Three-sided stable matchings with cyclic preferences and the kidney exchange problem[1]

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Abstract

Knuth [14] asked whether the stable matching problem can be generalised to three dimensions i. e., for families containing a man, a woman and a dog. Subsequently, several authors considered the three-sided stable matching problem with cyclic preferences, where men care only about women, women only about dogs, and dogs only about men. In this paper we prove that if the preference lists may be incomplete, then the problem of deciding whether a stable matching exists, given an instance of three-sided stable matching problem with cyclic preferences is NP-complete. Considering an alternative stability criterion, strong stability, we show that the problem is NP-complete even for complete lists. These problems can be regarded as special types of stable exchange problems, therefore these results have relevance in some real applications, such as kidney exchange programs.

1 Introduction

An instance of the Stable Marriage problem (SM) comprises a set of n men $a_1, \ldots, a_n$ and a set of n women $b_1, \ldots, b_n$. Each person has a complete preference list consisting of the members of the opposite sex. If b_j precedes b_k on a_i's list then a_i is said to *prefer* b_j to b_k. The problem is to find a matching that is *stable* in the sense that no man and woman both prefer each other to their current partner in the matching. The Stable Marriage problem was introduced by Gale and Shapley [9]. They constructed a linear time algorithm that always finds a stable matching for an SM instance.

Considering the Stable Marriage problem with Incomplete Lists (SMI), the only difference is that the numbers of men and women are not necessarily equal and each preference list consist of a subset of the members of the opposite sex, i.e., each person lists his or her *acceptable partners*. Here, a matching $\mathcal{M}$ is a set of acceptable pairs, and $\mathcal{M}$ is stable if for every pair $(a_i, b_j) \notin \mathcal{M}$, either a_i prefers his matching partner $\mathcal{M}(a_i)$ to b_j or b_j prefers her matching partner $\mathcal{M}(b_j)$ to a_i. We can model this problem by a bipartite graph $G = (A \cup B, E)$, where the sets of vertices, A and B, correspond to the sets of men and women, respectively, and the set of edges, E represents the acceptable pairs. An extended version of the Gale–Shapley algorithm always produces a stable matching for this setting too.

In an instance of the Stable Marriage problem with Ties and Incomplete Lists (SMTI) it is possible that an agent is indifferent between some acceptable agents from the opposite set; in such a case, these agents appear together in a *tie* in the preference list. Here, a matching $\mathcal{M}$ is stable if there is no blocking pair $(a_i, b_j) \notin \mathcal{M}$ such that a_i is either unmatched or prefers b_j to $\mathcal{M}(a_i)$, and simultaneously b_j is either unmatched or prefers a_i to $\mathcal{M}(b_j)$. Manlove et al. [15] proved that the problem of finding a stable matching of maximum cardinality for an instance of SMTI, the so-called MAX SMTI problem, is NP-hard.

The Three-Dimensional Stable Matching problem (3DSM), also referred to as the Three Gender Stable Marriage problem, was introduced by Knuth [14]. Here, we have three sets of agents: men, women and dogs, say, and each agent has preference over all pairs from

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the two other sets. A *matching* is a set of disjoint *families* i.e., triples of the form (man, woman, dog). A matching is *stable* if there exists no blocking family that is preferred by all its members to their current families in the matching.

Alkan [2] gave the first example of an instance of 3DSM where no stable matching exists. Ng and Hirschberg [17] proved that the problem of deciding whether a stable matching exists, given an instance of 3DSM, is NP-complete; later Subramanian [26] gave an alternative proof for this. Recently, Huang [10] proved that the problem remains NP-complete even if the preference lists are "consistent". (A preference list is inconsistent if, for example, man m ranks (w_1, d_1) higher than (w_2, d_1), but he also ranks (w_2, d_2) higher than (w_1, d_2), so he does not consistently prefer woman w_1 to woman w_2.)

As an open problem, Ng and Hirschberg [17] mentioned the cyclic 3DSM, defined formally in Section 2, where men only care about women, women only care about dogs and dogs only care about men. Boros et al. [5] showed that if the number of agents n, is at most 3 in every set, then a stable matching always exists. Eriksson et al. [8] proved that this also holds for $n = 4$ and conjectured that a stable matching exists for every instance of cyclic 3DSM.

In Section 2, we study the cyclic 3DSM problem with Incomplete Lists (cyclic 3DSMI). Here, each preference list may consist of a subset of the members of the next gender, i.e. his, her or its *acceptable partners*, and the cardinalities of the sets are not necessarily the same, a matching is a set of acceptable families. Thus cyclic 3DSMI is obtained via a natural generalisation of cyclic 3DSM in a way analogous to the extension SMI of SM. First we give an instance of cyclic 3DSMI for $n = 6$ where no stable matching exists. Then, by using this instance as a gadget, we show that the problem of deciding whether a stable matching exists in an instance of cyclic 3DSMI is NP-complete. We reduce from MAX SMTI.

In Section 3, we study the cyclic 3DSM problem under *strong stability*. A matching is strongly stable if there exists no *weakly blocking family*. This is a family not in the matching that is weakly preferred by all its members (i.e. no member prefers his original family to the new blocking family). We show that the problem of deciding whether a strongly stable matching exists in an instance of cyclic 3DSM is NP-complete.

In Section 4, we describe the correspondence between the cyclic 3DSMI problem and the so-called stable exchange problem with restrictions, defined in Section 4. More precisely, we show that the 3-way stable 3-way exchange problem for tripartite cyclic graphs is equivalent to cyclic 3DSMI. Therefore, the complexity result for cyclic 3DSMI applies also to the 3-way stable 3-way exchange problem, which is an important model for the kidney exchange problem (this application is described in further detail in Section 4).

We remark that all of these problems (namely, SM, SMI, 3DSM, 3DSMI, cyclic 3DSM and cyclic 3DSMI) can be considered as special *coalition formation games*, where the notion of a stable matching is equivalent to the notion of a *core element* in the corresponding NTU-game. Those games, where the set of *basic coalitions* contain all singletons (i.e. where every player has the right not to cooperate with the others) correspond to the stable matching problems with incomplete lists. See more about this correspondence in [3].

2 Cyclic 3DSMI is NP-complete

Problem definition

We consider three sets of agents: M, W, D (men, women and dogs). Every man has a strict preference list over the women that are acceptable to him. Analogously, every woman has a strict preference list over her acceptable dogs, and every dog has a strict preference list over its acceptable men. The list of an agent x is denoted by $P(x)$. A *matching* $\mathcal{F}$ is a set of disjoint families, i.e., triples from $M \times W \times D$, such that for each family $(m, w, d) \in \mathcal{F}$, w is acceptable to m, d is acceptable to w and m is acceptable to d. Formally, if $(m, w, d) \in \mathcal{F}$, then we say that $\mathcal{F}(m) = w$, $\mathcal{F}(w) = d$ and $\mathcal{F}(d) = m$, thus in a matching, $\mathcal{F}(x) \in P(x) \cup \{x\}$ holds for every agent x, where $\mathcal{F}(x) = x$ means that agent x is unmatched in $\mathcal{F}$. Note that agent x prefers y to being unmatched if $y \in P(x)$.

A matching $\mathcal{F}$ is said to be *stable* if there exists no *blocking family*, that is a triple $(m, w, d) \notin \mathcal{F}$ such that m prefers w to $\mathcal{F}(m)$, w prefers d to $\mathcal{F}(w)$ and d prefers m to $\mathcal{F}(d)$.

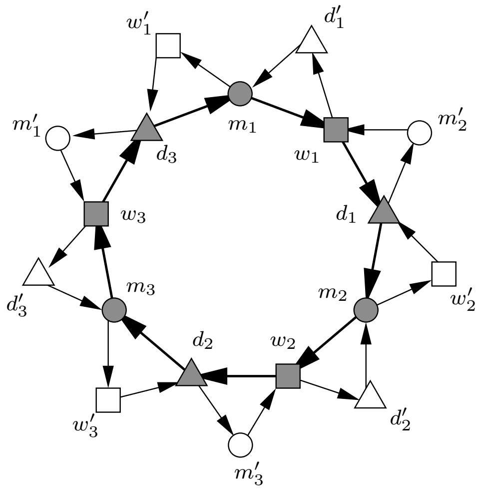
We define the underlying directed graph $D_I = (V, A)$ of an instance I of cyclic 3DSMI as follows. The vertices of D_I correspond to the agents, so $V(D_I) = M \cup W \cup D$, and we have an arc (x, y) in D_I if $y \in P(x)$. This type of directed graph where $A(D_I) \subseteq (M \times W) \cup (W \times D) \cup (D \times M)$ is called a *tripartite cyclic digraph.* Therefore, a matching of I corresponds to a disjoint packing of directed 3-cycles in D_I.

An unsolvable instance of cyclic 3DSMI

We give an instance of cyclic 3DSMI with $n = 6$, denoted by $R6$, where no stable matching exists.

Example 1. *The preference lists and underlying graph of* $R6$ *are as shown below. Here, the thickness of arrows correspond to preferences.*

$m_1 : w_1, w'_1$	$w_1 : d_1, d'_1$	$d_1 : m_2, m'_2$
$m_2 : w_2, w'_2$	$w_2 : d_2, d'_2$	$d_2 : m_3, m'_3$
$m_3 : w_3, w'_3$	$w_3 : d_3, d'_3$	$d_3 : m_1, m'_1$
$m'_1 : w_3$	$w'_1 : d_3$	$d'_1 : m_1$
$m'_2 : w_1$	$w'_2 : d_1$	$d'_2 : m_2$
$m'_3 : w_2$	$w'_3 : d_2$	$d'_3 : m_3$



We refer to the agents $\{m_i, w_i, d_i : 1 \leq i \leq 3\} = I$ as the *inner agents* of $R6$ and the agents $\{m'_i, w'_i, d'_i : 1 \leq i \leq 3\} = O$ as the *outer agents* of $R6$.

Lemma 1. *The instance* $R6$ *of cyclic 3DSMI admits no stable matching.*

Proof. By inspection of the underlying graph of $R6$, we can observe that the only acceptable families are of the form (m_i, w'_i, d_{i-1}), (m_i, w_i, d'_i) and (m'_i, w_{i-1}, d_{i-1}), so that any acceptable family contains exactly two inner agents. It is clear that for any matching $\mathcal{F}$, it must be the case that at least one inner agent is unmatched in $\mathcal{F}$. By the symmetry of the instance we may suppose without loss of generality that the inner agent m_1 is unmatched in $\mathcal{F}$. Then, the family (m_1, w'_1, d_3) is a blocking family for $\mathcal{F}$. $\square$

We note that the 9 acceptable families of $R6$ have a natural cyclic order, the same order that the directed 9-cycle has which is formed by the 9 inner agents in the underlying graph, such that if an acceptable family is not in a stable matching $\mathcal{F}$ then the successor family must be in $\mathcal{F}$. For example, if $(m_1, w_1, d'_1) \notin \mathcal{F}$ then $(m'_2, w_1, d_1) \in \mathcal{F}$, since (m_1, w_1, d'_1) would be blocking otherwise. This argument gives an alternative proof for the above Lemma.

The instance created by removing the inner agent m_1 from $R6$, denoted by $R6 \setminus m_1$, becomes solvable, since $\mathcal{F}^* = \{(m'_2, w_1, d_1), (m_2, w_2, d'_2), (m_3, w'_3, d_2), (m'_1, w_3, d_3)\}$ is a stable matching for $R6 \setminus m_1$. In fact, $\mathcal{F}^*$ is the unique stable matching for $R6 \setminus m_1$, so we denote it by $\mathcal{F}_{R6\setminus m_1}$. This is because in $R6 \setminus m_1$ we have 7 acceptable families in a row with the property discussed above: if an acceptable family is not in a stable matching $\mathcal{F}$ then the subsequent family must be in $\mathcal{F}$. We state this claim formally below; its proof follows from the symmetry of the instance.

Lemma 2. *Let a_i be an inner agent of $R6$. Then, $R6 \setminus a_i$ admits a unique stable matching, denoted by $\mathcal{F}_{R6\setminus a_i}$.*

The instance $R6$ will also be of use to us as a gadget in the NP-completeness proofs of the subsequent sections.

The NP-completeness proof

In [15], Manlove et al. proved that determining if an instance of SMTI admits a complete stable matching is NP-complete, even if the ties appear only on the women's side, and each woman's preference list is either strictly ordered or consists entirely of a tie of size two (these conditions holding simultaneously).

We refer to the MAX SMTI problem under the above restrictions as Restricted SMTI. The underlying graph $G = (A \cup B, E)$ of a Restricted SMTI instance is such that the set $A = \{a_1, a_2, \ldots, a_n\}$ consists of men a_i, all of whom have strictly ordered preference lists, while the set B of women can be partitioned into two sets $B_1 \cup B_2 = \{b_1, \ldots, b_{n_1}\} \cup \{b_1^T, \ldots, b_{n_2}^T\}$ where $n_1 + n_2 = n$, each woman $b_j \in B_1$ has a strictly ordered preference list, and each woman $b_j^T \in B_2$ has a preference list consisting solely of a tie of length 2. We denote a woman who can either be a member of B_1 or B_2 by $b_i^{(T)}$.

In the remainder of this section we describe a polynomial-time reduction from Restricted SMTI to cyclic 3DSMI. Let I be an instance of Restricted SMTI with the underlying graph $G = (A \cup B, E)$. We construct an instance I' of cyclic 3DSMI with sets M, W, and D of men, women, and dogs as follows.

The sets of men and women of I' are created in direct correspondence to the men and women in I, so let $M = \{m_1, \ldots, m_n\}$ and $W = W_1 \cup W_2 = \{w_1, \ldots, w_{n_1}\} \cup \{w_1^T, \ldots, w_{n_2}^T\}$. The set of dogs of I' consists of two parts $D_1 \cup D_2 = D$, defined by creating a dog $d_{j,i}$ in D_1 if $a_i \in P(b_j)$, and creating a dog d_j^T in D_2 if $b_j^T \in B_2$.

Let us now describe the construction of the strictly ordered preference lists of I'. We let $P(x)[l]$ denote the lth entry in agent x's preference list, and a tie in the preference list of an agent is indicated by parentheses. The preference lists of I' are defined by the following cases:

1. If $P(a_i)[l] = b_j^{(T)}$ then let $P(m_i)[l] = w_j^{(T)}$ ($1 \leq l \leq r$, where r is the length of a_i's list).

2. If $P(b_j)[l] = a_i$ then let $P(w_j)[l] = d_{j,i}$ and $P(d_{j,i}) = m_i$ ($1 \leq l \leq r$, where r is the length of b_j's list).

3. If $P(b_j^T) = (a_p, a_q)$ then let $P(w_j^T) = d_j^T$ and $P(d_j^T) = m_p m_q$ (in arbitrary order).

This is the *proper part* of the instance. Next we construct the *additional part* of the instance by creating $n = |M|$ copies of $R6$, such that the t-th copy of $R6$ consists of inner agents $\{m_{t_i}, w_{t_i}, d_{t_i} : 1 \leq i \leq 3\}$ and outer agents $\{m'_{t_i}, w'_{t_i}, d'_{t_i} : 1 \leq i \leq 3\}$ with preference lists as described in Example 1. We add these n copies of $R6$ to the instance in the following way. In the t-th added copy of $R6$, denoted by $R6_t$, replace the inner agent m_{t_1} in $R6_t$ with man $m_t \in M$ by replacing each occurrence of m_{t_1} in the preference lists of each agent in $R6_t$ with m_t. Also, let m_{t_1}'s acceptable partners in $R6_t$, namely w_{t_1} and w'_{t_1} be appended in this order to the end of m_t's list. The final preference list of man m_t along with $R6_t$ is shown below. The portion of m_t's preference list consisting of women from the proper part of the instance is denoted by P_t.

$$
\begin{array}{lcl@{\qquad\qquad}lcl@{\qquad\qquad}lcl}
m_t & : & P_t\ w_{t_1}\ w'_{t_1} & w_{t_1} & : & d_{t_1}\ d'_{t_1} & d_{t_1} & : & m_{t_2}\ m'_{t_2} \\
m_{t_2} & : & w_{t_2}\ w'_{t_2} & w_{t_2} & : & d_{t_2}\ d'_{t_2} & d_{t_2} & : & m_{t_3}\ m'_{t_3} \\
m_{t_3} & : & w_{t_3}\ w'_{t_3} & w_{t_3} & : & d_{t_3}\ d'_{t_3} & d_{t_3} & : & m_t\ m'_{t_1} \\
m'_{t_1} & : & w_{t_3} & w'_{t_1} & : & d_{t_3} & d'_{t_1} & : & m_t \\
m'_{t_2} & : & w_{t_1} & w'_{t_2} & : & d_{t_1} & d'_{t_2} & : & m_{t_2} \\
m'_{t_3} & : & w_{t_2} & w'_{t_3} & : & d_{t_2} & d'_{t_3} & : & m_{t_3}
\end{array}
$$

This ends the reduction, which plainly can be computed in polynomial time. Now, we prove that there is a one-to-one correspondence between the complete stable matchings in I and the stable matchings in I'.

First we show that there is a one-to-one correspondence between the matchings of I and the matchings in the proper part of I'. This comes from the natural one-to-one correspondence between the edges of I and the families in the proper part of I'. More precisely, if $\mathcal{M}$ is a matching in I, then the corresponding matching $\mathcal{F}_p$ in the proper part of I is created as follows: $(a_i, b_j) \in \mathcal{M} \iff (m_i, w_j, d_{j,i}) \in \mathcal{F}_p$ and $(a_i, b_j^T) \in \mathcal{M} \iff (m_i, w_j^T, d_j^T) \in \mathcal{F}_p$. To prove this, it is enough to observe that two edges in I are disjoint if and only if the two corresponding families in I' are also disjoint. Next, we show that stability is preserved by this correspondence.

Lemma 3. *A matching $\mathcal{M}$ of I is stable if and only if the corresponding matching $\mathcal{F}_p$ in the proper part of I' is stable.*

Proof. It is enough to show that an edge (a_i, b_j) is blocking in I if and only if the corresponding family $(m_i, w_j, d_{j,i})$ is also blocking in I'; and similarly, an edge (a_i, b_j^T) is blocking in I if and only if the corresponding family (m_i, w_j^T, d_j^T) is also blocking in I'.

Suppose first that (a_i, b_j) is blocking in I, which means that a_i is either unmatched or prefers b_j to $\mathcal{M}(a_i)$ and b_j is either unmatched or prefers a_i to $\mathcal{M}(b_j)$. This implies that m_i prefers w_j to $\mathcal{F}_p(m_i)$, w_j prefers $d_{j,i}$ to $\mathcal{M}(w_j)$, and $d_{j,i}$ is unmatched in $\mathcal{F}_p$, i.e. $(m_i, w_j, d_{j,i})$ is blocking in I'. Similarly, if (a_i, b_j^T) is blocking then a_i is either unmatched or prefers b_j^T to $\mathcal{M}(a_i)$ and b_j^T is unmatched in $\mathcal{M}$. This implies that m_i prefers w_j^T to $\mathcal{F}_p(m_i)$, w_j^T and d_j^T are both unmatched in $\mathcal{F}_p$, and hence (m_i, w_j^T, d_j^T) is blocking in I'.

In the other direction, if $(m_i, w_j, d_{j,i})$ is blocking in I', then m_i prefers w_j to $\mathcal{F}_p(m_i)$, w_j prefers $d_{j,i}$ to $\mathcal{F}_p(w_j)$, and $d_{j,i}$ is unmatched in $\mathcal{F}_p$. This implies that a_i is either unmatched or prefers b_j to $\mathcal{M}(a_i)$ and b_j is either unmatched or prefers a_i to $\mathcal{M}(b_j)$, so (a_i, b_j) is blocking in I. Similarly, if (m_i, w_j^T, d_j^T) is blocking in I', then w_j^T and d_j^T are both unmatched in $\mathcal{F}_p$ and m_i prefers w_j^T to $\mathcal{F}_p(m_i)$. This implies that a_i is either unmatched or prefers b_j^T to $\mathcal{M}(a_i)$ and b_j^T is unmatched in $\mathcal{M}$, so (a_i, b_j^T) is blocking in I. □

Furthermore, if the matching $\mathcal{M}$ is complete, then we can enlarge the corresponding matching to the additional part of I' by matching every $R6_t \setminus m_t$ in the unique stable way, so by adding $\mathcal{F}_{R6_t \setminus m_t}$ to $\mathcal{F}_p$ for every t. This leads to the following one-to-one correspondence between the complete stable matchings of I and the stable matching of I'.

Lemma 4. *The instance I admits a complete stable matching $\mathcal{M}$ if and only if the reduced instance I' admits a stable matching $\mathcal{F}$, where $\mathcal{F}$ is the corresponding matching of $\mathcal{M}$.*

Proof. The stability of $\mathcal{M}$ implies that $\mathcal{F}$ is stable in the proper part of I' by Lemma 3. The completeness of $\mathcal{M}$ and Lemma 2 implies that $\mathcal{F}$ is also stable in the additional part of I'.

In the other direction, if $\mathcal{F}$ is stable then every man in M must be matched in a proper family, since otherwise, if a proper man m_t does not have a proper partner in $\mathcal{F}$ then $R6_t$ would contain a blocking family, by Lemma 1. This implies that the corresponding matching $\mathcal{M}$, defined in Lemma 3, is complete. The stability of $\mathcal{M}$ is a consequence of Lemma 3. Finally, we note that the additional part has a unique stable matching, since every $R6_t \setminus a_t$ must be matched in the unique stable way indicated by Lemma 2, which implies the one-to-one correspondence. □

The following Theorem is a direct consequence of Lemma 4.

Theorem 1. *Determining the existence of a stable matching in a given instance of cyclic* 3DSMI *is NP-complete.*

3 Cyclic 3DSM under strong stability is NP-complete

Problem definition

For an instance of cyclic 3DSM, a matching $\mathcal{F}$ is *strongly stable* if there exists no *weakly blocking family*, that is a family $(m, w, d) \notin \mathcal{F}$ such that m prefers w to $\mathcal{F}(m)$ or $w = \mathcal{F}(m)$, w prefers d to $\mathcal{F}(w)$ or $d = \mathcal{F}(w)$, and d prefers m to $\mathcal{F}(d)$ or $m = \mathcal{F}(d)$. We note that in a weakly blocking family at least two members obtain a better partner, since the preference lists are strictly ordered.

An unsolvable instance

We firstly show that, by completing the preference lists of $R6$ in an arbitrary way (i.e. by appending agents not on the lists in an arbitrary order to the tail of the original lists), the resulting instance of cyclic 3DSM, denoted by $\overline{R6}$, does not admit any strongly stable matching. The subinstance $R6$ of $\overline{R6}$ is called the *suitable part* of $\overline{R6}$, the original entries of an agent x in $R6$ are the *suitable partners of* x and the families of $R6$ are called *suitable families.*

Lemma 5. *The instance* $\overline{R6}$ *of cyclic 3DSM admits no strongly stable matching.*

Proof. Suppose for contradiction that $\mathcal{F}$ is a strongly stable matching. As the 9 inner agents form a 9-cycle in the underlying directed graph, the 9 suitable families have a natural cyclic order. We show that if a suitable family, say (m_1, w_1, d_1') is not in $\mathcal{F}$, then the successor suitable family (m_2', w_1, d_1) must be in $\mathcal{F}$, which would imply a contradiction given that the number of these suitable families is odd. If $(m_1, w_1, d_1') \notin \mathcal{F}$ then $\mathcal{F}(w_1) = d_1$, since otherwise (m_1, w_1, d_1') would be weakly blocking. Similarly, $(m_2', w_1, d_1) \notin \mathcal{F}$ implies $\mathcal{F}(d_1) = m_2$. But this means that $(m_2, w_1, d_1) \in \mathcal{F}$, so (m_2, w_2', d_1) is weakly blocking. □

Recall that $\mathcal{F}_{R6\setminus a_t}$ is the unique stable matching for $R6 \setminus a_t$. Let $\overline{R6} \setminus a_t$ denote the instance created by removing an inner agent a_t from $\overline{R6}$. We denote by $C_{R6\setminus a_t}$ the subset of agents of $\overline{R6} \setminus a_t$ that are covered by $\mathcal{F}_{R6\setminus a_t}$, and by $U_{R6\setminus a_t}$ those who are uncovered by $\mathcal{F}_{R6\setminus a_t}$, respectively.

Lemma 6. *Let* a_t *be an inner agent of* $\overline{R6}$. *For every matching* $\mathcal{F}^* \supseteq \mathcal{F}_{R6\setminus a_t}$ *of* $\overline{R6} \setminus a_t$, *no suitable family can be weakly blocking, and therefore no agent from* $C_{R6\setminus a_t}$ *can be involved in a weakly blocking family. For any other matching, at least one suitable family is weakly blocking.*

Proof. It is straightforward to verify that $\mathcal{F}_{R6\setminus a_t}$ is a strongly stable matching for $R6 \setminus a_t$, so no suitable family in $\overline{R6} \setminus a_t$ can weakly block $\mathcal{F}^* \supseteq \mathcal{F}_{R6\setminus a_t}$. Moreover, no agent x of $C_{R6\setminus a_t}$ can be involved in a non-suitable weakly blocking family either, since x has a suitable partner in $\mathcal{F}^*$.

Suppose that $\mathcal{F}'$ is a matching of $\overline{R6} \setminus a_t$ which is not a superset of $\mathcal{F}_{R6\setminus a_t}$. As in the proof of Lemma 5, we use the fact that if a suitable family is not in $\mathcal{F}'$, then the successor suitable family is either in $\mathcal{F}'$ or weakly blocking. Therefore, if we do not include four from the seven suitable families of $\overline{R6} \setminus a_t$ in a matching then one of them would be weakly blocking. □

The NP-completeness proof

The reduction we describe in this section again begins with an instance of Restricted SMTI, only we assume without loss of generality the role of the men and women of the instance to be "reversed". To be precise, we assume a given instance of Restricted SMTI I that its vertex set $((A_1 \cup A_2) \cup B)$ consists of a set $A_1 = \{a_1, a_2, \ldots, a_{n_1}\}$ of men with strictly ordered preference lists, and $A_2 = \{a_1^T, a_2^T, \ldots, a_{n_2}^T\}$ of men with preference lists consisting of a single tie of length 2, and $n_1 + n_2 = n$. The set $B = \{b_1, b_2, \ldots, b_n\}$ consists entirely of women with strictly ordered preference lists.

Given an instance I of Restricted SMTI as defined above, we create an instance I' of cyclic 3DSM. First we create a *proper instance* I'_p of cyclic 3DSMI as a subinstance of I' with agents $M_p \cup W_p \cup D_p$ in the following way.

First we create a set W_p of n women $\{w_1, w_2, \ldots, w_n\}$ such that the preference list of woman w_j is a single entry, dog $d_j \in D_p$. The preference list of d_j is such that if $P(b_j)[l] = a_i$, then $P(d_j)[l] = m_i$, otherwise if $P(b_j)[l] = a_i^T$, then $P(d_j)[l] = m'_{i,j}$ for $1 \leq l \leq r$, where r is the length of b_j's list. So the preference list of dog d_j is essentially the "same" as that of woman b_j, only with men in M_p rather than A.

For each man $a_i \in A_1$, create a man $m_i \in M_p$, such that if $P(a_i)[l] = b_j$, then let $P(m_i)[l] = w_j$ for $1 \leq l \leq r$, where r is the length of a_i's list. So the preference list of man m_i is essentially the "same" as that of man a_i. For each man $a_i^T \in A_2$, with a preference list consisting of a single tie of length two, say (b_r, b_s), we create five men $m_i^T, m'_{i,r}, m''_{i,r}, m'_{i,s}, m''_{i,s}$, four women $w'_{i,r}, w''_{i,r}, w'_{i,s}, w''_{i,s}$ and four dogs $d'_{i,r}, d''_{i,r}, d'_{i,s}, d''_{i,s}$ where the preference list of m_i^T contains $w'_{i,r}$ and $w'_{i,s}$ in an arbitrary order, and the other preference lists are as shown below.

$$
\begin{array}{lll@{\qquad}lll@{\qquad}lll}
m'_{i,r} : & w'_{i,r} & w_r & w'_{i,r} : & d'_{i,r} & d''_{i,r} & d'_{i,r} : & m''_{i,r} & m_i^T \\
m''_{i,r} : & w''_{i,r} & & w''_{i,r} : & d''_{i,r} & d'_{i,r} & d''_{i,r} : & m'_{i,r} & m''_{i,r} \\
m'_{i,s} : & w'_{i,s} & w_s & w'_{i,s} : & d'_{i,s} & d''_{i,s} & d'_{i,s} : & m''_{i,s} & m_i^T \\
m''_{i,s} : & w''_{i,s} & & w''_{i,s} : & d''_{i,s} & d'_{i,s} & d''_{i,s} : & m'_{i,s} & m''_{i,s}
\end{array}
$$

We also add these agents to M_p, W_p and D_p, respectively. Note that in I'_p every set of agents has the same cardinality: $n_p = |M_p| = |W_p| = |D_p| = n + 4n_2$. The notions of *proper agent, proper partner* and *proper family* are defined in the obvious way.

The *additional part* of instance I' contains three subinstances. The *suitable part* of I' is the disjoint union of $3n_p$ copies of $R6$, such that the ith copy of $R6$, denoted $R6_i$, incorporates the ith agent of I'_p, as described in the previous reduction in the proof of Theorem 1 (we omit the full description of this process again). The new agents are referred to as *additional agents*.

Let $\mathcal{F}_s = \cup_{i\in\{1,\dots 3n_p\}}\mathcal{F}_{R6_i\setminus a_i}$ be the so-called *suitable matching* of the additional part, where a_i is the proper agent of $R6_i$. We call the set $C = \cup_{i\in\{1,\dots 3n_p\}}C_{R6_i\setminus a_i}$ *covered additional agents*, as these additional agents are covered by $\mathcal{F}_s$, and we call the set $U = \cup_{i\in\{1,\dots 3n_p\}}U_{R6_i\setminus a_i}$ *uncovered additional agents*, as these additional agents are not covered by $\mathcal{F}_s$.

The *fitting part* of I' is constructed on U as follows. Note that U has equal numbers of men, women and dogs. The fitting part consists of disjoint families that covers U, so that every agent has exactly one agent in his/her/its list, i.e. the fitting part is a complete matching of U, denoted by $\mathcal{F}_f$.

Finally, the *dummy part* is obtained by an arbitrary extension of the preference lists, so that by putting together the four subinstances, the proper and the three additional parts, we get the complete instance I'. The preferences of the agents over the partners in different parts respect the order in which we defined these parts: the list of a proper agent contains the proper partners first, then the suitable partners, and finally the dummy partners; the list of a covered additional agent contains the suitable partners first, then the dummy partners; the list of an uncovered additional agent contains the suitable partners first, then the fitting partner, and finally the dummy partners.

First we show that there is a one-to-one correspondence between the complete stable matchings of I and the complete strongly stable matchings of I'_p. The stability is preserved via the following one-to-one correspondence between the complete matchings of I and complete matchings of I':

$$(a_i, b_j) \in \mathcal{M} \iff (m_i, w_j, d_j) \in \mathcal{F}_p$$

$$(a_i^T, b_s) \in \mathcal{M} \iff (m_i^T, w'_{i,s}, d'_{i,s}), (m''_{i,s}, w''_{i,s}, d''_{i,s}), (m'_{i,s}, w_s, d_s) \in \mathcal{F}_p$$

$$(a_i^T, b_s) \notin \mathcal{M} \iff (m'_{i,s}, w'_{i,s}, d''_{i,s}), (m''_{i,s}, w''_{i,s}, d'_{i,s}) \in \mathcal{F}_p$$

Lemma 7. *A complete matching $\mathcal{M}$ of I is stable if and only if the corresponding complete matching $\mathcal{F}_p$ of I'_p is strongly stable.*

Proof. As a man a_i^T cannot belong to a blocking pair in I, it may be verified that his corresponding copy m_i^T cannot belong to a weakly blocking family in I_p either. Therefore, it is enough to show that a pair (a_i, b_j) is blocking for $\mathcal{M}$ if and only if the corresponding family (m_i, w_j, d_j) is blocking for $\mathcal{F}_p$. But this is obvious, because the preference lists of a_i and m_i are essentially the same, and the preference lists of b_j and d_j are also essentially the same. □

Now, given a matching $\mathcal{M}$ of I let us create the corresponding matching $\mathcal{F}$ of I' by adding $\mathcal{F}_s$ and $\mathcal{F}_f$ to $\mathcal{F}_p$, so $\mathcal{F} = \mathcal{F}_p \cup \mathcal{F}_s \cup \mathcal{F}_f$.

Lemma 8. *The instance I admits a complete stable matching $\mathcal{M}$ if and only if the reduced instance I' admits a strongly stable matching $\mathcal{F}$, where $\mathcal{F}$ is the corresponding matching of $\mathcal{M}$.*

Proof. Suppose that we have a complete stable matching $\mathcal{M}$ of I, and $\mathcal{F}$ is the corresponding matching in I'. Lemma 7 implies that every proper agent has a proper partner in $\mathcal{F}$ and no proper family is weakly blocking. Therefore, no proper agent can be involved in any weakly blocking family either. By construction of $\mathcal{F}_s$, every covered additional agent has a suitable partner in $\mathcal{F}$ and by Lemma 6, no suitable family is weakly blocking. Therefore, no such agent can be part of any weakly blocking family. Finally, every uncovered additional agent has a fitting partner in $\mathcal{F}$, so these agent cannot form a weakly blocking family either, since an uncovered additional agent prefers only suitable partners to fitting partners, which cannot be involved in a weakly blocking family. Hence $\mathcal{F}$ is strongly stable.

In the other direction, suppose that $\mathcal{F}$ is a strongly stable matching of I'. Every proper agent must have a proper partner, since otherwise if a_t had no proper partner in $\mathcal{F}$, then $\overline{R6_t}$ would contain a suitable weakly blocking family by Lemma 5. So the corresponding matching $\mathcal{M}$ in I is complete. The stability of $\mathcal{M}$ is a consequence of Lemma 7. Finally, we note that the additional agents must be matched in the unique strongly stable way in $\mathcal{F}$, namely, the covered additional agents must be covered by matching $\mathcal{F}_s$ by Lemma 6, and the uncovered additional agents must be covered by $\mathcal{F}_f$, since otherwise a fitting family would weakly block $\mathcal{F}$. Therefore, we have a one-to-one correspondence as was claimed. $\square$

Theorem 2. *Determining the existence of a strongly stable matching in a given instance of cyclic* 3DSM *is NP-complete.*

4 Stable exchanges with restrictions

Problem definition

Given a simple digraph $D = (V, A)$, where V is the set of agents, suppose that each agent has exactly one indivisible good, and $(i, j) \in A$ if the good of agent j is suitable for agent i. An *exchange* is a permutation π of V such that, for each $i \in V$, $i \neq \pi(i)$ implies $(i, \pi(i)) \in A$. Alternatively, an exchange can be considered as a disjoint packing of directed cycles in D.

Let each agent have strict preferences over the goods, that are suitable for him. These orderings can be represented by preference lists. In an exchange π, the agent i receives the good of his *successor*, $\pi(i)$; therefore the agent i prefers an exchange π to another exchange σ if he prefers $\pi(i)$ to $\sigma(i)$. An exchange π is *stable* if there is no *blocking coalition* B, i.e. a set B of agents and a permutation σ of B where every agent $i \in B$ prefers σ to π. An exchange is *strongly stable* is there exists no *weakly blocking coalition* B with a permutation σ of B where for every agent $i \in B$, either $\sigma(i) = \pi(i)$ or i prefers σ to π, and $\sigma(i) \neq \pi(i)$ for at least one agent $i \in B$.

Complexity results about stable exchanges

Shapley and Scarf [25] showed that the stable exchange problem is always solvable and a stable exchange can be found in polynomial time by the Top Trading Cycle (TTC) algorithm, proposed by Gale. Moreover, Roth and Postlewaite [18] proved that the exchange obtained by the TTC algorithm is strongly stable and this is the only such solution. We note that they considered this problem as a so-called *houseswapping game*, where a *core element* corresponds to a stable solution. (For further details about these connections with Game Theory, see [3].)

In some applications the length of the possible cycles is bounded by some constant l. In this case we consider an *l-way exchange problem.* Furthermore, the size of the possible blocking coalitions can also be restricted. We say that an exchange is *b-way stable* if there exists no blocking coalition of size at most b. Because of some applications, the most relevant problems are for constants 2 and 3. Henceforth we also refer to "2-way" as "pairwise" in the context of cycle lengths and blocking coalitions sizes. We remark that if $b = l$ then a stable exchange corresponds to a core-solution of some related NTU-game, because the possible coalitions, those that can form and those that can block, are the same (see [3] for details).

For $l = b = 2$, the pairwise stable pairwise exchange problem is in fact, equivalent to the *stable roommates problem.* Therefore, a stable solution may not exist [9], but there is a polynomial-time algorithm that finds a stable solution if one does exist [11] or reports that none exists. For $l = b = 3$, the 3-way stable 3-way exchange problem is NP-hard, even for three-sided directed graphs, as is stated by the following theorem.

Theorem 3. *The 3-way stable 3-way exchange problem for tripartite directed graphs is equivalent to the cyclic 3DSMI problem, and is therefore NP-complete.*

Finally, we note that Irving [12] proved recently that the stable pairwise exchange and the 3-way stable pairwise exchange problems are NP-hard. The pairwise stable 3-way exchange problem is open. This particular problem can be a relevant regarding the application of kidney exchanges, next described.

Kidney exchange problem

Living donation is the most effective treatment that is currently known for kidney failure. However a patient who requires a transplant may have a willing donor who cannot donate to them for immunological reasons. So these incompatible patient-donor pairs may want to exchange kidneys with other pairs. Kidney exchange programs have already been established in several countries such as the Netherlands [13] and the USA [20].

In most of the current programs the goal is to maximise the number of patients that receive a suitable kidney in the exchange [21, 22, 23, 1] by regarding only the eligibility of the grafts. Some more sophisticated variants consider also the difference between suitable kidneys. Sometimes the "total benefit" is maximised [24], whilst other models [19, 6, 7, 4] require first the stability of the solution under various criteria.

The length of the cycles in the exchanges is bounded in the current programs, because all operations along a cycle have to be carried out simultaneously. Most programs allow only pairwise exchanges. But sometimes 3-way exchanges are also possible, like in the New England Program [16] and in the National Matching Scheme of the UK [27][2]. In these kind of applications, if one considers stability as the first priority of the solution, then we obtain a 3-way stable 3-way exchange problem, where the incompatible patient-donor pairs are the agents and their preferences are determined according to the special parameters of the suitable kidneys.

Finally, we remark that although the induced digraph of a real kidney exchange instance may have special properties (see e.g. [22] about the effect of the blood-types on the digraph) the problem remains hard, even for realistic situations. For example, if we have three sets of patient-donor pairs with blood types O-A, A-B and B-O, then the digraph may appear to be tripartite. But this particular case of the 3-way stable 3-way exchange problem is also hard by Theorem 3.

5 Further questions

For cyclic 3DSMI, the smallest instance that admits no stable matching given here satisfies $n = 6$. Is there an even smaller counterexample? In the case of strong stability, we are aware of instances of cyclic 3DSM for $n = 4$ that admit no strongly stable matching.

The main questions that remain unsolved are (i) whether there exists an instance of cyclic 3DSM that admits no stable matching, and (ii) whether there is a polynomial-time algorithm to find such a matching or report that none exists, given an instance of cyclic 3DSM.

[2] 3-way exchanges may be also allowed in the national program of the USA (as it is declared to be a goal of the system in the future in the Proposal for National Paired Donation Program [28]).

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