 'left'-'right' orders of the individuals. Important conditions in this family are Grandmont's intermediateness [17] and Rothstein's order restriction ([34], [35]) with its special case of single-crossingness (e.g., Saporiti and Tohmé [36]). To illustrate, a profile of individual preferences is order-restricted if the individuals – rather than the alternatives – can be ordered from 'left' to 'right' such that, for each pair of alternatives x and y, the individuals preferring x to y are either all to the left, or all the right, of those preferring y to x.

Empirically, domain restrictions are important as many political and economic contexts induce a natural structure in preferences. For example, domain restrictions based on a 'left'-'right' order – whether of the alternatives or of the individuals – can capture situations in which preferences are structured by one normative or cognitive dimension, such as from socialist to libertarian, from urban to rural, or from secular to religious.

In the theory of judgment aggregation, by contrast, domain restrictions have received much less attention (the only exception is the work on unidimensional alignment, e.g., List [22]). This is an important gap in the literature since, here too, majority voting with unrestricted but consistent individual inputs may generate inconsistent collective outputs, while on a suitably restricted domain such inconsistencies can be avoided. As illustrated by the much-discussed discursive paradox (e.g., Pettit [31]), if one individual judges that a, $a \to b$ and b, a second that a, but not $a \to b$ and not b, and a third that $a \to b$, but not a and not b, there are majorities for a, for $a \to b$ and yet for not b, an inconsistency. But if no individual rejects $a \to b$, for example, this problem can never arise.

Surprisingly, however, despite the abundance of impossibility results generalizing the discursive paradox as reviewed below, very little is known about the domains of individual judgments on which discursive paradoxes can occur (as opposed to agendas of propositions susceptible to such problems, which have been extensively characterized in the literature). If we can find compelling domain restrictions to ensure majority consistency, this allows us to refine and possibly ameliorate the lessons of the discursive paradox. Going beyond the standard impossibility results, which all assume an unrestricted domain, we can then ask: in what political and economic contexts do the identified domain restrictions hold, so that majority voting becomes safe, and in what contexts are they violated, so that majority voting becomes problematic?

This paper introduces several conditions on profiles of individual judgments that guarantee consistent majority judgments. As explained in a moment, these can be distinguished in at least two respects: first, in terms of whether they are based on orders of propositions, on orders of individuals, or not on orders at all; and second, if they are based on orders, in terms of whether these are 'global' or 'local'. We also discuss parallels and disanalogies with domain-restriction conditions on preferences.

Let us briefly comment on the two distinctions underlying our discussion. First, our conditions based on orders of the individuals are analogous to, and in fact generalize, some of the conditions on preferences reviewed above, particularly intermediateness and order restriction. By contrast, those conditions based on orders of the propositions are not obviously analogous to any conditions on preferences. While an order of individuals can be interpreted similarly in judgment and preference aggregation – namely in terms of the individuals' positions on a normative or cognitive dimension – an order of propositions in judgment aggregation is conceptually distinct from an order of alternatives in preference aggregation. Propositions, unlike alternatives, are not mutually exclusive. It is therefore surprising that sufficient conditions for consistent majority judgments can be given even based on orders of propositions. We also introduce a very general domain-restriction condition not based on orders at all: it generalizes Sen's condition of triplewise value-restriction. In concluding the paper, we characterize the maximal domain on which majority voting yields consistent collective judgments.

Secondly, our domain-restriction conditions based on orders admit global and local variants. In the global case, the individuals' judgments on all propositions on the agenda are constrained by the same 'left'-'right' order of propositions or individuals, whereas in the local case, that order may differ across subsets of the agenda. To give an illustration from the more familiar context of preference aggregation, single-peakedness and single-cavedness are global conditions, whereas the restriction of these conditions to triples of alternatives yields local ones. But while in preference aggregation local conditions result from the restriction of global conditions to triples of alternatives, the picture is more general in judgment aggregation. Here different 'left'-'right' orders may apply to different subagendas, which correspond to different semantic fields. We give precise criteria for selecting appropriate subagendas. An individual can be left-wing on a 'social' subagenda and right-wing on an 'economic' one, for example.

As already noted, some of our conditions generalize existing conditions in preference aggregation, notably Grandmont's intermediateness, Rothstein's order restriction and Sen's triplewise value-restriction, and reduce to them when the agenda of propositions under consideration contains binary ranking propositions suitable for representing preferences (such as xPy, yPz, xPz etc.).[1]

We state our results for the general case in which individual and collective judgments are only required to be consistent; they need not be complete (i.e., they need not be opinionated on every proposition-negation pair). But whenever this is relevant, we also consider the important special case of full rationality, i.e., the conjunction of consistency and completeness.

A few remarks about the literature on judgment aggregation are due. The recent field of judgment aggregation emerged from the areas of law and political philosophy (e.g., Kornhauser and Sager [20] and Pettit [31]) and was formalized social-choice-theoretically by List and Pettit [24]. The literature contains several impossibility results generalizing the observation that on an unrestricted domain majority judgments can be logically inconsistent (e.g., List and Pettit [24] and [25], Pauly and van Hees [30], Dietrich [2], Gärdenfors [15], Nehring and Puppe [29], van Hees [38], Mongin [26], Dietrich and List [7], and Dokow and Holzman [13]). Other impossibility results follow from Nehring and Puppe's [27] strategy-proofness results on property spaces. Earlier precursors include works on abstract aggregation (Wilson [39], Rubinstein and Fishburn [33]). A liberal-paradox-type impossibility was derived in Dietrich and List [12]. Giving up propositionwise aggregation, possibility results were obtained, for example, by using sequential rules (List [23]) and fusion operators (Pigozzi [32]). Voter manipulation in the judgment-aggregation model was analysed in Dietrich and List [8]. But so far the only domain-restriction condition known to guarantee consistent majority judgments is List's unidimensional alignment ([21], [22]), a global domain condition based on orders of individuals. Here we use Dietrich's generalized model [3], which allows propositions to be expressed in rich logical languages.

This paper summarizes results from our working paper [6], in which we also give proofs; these are omitted here for brevity. We are grateful to the ComSoc referees for comments and suggestions.

2 The model

We consider a group of individuals $N = \{1, 2, \ldots, n\}$ ($n \geq 2$) making judgments on some propositions represented in logic (Dietrich [3], generalizing List and Pettit [24], [25]).

Logic. A *logic* is given by a *language* and a notion of *consistency*. The *language* is a non-empty set $\mathbf{L}$ of sentences (called *propositions*) closed under negation (i.e., $p \in \mathbf{L}$ implies $\neg p \in \mathbf{L}$, where $\neg$ is the negation symbol). For example, in standard propositional logic, $\mathbf{L}$ contains propositions such as a, b, $a \wedge b$, $a \vee b$, $\neg(a \rightarrow b)$ (where $\wedge$, $\vee$, $\rightarrow$ denote 'and', 'or', 'if-then', respectively). In other logics, the language may involve additional connectives, such as modal operators ('it is necessary/possible that'), deontic operators ('it is obligatory/permissible that'), subjunctive conditionals ('if p were the case, then q would be the case'), or quantifiers ('for all/some'). The notion of *consistency* captures the logical connections between propositions by stipulating that some sets of propositions $S \subseteq \mathbf{L}$ are *consistent* (and the others *inconsistent*), subject to some regularity axioms.[2] A proposition $p \in \mathbf{L}$ is a *contradiction* if $\{p\}$ is inconsistent and a *tautology* if $\{\neg p\}$ is inconsistent. For example,

[1] The fact that these three existing conditions are already very general representatives of their respective families underlines the generality of our new conditions here.

[2] Self-entailment: Any pair $\{p, \neg p\} \subseteq \mathbf{L}$ is inconsistent. Monotonicity: Subsets of consistent sets $S \subseteq \mathbf{L}$ are consistent. Completability: $\emptyset$ is consistent, and each consistent set $S \subseteq \mathbf{L}$ has a consistent superset $T \subseteq \mathbf{L}$ containing a member of each pair $p, \neg p \in \mathbf{L}$. See Dietrich [3].

in standard logics, $\{a, a \to b, b\}$ and $\{a \wedge b\}$ are consistent and $\{a, \neg a\}$ and $\{a, a \to b, \neg b\}$ inconsistent; $a \wedge \neg a$ is a contradiction and $a \vee \neg a$ a tautology.

Agenda. The *agenda* is the set of propositions on which judgments are to be made. It is a non-empty set $X \subseteq \mathbf{L}$ expressible as $X = \{p, \neg p : p \in X_+\}$ for some set X_+ of unnegated propositions (this avoids double-negations in X). In our introductory example, the agenda is $X = \{a, \neg a, a \to b, \neg(a \to b), b, \neg b\}$. For convenience, we assume that X is finite.[3] As a notational convention, we cancel double-negations in front of propositions in X.[4] Further, for any $Y \subseteq X$, we write $Y^{\pm} = \{p, \neg p : p \in Y\}$ to denote the (single-)negation closure of Y.

Judgment sets. An individual's *judgment set* is the set $A \subseteq X$ of propositions in the agenda that he or she accepts (e.g., 'believes'). A *profile* is an n-tuple $(A_1, \ldots, A_n)$ of judgment sets across individuals. A judgment set is *consistent* if it is consistent in **L;** it is *complete* if it contains at least one member of each proposition-negation pair $p, \neg p \in X$; it is *opinionated* if it contains precisely one such member. Our results mostly do not require completeness, in line with several works on the aggregation of incomplete judgments (Gärdenfors [15]; Dietrich and List [9], [10], [11]; Dokow and Holzman [14]; List and Pettit [24]). This strengthens our possibility results as the identified possibilities hold on larger domains of profiles. But we also consider the complete case.

Aggregation functions. A *domain* is a set D of profiles, interpreted as admissible inputs to the aggregation. An *aggregation function* is a function F that maps each profile $(A_1, \ldots, A_n)$ in a given domain D to a collective judgment set $F(A_1, \ldots, A_n) = A \subseteq X$. While the literature focuses on the *universal domain* (which consists of all profiles of consistent and complete judgment sets), we here focus mainly on domains that are less restrictive in that they allow for incomplete judgments, but more restrictive in that we impose some structural conditions. We call an aggregation function *consistent* or *complete*, respectively, if it generates a consistent or complete judgment set for each profile in its domain. The *majority outcome on a profile* $(A_1, ..., A_n)$ is the judgment set $\{p \in X :$ there are more individuals $i \in N$ with $p \in A_i$ than with $p \notin A_i\}$. The aggregation function that generates the majority outcome on each profile in its domain D is called *majority voting on* D.[5]

Preference aggregation as a special case. To relate our results to existing results on preference aggregation, we must explain how preference aggregation can be represented in our model.[6] Since preference relations are binary relations on some set, they allow a logical representation. Take a simple predicate logic $\mathbf{L}$ with a set of two or more constants $K = \{x, y, ...\}$ representing alternatives and a two-place predicate P representing (strict) preference. For any $x, y \in K$, xPy means 'x is preferable to y'. Define any set $S \subseteq \mathbf{L}$ to be *consistent* if $S \cup Z$ is consistent in the standard sense of predicate logic, where Z is the set of rationality conditions on strict preferences.[7] Now the *preference agenda* is $X_K = \{xPy \in \mathbf{L} : x, y \in K\}^{\pm}$. Preference relations and opinionated judgment sets stand in a bijective correspondence:

[3] For infinite X, our results hold either as stated or under compactness of the logic.

[4] More precisely, if $p \in X$ is already of the form $p = \neg q$, we write $\neg p$ to mean q rather than $\neg\neg q$. This ensures that, whenever $p \in X$, then $\neg p \in X$.

[5] Other widely discussed aggregation functions include dictatorships, supermajority functions, and premise-based or conclusion-based functions.

[6] For details of the construction, see Dietrich and List [7], extending List and Pettit [25].

[7] Z consists of $(\forall v_1)(\forall v_2)(v_1 P v_2 \to \neg v_2 P v_1)$ (asymmetry), $(\forall v_1)(\forall v_2)(\forall v_3)((v_1 P v_2 \wedge v_2 P v_3) \to v_1 P v_3)$ (transitivity), $(\forall v_1)(\forall v_2)(\neg\, v_1 = v_2 \to (v_1 P v_2 \vee v_2 P v_1))$ (connectedness) and, for each pair of distinct constants $x, y \in K$, $\neg\, x = y$ (exclusiveness of alternatives).

- to any preference relation (arbitrary binary relation) $\succ$ on K corresponds the opinionated judgment set $A_{\succ} \subseteq X_K$ with $A_{\succ} = \{xPy : x, y \in K \& x \succ y\} \cup \{\neg xPy : x, y \in K \& x \not\succ y\}$;
- conversely, to any opinionated judgment set $A \subseteq X_K$ corresponds the preference relation $\succ_A$ on K with $x \succ_A y \Leftrightarrow xPy \in A \ \forall x, y \in K$.

A preference relation $\succ$ is fully rational (i.e., asymmetric, transitive and connected) if and only if $A_{\succ}$ is consistent, because we have built the rationality conditions on preferences into the logic. Under this construction, a judgment aggregation function (for opinionated judgment sets) represents a preference aggregation function, and majority voting as defined above corresponds to pairwise majority voting in the standard Condorcetian sense.

3 Conditions for majority consistency based on global orders

On which domains of profiles is majority voting consistent? We already know from the discursive paradox that without any domain restriction it is not (unless the agenda is trivial).[8] However, we now show that there exist many compelling domains on which majority voting is consistent.

3.1 Conditions based on orders of propositions

We begin with two conditions based on 'global' orders of the propositions. An *order of the propositions (in X)* is a linear order $\leq$ on X.[9]

Single-plateauedness. A judgment set A is *single-plateaued relative to* $\leq$ if $A = \{p \in X : p_{\text{left}} \leq p \leq p_{\text{right}}\}$ for some $p_{\text{left}}, p_{\text{right}} \in X$, and a profile is $(A_1, ..., A_n)$ is *single-plateaued relative to* $\leq$ if every A_i is *single-plateaued* relative to $\leq$.

Single-canyonedness. A judgment set A is *single-canyoned relative to* $\leq$ if $A = X \backslash \{p \in X : p_{\text{left}} \leq p \leq p_{\text{right}}\}$ for some $p_{\text{left}}, p_{\text{right}} \in X$, and a profile is $(A_1, ..., A_n)$ is *single-canyoned relative to* $\leq$ if every A_i is *single-canyoned* relative to $\leq$.[10]

An order $\leq$ that renders a profile single-plateaued or single-canyoned is called a *structuring order*; it need not be unique. If a profile is single-plateaued or single-canyoned relative to some $\leq$, we also call it *single-plateaued* or *single-canyoned* simpliciter. Both conditions are illustrated in Figure 1.

The order $\leq$ may represent a normative or cognitive dimension on which propositions are located. If the agenda contains scientific propositions about global warming, for example, individuals may hold single-plateaued judgment sets relative to an order of the propositions from 'most pessimistic' to 'most optimistic', and the location of each individual's plateau may reflect his or her scientific position. If the agenda contains propositions about the effects of various tax or budget policies, the propositions may be ordered from 'socialist' to 'libertarian'. If the agenda contains propositions concerning biological issues, the order may

[8] Majority inconsistencies can arise whenever the agenda has a minimal inconsistent subset of three or more propositions. For a proof of this fact under consistency alone, see Dietrich and List [9]; under full rationality, see Nehring and Puppe [28].

[9] Thus $\leq$ is reflexive ($x \leq x \ \forall x$), transitive ($[x \leq y$ and $y \leq z] \Rightarrow x \leq z \ \forall x, y, z$), connected ($x \neq y \Rightarrow [x \leq y$ or $y \leq x] \ \forall x, y$) and antisymmetric ($[x \leq y$ and $y \leq x] \Rightarrow x = y \ \forall x, y$).

[10] In the definitions of single-plateauedness and single-canyonedness, we do not require $p_{\text{left}} \leq p_{\text{right}}$, i.e., $\{p : p_{\text{left}} \leq p \leq p_{\text{right}}\}$ may be empty.

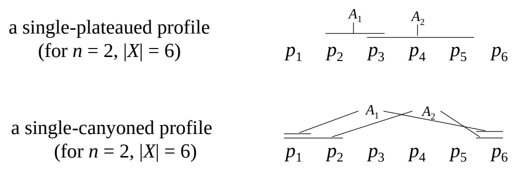


Figure 1: Single-plateauedness and single-canyonedness

range from 'closest to theory X' (e.g., evolutionary theory) to 'closest to theory Y' (e.g., creationism).

We first observe that every single-canyoned profile is single-plateaued.

Proposition 1 *Every single-canyoned profile* $(A_1, ..., A_n)$ *of consistent judgment sets is single-plateaued.*

As anticipated, majority voting preserves consistency on single-plateaued profiles. On single-canyoned profiles, it does even more: it also preserves single-canyonedness.

Proposition 2 *For any profile* $(A_1, ..., A_n)$ *of consistent judgment sets,*
(a) if $(A_1, ..., A_n)$ *is single-plateaued, the majority outcome is consistent;*
(b) if $(A_1, ..., A_n)$ *is single-canyoned, the majority outcome is consistent and single-canyoned (relative to the same structuring order).*

3.2 Conditions based on orders of individuals

Let us now turn to two conditions based on 'global' orders of the individuals. An *order of the individuals (in N)* is linear order Ω on N. For any sets of individuals $N_1, N_2 \subseteq N$, we write $N_1 \Omega N_2$ if $i \Omega j$ for all $i \in N_1$ and $j \in N_2$.

Unidimensional orderedness.[11] A profile $(A_1, ..., A_n)$ is *unidimensionally ordered relative to* Ω if, for all $p \in X$, $\{i \in N : p \in A_i\} = \{i \in N : i_{\text{left}} \Omega i \Omega i_{\text{right}}\}$ for some $i_{\text{left}}, i_{\text{right}} \in N$.

Unidimensional alignment. (List [22]) A profile $(A_1, ..., A_n)$ is *unidimensionally aligned relative to* Ω if, for all $p \in X$, $\{i \in N : p \in A_i\} \Omega \{i \in N : p \notin A_i\}$ or $\{i \in N : p \notin A_i\} \Omega \{i \in N : p \in A_i\}$.

In analogy to the earlier definition, an order Ω that renders a profile unidimensionally ordered or unidimensionally aligned is called a *structuring order*; again, it need not be unique. If a profile is unidimensionally ordered or unidimensionally aligned relative to some Ω, we also call it *unidimensionally ordered* or *unidimensionally aligned* simpliciter. Both conditions are illustrated in Figure 2.

Unidimensional alignment is a special case of unidimensional orderedness: it is the case in which, for every $p \in X$, at least one of $i_{\text{left}}, i_{\text{right}}$ is 'extreme', i.e., the left-most or right-most individual in the structuring order Ω.

Proposition 3 *Every unidimensionally aligned profile* $(A_1, ..., A_n)$ *is unidimensionally ordered.*

How can we interpret the two conditions? A profile is unidimensionally ordered if the individuals can be ordered from 'left' to 'right' such that, for each proposition, the individuals accepting it are all adjacent to each other; a profile is unidimensionally aligned if, in

[11] In this definition, we do not require $i_{\text{left}} \Omega i_{\text{right}}$, i.e., $\{i : i_{\text{left}} \Omega i \Omega i_{\text{right}}\}$ may be empty.

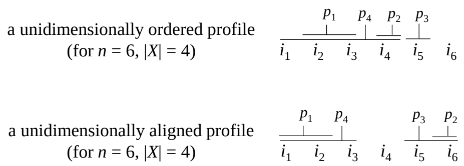


Figure 2: Unidimensional orderedness and unidimensional alignment

addition, the individuals accepting each proposition are either all to the left or all to right of those rejecting it. The order of the individuals can be interpreted as reflecting their location on some underlying normative or cognitive dimension. The idea underlying unidimensional orderedness is that each proposition, like each individual, is located somewhere on the dimension and is accepted by those individuals whose location is 'close' to it, hence by some interval of individuals 'around' it. In a decision problem about climate policies, for example, the proposition 'taxation on emissions should be moderately increased' might have a central location and might therefore be accepted by a 'central' interval of individuals. In the case of unidimensional alignment, the extreme positions on the given dimension correspond to either clear acceptance or clear rejection of each proposition, and, for each proposition, there is a threshold between these extremes (which may vary across propositions) that divides the 'acceptance-region' from the 'rejection-region'.[12]

On unidimensionally ordered profiles, majority voting preserves consistency, and we can say something about the nature of its outcome: it is a subset of the middle individual's judgment set (or, for even n, a subset of the intersection of the two middle individuals' judgment sets). If the profile is unidimensionally aligned, the majority outcome is not just included in that set but coincides with it.

Proposition 4 *For any profile $(A_1, ..., A_n)$ of consistent judgment sets,*
(a) if $(A_1, ..., A_n)$ is unidimensionally ordered, the majority outcome A is consistent and

$$A \subseteq \begin{cases} A_m & \text{if } n \text{ is odd,} \\ A_{m_1} \cap A_{m_2} & \text{if } n \text{ is even,} \end{cases}$$

where m is the middle individual (if n is odd) and m_1, m_2 the middle pair of individuals (if n is even) in any structuring order Ω;
(b) (List [22]) if $(A_1, ..., A_n)$ is unidimensionally aligned, the majority outcome is as stated in part (a) with $\subseteq$ replaced by $=$.

3.3 The logical relationships between the four conditions

We have already seen that single-canyonedness implies single-plateauedness, and that unidimensional alignment implies unidimensional orderedness. A natural question is how the first two conditions, which are based on orders of the propositions, are related to the second two, which are based on orders of the individuals. The following result answers this question.

Proposition 5 *(a) Restricted to profiles of consistent judgment sets,*
- *unidimensional alignment implies any of the other three conditions;*
- *single-canyonedness implies single-plateauedness;*
- *there are no other pairwise implications between the four conditions.*

(b) Restricted to profiles of consistent and complete (or just of opinionated) judgment sets, the four conditions are equivalent.

[12] In List [21], unidimensional alignment is interpreted in terms of 'meta-agreement'.

3.4 Applications to preference aggregation: order restriction and intermediateness

As we explain precisely in Dietrich and List [6], the conditions based on orders of the individuals reduce to classic conditions if applied to the preference agenda:

Remark 6 (a) *A preference profile* $(\succ_1, ..., \succ_n)$ *is order restricted (relative to some* Ω*) if and only if the corresponding judgment profile* $(A_{\succ_1}, ..., A_{\succ_n})$ *is unidimensionally aligned (relative to the same* Ω*).*

(b) *An opinionated preference profile* $(\succ_1, ..., \succ_n)$ *is intermediate (relative to some* Ω*) if and only if the corresponding judgment profile* $(A_{\succ_1}, ..., A_{\succ_n})$ *is unidimensionally ordered (relative to the same* Ω*), where opinionation means that, for each* $i \in N$ *and all distinct* $x, y \in K$*, precisely one of* $x \succ_i y$ *or* $y \succ_i x$ *holds.*

4 Conditions for majority consistency based on local orders

For many agendas, the four domain-restriction conditions discussed so far are stronger than necessary for achieving majority consistency. As developed in detail in our working paper [6], it suffices to apply our conditions to the judgments on various subagendas of X, thereby allowing the relevant structuring order of individuals or propositions to vary across different subagendas. This move parallels the move in preference aggregation from single-peakedness to single-peakedness restricted to triples of alternatives.

Formally, a *subagenda (of* X*)* is a subset $Y \subseteq X$ that is itself an agenda (i.e., non-empty and closed under single negation). For each of our four global domain-restriction conditions, we say that a profile $(A_1, ..., A_n)$ satisfies the given condition *on a subagenda* $Y \subseteq X$ if the restricted profile $(A_1 \cap Y, ..., A_n \cap Y)$, viewed as a profile of judgment sets on the agenda Y, satisfies it. The relevant structuring order is then called a *structuring order on* Y and denoted $\leq_Y$ (if it is an order of propositions) or Ω_Y (if it is an order of individuals). Whenever one of the conditions is satisfied globally, then it is also satisfied on every $Y \subseteq X$. But we now define a local counterpart of each global condition. Let $\mathcal{Y}$ be some set of subagendas.

Local single-plateauedness / single-canyonedness / unidimensional orderedness / unidimensional alignment. A profile $(A_1, ..., A_n)$ satisfies *the local counterpart of each global condition* (with respect to a given set of subagendas $\mathcal{Y}$) if it satisfies the global condition on every $Y \in \mathcal{Y}$.

Provided the set of subagendas $\mathcal{Y}$ is suitably chosen, these local conditions are sufficient to ensure consistent majority outcomes (if individuals hold consistent judgments). In our working paper [6], we discuss two choices of subagendas; according to the first specification,

$$\mathcal{Y} = \{Y^{\pm} : Y \text{ is a minimal inconsistent subset of } X\}.$$

The second specification uses so-called irreducible sets and generalizes the classic local conditions of *intermediateness on triples* and *order restriction on triples* in preference aggregation.

5 Conditions for majority consistency not based on orders

Although our domain-restriction conditions based on local orders are already much less restrictive than those based on global orders, it is possible to weaken them further. Just as the various conditions based on orders in preference aggregation – single-peakedness, single-cavedness etc. – can be generalized to a weaker, but less easily interpretable, condition – namely Sen's triplewise value-restriction [37] – so in judgment aggregation the conditions based on orders can be weakened to a more abstract condition, to be called *value-restriction.* When applied to the preference agenda, this condition becomes non-trivially equivalent to Sen's condition. But despite generalizing Sen's condition, our condition is simpler to state; we thus also hope to offer a new perspective on Sen's condition.

5.1 Value-restriction

For any inconsistent set $Y \subseteq X$, we call another inconsistent set $Z \subseteq X$ a *reduction* of Y if

$$|Z| < |Y| \text{ and each } p \in Z\backslash Y \text{ is entailed by some } V \subseteq Y \text{ with } |Y\backslash V| > 1,$$

and we call Y *irreducible* if it has no reduction. For instance, the inconsistent set $\{a, a \rightarrow b, b \rightarrow c, \neg c\}$ (where a, b, c are distinct atomic propositions) is reducible to $Z = \{b, b \rightarrow c, \neg c\}$, since b is entailed by $\{a, a \rightarrow b\}$, whereas Z is irreducible.

We state two variants of our condition, one based on minimal inconsistent sets, the other based on irreducible sets.

Value-restriction. A profile $(A_1, ..., A_n)$ is *value-restricted* if every (non-singleton[13]) minimal inconsistent set $Y \subseteq X$ has a two-element subset $Z \subseteq Y$ that is not a subset of any A_i.

Weak value-restriction. A profile $(A_1, ..., A_n)$ is *weakly value-restricted* if every (non-singleton) irreducible set $Y \subseteq X$ has a two-element subset $Z \subseteq Y$ that is not a subset of any A_i.

Informally, value-restriction reflects a particular kind of agreement: for every minimal inconsistent (or irreducible in the weak case) subset of the agenda, there exists a particular conjunction of two propositions in this subset that no individual endorses. Like our previous domain-restriction conditions, the two new conditions are each sufficient for consistent majority outcomes (the weaker condition in the important special case of individual completeness).

Proposition 7 *For any profile $(A_1, ..., A_n)$ of consistent judgment sets,*
(a) if $(A_1, ..., A_n)$ is value-restricted, the majority outcome is consistent;
(b) if $(A_1, ..., A_n)$ is weakly value-restricted and each A_i is complete, the majority outcome is consistent.

How general are our two value-restriction conditions? The following proposition answers this question.

Proposition 8 *(a) Each of our four conditions based on global orders implies value-restriction.*

[13]The qualification 'non-singleton' in this definition and the next is unnecessary if X contains only contingent propositions, since this rules out singleton inconsistent sets.

(b) Each of our four conditions based on local orders, with respect to $\mathcal{Y}$ defined in terms of minimal inconsistent sets, implies value-restriction.
(c) Each of our four conditions based on local orders, with respect to $\mathcal{Y}$ defined in terms of irreducible sets, implies weak value-restriction.

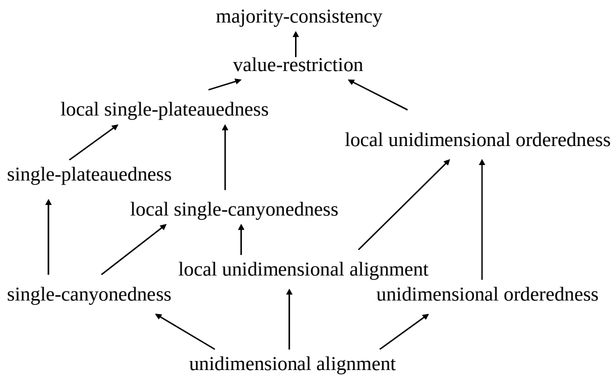
5.2 Applications to preference aggregation: triplewise value-restriction

When applied to the preference agenda, our two value-restriction conditions surprisingly both collapse into Sen's ([37]) triplewise value-restriction.

Proposition 9 *For any profile $(A_1, ..., A_n)$ of consistent and complete judgment sets on the preference agenda, the following are equivalent:*
(a) $(A_1, ..., A_n)$ is value-restricted,
(b) $(A_1, ..., A_n)$ is weakly value-restricted,
(c) the associated preference profile $(\succ_{A_1}, ..., \succ_{A_n})$ is triplewise value-restricted.

6 Conclusion

The following figure summarizes the logical relationship between all the domain-restriction conditions discussed in this paper, in each case applied to profiles of consistent individual judgment sets.



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