A geometric approach to judgment aggregation

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Abstract

Don Saari has developed a geometric approach to the analysis of paradoxes of preference aggregation such as the Condorcet paradox or Arrow's general possibility theorem. In this paper we extend this approach to judgment aggregation. In particular we use Saari's representation cubes to provide a geometric representation of profiles and majority rule outcomes. We then show how profile decompositions can be used to derive restrictions on profiles that guarantee logically consistent majority outcomes. Moreover, we use our framework to determine the likelihood of inconsistencies. Finally, current distance-based approaches in judgment aggregation are discussed within our framework.

1 Introduction

The problem of judgment aggregation consists in aggregating individual judgments on an agenda of logically interconnected propositions into a collective set of judgments on these propositions. This relatively new literature (see List and Puppe [6] for a survey) is centred on problems like the discursive dilemma which are structurally similar to paradoxes and problems in social choice theory like the Condorcet paradox and Arrow's general possibility theorem. For the analysis of such paradoxes Saari [9] has successfully introduced a geometric approach, the extension of which to judgment aggregation seems promising.

A major difference of judgment aggregation to social choice theory lies in the representation of the information involved. While binary relations over a set of alternatives are a canonical representation of preferences, a natural representation of judgments are binary valuations over a set of propositions, where the logical interconnections between these propositions determine the set of admissible valuations. E.g. the agenda of the famous discursive dilemma $\{p, q, p \wedge q\}$ is associated the set of admissible, i.e. logically consistent valuations $\{(0,0,0),(1,0,0),(0,1,0),(1,1,1)\}$.

In this paper we want to use Saari's tools to derive and analyse results in judgment aggregation. In section 2 we introduce the formal framework. Section 3 uses Saari's representation cubes to provide a unified geometric representation of profiles and majority rule outcomes. This will clarify which problems can occur in judgment aggregation using majority rule on certain domains. Applying Saari's idea of a profile decomposition, we also show how majority inconsistencies can be avoided with the help of restrictions on the distribution of individual valuations, i.e. a kind of generalized domain restriction. This leads us to the determination of the likelihood of inconsistencies under majority rule for different agendas in section 4. In section 5, we apply our approach to illuminate current results on distance-based judgment aggregation. Finally, section 6 concludes the paper.

2 Formal Framework

Let J be the set of propositions on which judgments have to be made. Most problems in the literature on judgment aggregation can be formulated with the help of vectors of binary valuations $x = (x_1, x_2, ..., x_{|J|}) \in X \subseteq \{0,1\}^{|J|}$, where $x^j = 1$ means that proposition j is believed and X denotes the set of all admissible (logically consistent) valuations (see Dokow and Holzman [2]).

A profile of individual valuations is represented by a vector $\mathbf{p} = (p_1, p_2, ..., p_{|X|})$ which associates with every binary valuation $x_k \in X$ the fraction p_k of individuals with this valuation. This is an anonymous representation of voters' valuations as only the distribution of the valuations matters.

A judgment aggregation rule is a mapping f that associates with every profile $\mathbf{p} = (p_1, p_2, ..., p_{|X|})$ a valuation $f(\mathbf{p}) \in \{0,1\}^{|J|}$. If $f(\mathbf{p}) \in X$ for all $\mathbf{p}$ we will call its domain f-consistent.

Saari [9] analyses preference aggregation using a geometric approach. For the simplest setting consider three alternatives a, b, c. This gives rise to three issues, i.e. pairwise comparisons, namely between a and b, b and c and c and a. A "1" for the first issue (i.e. the comparison between a and b) means that a is preferred to b, written $a \succ b$. On the other hand, a "0" indicates the opposite preference, i.e. $b \succ a$. Defining the average support for issue j by $x_j = \frac{\sum_{i \in N} x_j^i}{n}$, where N denotes the set of individuals, a preference profile maps into a point $x \in [0,1]^{|J|}$ in the hypercube with dimension $|J| = 3$ (the number of issues, i.e. pairwise comparisons). See figure 1.

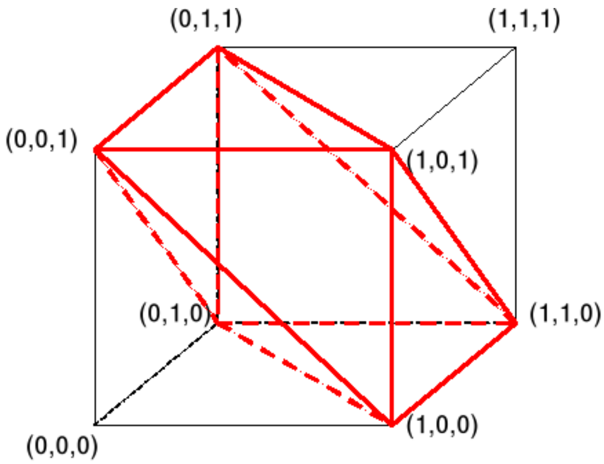

Figure 1: Saari's representation cube

In figure 1, the vertex $(0,0,1)$ thus represents the preference where $b \succ a$, $c \succ b$ and $c \succ a$ or - for simplicity - the ranking *cba*. As there are eight vertices but only six transitive rankings of the three alternatives, there are two vertices representing irrational voters with cyclic preferences, namely $(0,0,0)$ and $(1,1,1)$. If we exclude those vertices, we see that the convex hull of the remaining six vertices is the representation polytope, i.e. every preference profile maps into a point in this polytope.

3 Majority (In)consistency and Domain Restrictions

The same 3-dimensional hypercube can be used for a simple judgment aggregation problem with 3 propositions (issues), i.e. $|J| = 3$. For simple majority voting on the issues, every profile $\mathbf{p}$ of individual judgments on J is mapped into a point $x(\mathbf{p})$ in the hypercube. Its Euclidean distance to the respective vertices determines the majority outcome. This means that the hypercube can be partitioned into 8 equally sized subcubes each determining the majority outcome for profiles mapped into those subcubes. E.g. in figure 2 the shaded subcube, determined by the diagonal $\left[(\frac{1}{2},\frac{1}{2},\frac{1}{2}),(1,0,1)\right]$ consists of all points that are of closest Euclidean distance to the vertex $(1,0,1)$ and hence any $x(\mathbf{p})$ in that subcube leads to a majority outcome of $(1,0,1)$. For $d_E(x,y)$ denoting the Euclidean distance between $x,y \in \{0,1\}^{|J|}$, we can also think of the majority valuation x^M as the

$$argmin_{x\in\{0,1\}^{|J|}} \sum_{k=1}^{|X|} p_k d_E(x^k,x)$$

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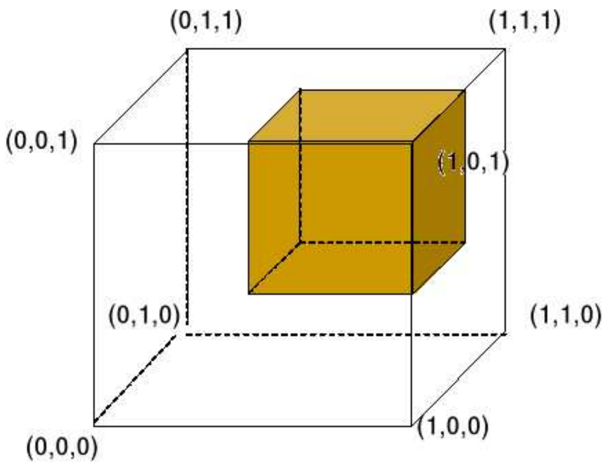

Figure 2: Majority subcube

Consider the agenda $\{p,q,p\wedge q\}$ of the discursive dilemma with associated domain of admissible valuations $X=\{(0,0,0),(1,0,0),(0,1,0),(1,1,1)\}$. The four admissible vertices in the hypercube determine the representation polytope as seen in figure 3.

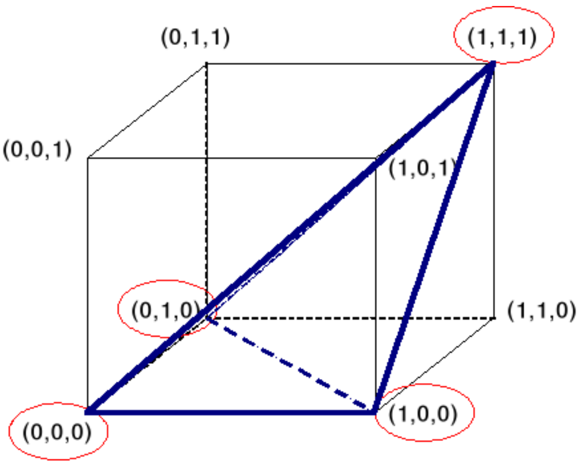

Figure 3: representation polytope

Given X, consider the profile $\mathbf{p}=(0,\frac{1}{3},\frac{1}{3},\frac{1}{3})$, i.e. no voter has valuation $(0,0,0)$, one third of the voters has valuation $(1,0,0)$, and so on. As this maps into the point $x=(\frac{2}{3},\frac{2}{3},\frac{1}{3})$ - a point whose closest vertex is $(1,1,0)$ - the representation polytope obviously passes through one subcube representing a majority outcome not in the domain. That such an inconsistency can easily occur in general is seen from the following lemma:

Lemma 1 *Given any vertex $x \in \{0,1\}^{|J|}$, there exist 3 vertices a, b, c such that for some linear combinations of those vertices there is a point in the x-subcube.*

For $|J| = 3$, these 3 vertices necessarily need to be the neigbors of that vertex, i.e. they are only allowed to differ from it in one issue. Given that, we can now provide a simple result for the occurrence of majority consistency, i.e. what X needs to look like to guarantee that the majority outcomes are themselves in X.

Proposition 1 *For $|J| = 3$, the set of valuations X is majority consistent iff for any triple of vertices in the domain with a common neighbor, this common neighbor is also contained in the domain.*

To analyse those paradoxical outcomes and suggest restrictions to overcome those, we will use a profile decomposition technique developed by Saari [9]. Consider two indviduals with the respective valations $(1,0,0)$ and $(0,1,1)$. They are exact opposites, so from a majority rule point of view those two valuations cancel out. Hence this implies that for any two opposite admissible valuations in X, we can cancel the valuation held by the smaller number of individuals. This leads to a reduced profile, the majority outcome of which is identical to the majority outcome of the original profile.

E.g. given $X = \{(0,0,0),(1,0,0),(0,1,0),(1,1,1)\}\}$, with (p_1, p_2, p_3, p_4) being the shares of individuals holding each of the respective valuations where $\sum_i p_i = 1$. As $(0,0,0)$ and $(1,1,1)$ are exact opposites, the reduced profile will have a share of 0 for the valuation held by the smaller number of individuals. In the case of $p_1 > p_4$ such a reduced profile will be $\mathbf{p}' = (\frac{p_1 - p_4}{p_1+p_2+p_3-p_4}, \frac{p_2}{p_1+p_2+p_3-p_4}, \frac{p_3}{p_1+p_2+p_3-p_4}, 0)$, in the case of $p_1 \leq p_4$ we can create the reduced profile accordingly. Hence the reduced profile maps into one of the following two planes represented in figure 4, namely either into the one determined by the vertices $(0,0,0)$, $(1,0,0)$ and $(0,1,0)$ or the one determined by the vertices $(1,0,0)$, $(0,1,0)$ and $(1,1,1)$.

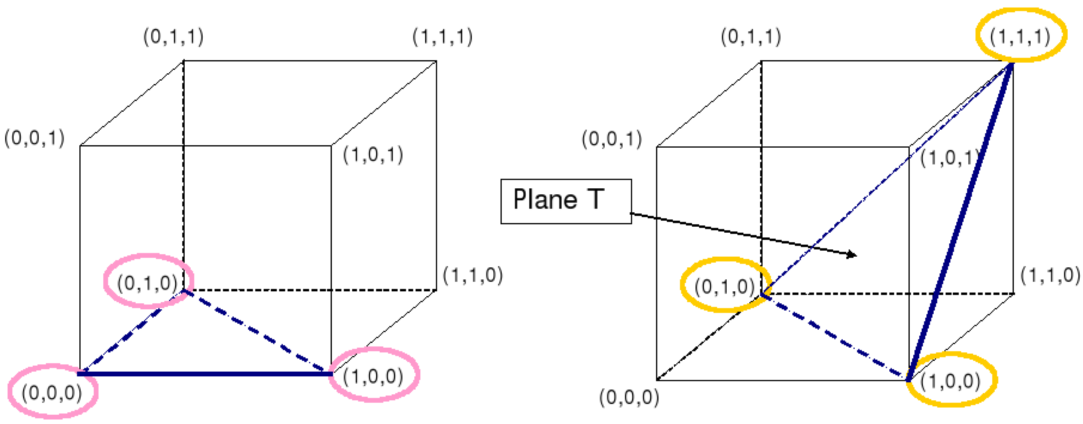

Figure 4: planes

Now we want to determine whether there are consistency conditions, i.e. what sort of profiles do guarantee majority consistency in the sense of a majority outcome being one of the valuations in X. Let $a_i = \frac{p_i}{\sum_{i=2}^{4} p_i}$, for $i = 2,3,4$. Then $\alpha = (a_2 + a_4, a_3 + a_4, a_4) \in T$. By definition, $\mathbf{p} = (1-p_1)(0, a_2, a_3, a_4) + p_1(1,0,0,0)$. By linearity, $x(\mathbf{p}) = (1-p_1)\alpha +$

$p_1(0,0,0) = (1-p_1)\alpha$, where $x(\mathbf{p}) \in [0,1]^3$ is a vector summarizing the average support for each issue.

So, geometrically any profile can be plotted via a point in the plane T, its connection to the $(0,0,0)$ vertex and a weight p_1. The following figure 5 shows plane T, the shaded area of which indicates the cut with the $(1,1,0)$-subcube and hence those points where a profile leads to an inadmissible majority outcome.

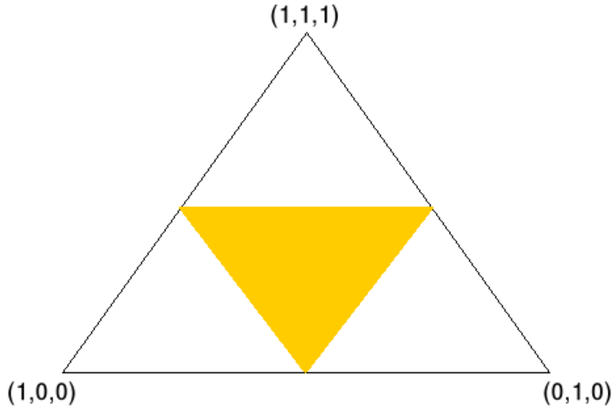

Figure 5: plane T

Now, we can state the following proposition:

Proposition 2 *Given* $X = \{(0,0,0),(1,0,0),(0,1,0),(1,1,1)\}\}$ *and* $\mathbf{p} = (p_1, p_2, p_3, p_4)$ *with* $\sum_{i=1}^4 = 1$*, satisfying one of the following restrictions on the distribution of* $\mathbf{p}$ *is necessary and sufficient for admissible majority outcomes:*

(a) $p_1 \geq p_4$

(b) $\frac{p_2}{\sum_{i=2}^4 p_i} > \frac{1}{2}$

(c) $\frac{p_3}{\sum_{i=2}^4 p_i} > \frac{1}{2}$

(d) $\frac{p_4}{\sum_{i=2}^4 p_i} > \frac{1}{2}$

Proof. If $p_1 \geq p_4$, the reduced profile is mapped into the plane indicated on the left of figure 4 which is closed under majority rule. Otherwise $p_1 < p_4$ and the reduced profile is mapped into plane T. As we see on the right side of figure 4, majority consistency is guaranteed in the white triangles, and a profile maps into one of them whenever one of restrictions $(b)-(d)$ is satisfied. On the other hand, given that for $|X| = 4$ and $|J| = 3$, elementary algebra shows that every profile maps into one and only one point in the representation polytope. Hence, majority consistency is satisfied only if one of the above restrictions is fulfilled. ∎

One interesting feature of those restrictions is that they are based on the space of profiles which is more general than restrictions on the space of valuations which is usually used in classical domain restrictions. E.g. List [5] introduces the unidimensional alignment domain which has a certain resemblence to Black's single peakedness condition in social choice theory. It requires individuals to be ordered from left to right such that on each issue there occurs only one switch from acceptance to non-acceptance (or vice versa). For $|J| = 3$ a unidimensional alignment domain does satisfy one of the above conditions for admissible majority outcomes.[1]

Moreover, this framework also opens the analysis of various paradoxical situations, e.g. strong support for one particular issue but still inadmissible majority outcomes. This is stated in the following proposition:

Proposition 3 *There exist profiles such that there is almost unanimous agreement on one issue and still an inadmissible majority outcome is obtained.*

Proof. Looking at figure 5 one observes, that points close to the edge connecting the vertices $(1,0,0)$ and $(1,1,1)$ have almost unanimous agreement on issue 1. However, at the midpoint of this edge, the shaded triangle comes arbitrarily close to the edge. Hence, there exist profiles which lie in the shaded triangle but imply almost unanimous agreement on one issue. The same argument applies to points close to the edge conncecting the vertices $(0,1,0)$ and $(1,1,1)$. ■

4 Likelihood of Inconsistency

The geometric framework can also be used to analyze the likelihood of inadmissible outcomes. The approach is based on the fact that only 4 vertices are admissible individual valuations, and hence any majority outcome in the representation cube is determined by a unique profile. Consider again the situation $X = \{(0,0,0),(1,0,0),(0,1,0),(1,1,1)\}\}$. Then for any vector of shares of individual valuations $\mathbf{p} = (p_1, p_2, p_3, p_4)$ we get the following for the average support on each issue: $x_1 = p_2 + p_4$, $x_2 = p_3 + p_4$, $x_3 = p_4$, $1 = p_1 + p_2 + p_3 + p_4$. As those are 4 equations with 4 unknowns there exists a unique solution. Thus, assuming every profile being equally likely - i.e. taking an impartial anonymous culture[2] - the volume of certain subspaces now indiciates the likelihood of occurrence of certain outcomes. Consider first the volume of the representation cube: $V = \frac{1}{2} \cdot 1 \cdot \frac{1}{3} = \frac{1}{6}$. On the other hand, points leading to inadmissible majority outcomes are located in the tetraeder determined by the points $\left[(\frac{1}{2},\frac{1}{2},0),(1,\frac{1}{2},\frac{1}{2}),(\frac{1}{2},1,\frac{1}{2}),(\frac{1}{2},\frac{1}{2},\frac{1}{2})\right]$. The volume of this tetraeder is $\frac{1}{48}$ (see figure 6).

So its volume relative to the volume of the whole representation polytope is $\frac{1}{8}$ and hence we can say that the probability of an outcome being inadmissible is 12,5 percent. This provides a different approach to derive the expected probability of paradoxical situations under the impartial anonymous culture compared to the non-geometric approach by List [5], leading to the same results.

Of course, different domains allow for different probabilities. E.g. consider the agenda $\{p, q, p \leftrightarrow q\}$ with $X = \{(0,0,1),(1,0,0),(0,1,0),(1,1,1)\}\}$. Then, for any point $x = (x_1, x_2, x_3)$ in the representation polytope we get $x_1 = p_2 + p_4$, $x_2 = p_3 + p_4$, $x_3 = p_1 + p_4$ and $p_1 + p_2 + p_3 + p_4 = 1$. Again, every profile maps into a unique point in the representation polytope. Making the same volume calculations as before, we get - under the impartial anonymous culture - a probability of inadmissible outcomes of 25 percent.

5 Codomain Restrictions and Distance-Based Aggregation

Besides restrictions on the space of profiles, there is an alternative way to guarantee logical consistency at the collective level, namely via restricting the set of collective outcomes

[1] For a more elaborated discussion on majority voting on restricted domains see also Dietrich and List [1].

[2] See Gehrlein [3] for a general discussion of the impartial anonymous culture.

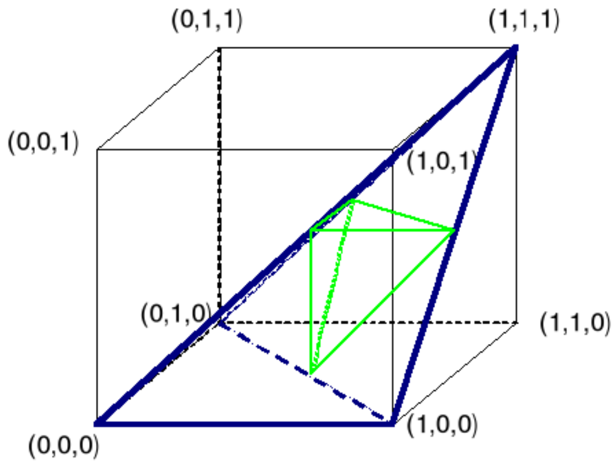

Figure 6: Distances

to admissible valuations. One way to work with such codomain restrictions is by using distance-based aggregation rules. In analogy to a well-known procedure in social choice theory (Kemeny [4]), Pigozzi [8] introduced such an approach to judgment aggregation. In principle a distance-based aggregation rule determines the collective valuation as the valuation that minimizes the sum of distances to the individual valuations. Formally, given the profile of individual valuations $(x^1, x^2, ..., x^n)$, the collective valuation is the admissible valuation $x \in X$ that minimizes the sum of distances to the individual valuations, i.e.

$$f(\mathbf{p}) = argmin_{x \in X} \sum_{i=1}^{n} d(x, x^i)$$

The most commonly used distance function is the Hamming distance, which counts the number of issues on which two valuations disagree, i.e. for $x = (1, 0, 0)$ and $x' = (1, 1, 1)$, $d(x, x') = 2$.

It is easily seen that on a majority consistent domain this distance-based aggregation rule coincides with majority voting on issues, thus providing a metric rationalization of majority voting.

Now given our geometric approach, there is a simple geometric explanation of this distance-based aggregation rule. As could be seen in figure 5, all problematic profiles lead to a point in the shaded triangle. However, one option is to divide the triangle into three sub-triangles as in figure 7.

The point in the middle is exactly the barycenter point of the triangle $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. Using these additional lines, we now have divided the triangle into three areas, points in which are characterized by being of smallest Euclidean distance to the vertex of the proper triangle w.r.t. the points within the shaded triangle. So points in the south-western part of the shaded triangle will be closest to the $(1, 0, 0)$ vertex. This, however, is identical to saying that for any point in the shaded triangle, switch the majority valuation on the issue which is closest to the 50-50 threshold (see Merlin and Saari [7]).

Example 1 *Let* $X = \{(0, 0, 0), (1, 0, 0), (0, 1, 0), (1, 1, 1)\}$ *and* $\mathbf{p} = (0.1, 0.35, 0.3, 0.25)$. *This leads to* $x(\mathbf{p}) = (0.6, 0.55, 0.25)$ *and hence an inadmissible majority outcome* $(1, 1, 0)$. *Looking at figure 5 we see that* $\alpha \in T$ *lies in the south-western shaded triangle. Thus, according to our distance-based aggregation rule, the outcome will be the admissible valuation* $(1, 0, 0)$

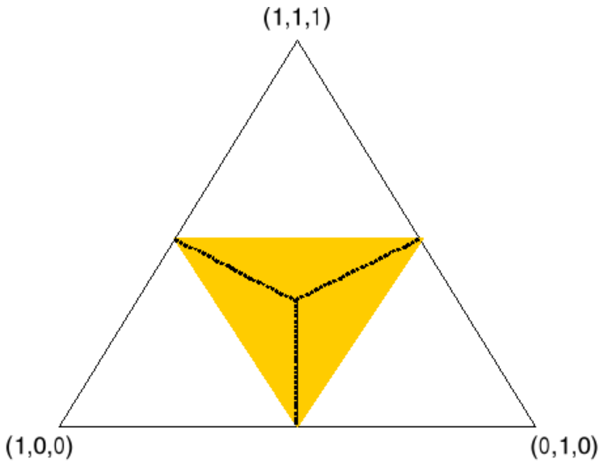

Figure 7: Distances

as α *is closest to the* $(1, 0, 0)$ *vertex. However, this can also be seen as switching the valuation on the issue which is closest to the 50-50 threshold, which - in* $x(\mathbf{p})$ *- is obviously issue 2.*

6 Conclusion

In this paper we have shown how geometry can be used to analyse results in judgment aggregation, such as majority inconsistencies and distance based aggregation rules, and to determine new results such as guaranteeing majority consistency via restrictions on distributions of individual valuations and determining the likelihood of inconsistencies.

Most of the stated results do not easily extend to more than three issues because of problems of dimensionality. E.g. an agenda with three propositions and their conjunction, like $\{p, q, r, p \wedge q \wedge r\}$, leads to eight admissible valuations, i.e. eight vertices out of the 16 vertices in the four-dimensional hypercube. The extensions of our (domain) restrictions and calculations of the likelihood of the occurrence of paradoxes to those higher dimensions are not obvious and need further work.

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