Dodgson's Rule
Approximations and Absurdity

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Abstract

With the Dodgson rule, cloning the electorate can change the winner, which Young (1977) considers an "absurdity". Removing this absurdity results in a new rule (Fishburn, 1977) for which we can compute the winner in polynomial time (Rothe et al., 2003), unlike the traditional Dodgson rule. We call this rule DC and introduce two new related rules (DR and D&). Dodgson did not explicitly propose the "Dodgson rule" (Tideman, 1987); we argue that DC and DR are better realizations of the principle behind the Dodgson rule than the traditional Dodgson rule. These rules, especially D&, are also effective approximations to the traditional Dodgson's rule. We show that, unlike the rules we have considered previously, the DC, DR and D& scores differ from the Dodgson score by no more than a fixed amount given a fixed number of alternatives, and thus these new rules converge to Dodgson under any reasonable assumption on voter behaviour, including the Impartial Anonymous Culture assumption.

1 Introduction

Finding the Dodgson winner to an election can be very difficult, Bartholdi et al. (1989) proved that determining whether an alternative is the Dodgson winner is an NP-hard problem. Later Hemaspaandra et al. (1997) refined this result by showing that the Dodgson winner problem was complete for parallel access to NP, and hence not in NP unless the polynomial hierarchy collapses. This result was of interest to computer science as the previously known problems in this complexity class were obscure by comparison.

For real world elections we do not want intractable problems. Tideman (1987) proposed a simple rule to approximate the Dodgson rule. The impartial culture assumption states that all votes are independent and equally likely. Under this assumption, it has been proven that the probability that Tideman's rule picks the Dodgson winner converges to one as the number of voters goes to infinity (M^{c}Cabe-Dansted et al., 2007). Our paper also showed that this growth was not exponentially fast. Two rules have been independently proposed for which this convergence is exponentially fast, our Dodgson Quick (DQ) rule and the `GreedyWinner` algorithm proposed by Homan and Hemaspaandra (2005). It is usually easy to verify that the DQ winner and GreedyWinner are the same as the Dodgson winner, a property that Homan and Hemaspaandra formalise as being "frequently self-knowingly correct". However the proofs of convergence depended heavily on the unrealistic impartial culture assumption. We shall show that under the Impartial Anonymous Culture (IAC) these rules do not converge.

The importance of ensuring that statistical results hold on reasonable assumptions on voter behaviour is considered by Procaccia and Rosenschein (2007). They define "deterministic heuristic polynomial time algorithm" in terms of both the problem to be solved and a probability distribution over inputs. However they do not consider the issue of whether a heuristic is self-knowingly correct. Thus we extend the concept of an algorithm being "frequently self-knowingly correct" to allow particular probability distributions to be specified.

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Procaccia and Rosenschein (2007) also propose "Junta" distributions; these distributions are intended to produce problems that are harder than would be produced under reasonable assumptions on voter behaviour. Hence if it is easy to solve a problem when input is generated according to a Junta distribution it is safe to assume that it will be easy under any reasonable assumption of voter behaviour. We will not use Junta distributions, but instead simply use that fact that even "neck-and-neck" national elections are won by thousands of votes. We will also show that the rules we have considered previously, Tideman, Dodgson Quick etc., do not converge to Dodgson's rule under IAC.

However, the reason that the Dodgson rule is so hard to compute is because cloning the electorate can change the winner. That is, if we replace each vote with two (or more) identical votes, this may change the winner. When discussing majority voting Young (1977) described this property as an "absurdity". Young suggested that such absurdities be fixed in majority voting by allowing fractions of a vote to be deleted. Fishburn (1977) proposed a similar modification to the traditional Dodgson rule, which we call Dodgson Clone. The Dodgson Clone scores can be computed by relaxing the integer constraints on the Integer Linear Program that Bartholdi et al. (1989) proposed to calculate the Dodgson score; normal (rational) Linear Programs can be solved in polynomial time, we can compute the Dodgson Clone score in polynomial time (Rothe et al., 2003).

The Dodgson Clone rule is also an effective approximation to the Dodgson rule. In computer science, an approximation typically refers to an algorithm that selects a value that is always accurate to within some error. This form of approximation is not meaningful when selecting a winner, although these rules can be used to approximate the frequency that Dodgson winner has some property. For example, Shah (2003) used Tideman's rule to approximate the frequency that the Dodgson winner matched the winners according to other rules, and so such Tideman, DQ etc. can be considered approximations of Dodgson's rule in a loose sense. However we can approximate the Dodgson score. We will show that for a fixed number of alternatives, the Dodgson Clone score approximates the Dodgson score to within a constant error. As it is implausible that the margin by which the winner wins the election will not grow with the size of the electorate, the Dodgson winner will converge to the Dodgson Clone winner under any reasonable assumption of voter behaviour. In particular we will show that they will converge under the Impartial Anonymous Culture assumption.

We propose two closer approximations Dodgson Relaxed (DR) and Dodgson Relaxed and Rounded (D&). These approximations, like the traditional Dodgson rule, are not resistant to cloning the electorate. This allows them to be closer to the Dodgson rule than Dodgson Clone, both rules converge to the Dodgson rule exponentially quickly under the Impartial Culture assumption. The DR rule is superior to the Dodgson rule in the sense that it can split ties in favour of alternatives that are fractionally better. The D& scores are rounded up, so the D& rule does not have this advantage. However it is exceptionally close to Dodgson. In 43 million elections randomly generated according to various assumptions on voter behaviour, the D& winner differed from the Dodgson winner in only one election.

The approximation proposed by Procaccia et al. (2007) is similar to these approximations in the sense that it involves a relaxation of the integer constraints. However their approximation is randomised, and thus quite different from our deterministic approximations. Using a randomised approximation as a voting rule would be unusual, and they do not discuss the merits of such a rule. Thus the focus of their paper is quite different, as we present rules that we argue are superior to the traditional formalisation of the Dodgson rule. Additionally, they do not discuss the issue of frequently self-knowing correctness.

Another approach to computing the Dodgson score has been to limit some parameter. Bartholdi et al. (1989) showed that computing the Dodgson scores and winner is polynomial when either the number of voters or alternatives is limited. It was shown that computing these from a voting situation is logarithmic with respect to the number of voters when the number of alternatives is fixed (M^{c}Cabe-Dansted, 2006), and hence Dodgson winner is Fixed Parameter Tractable (FPT) with number of alternatives as the fixed parameter. It is now also known that the Dodgson winner is FPT when the Dodgson score is taken as the fixed parameter (Betzleri et al., 2008).

Thus we will define the Dodgson based rules in terms of Condorcet-tie winners, rather than Condorcet winners. As we will discuss briefly, this does not affect convergence.

2 Preliminaries

In our results we use the term agent in place of voter and alternative in place of candidate, as not all elections are humans voting other humans into office. For example, in direct democracy, the citizens vote for laws rather than candidates.

We assume that agents' preferences are transitive, i.e. if they prefer a to b and prefer b to c they also prefer a to c. We also assume that agents' preferences are strict, if a and b are distinct they either prefer a to b or b to a. Thus we may consider each agent's preferences to be a ranking of each alternative from best to worst.

Let $\mathcal{A}$ and $\mathcal{N}$ be two finite sets of cardinality m and n respectively. The elements of $\mathcal{A}$ will be called alternatives, the elements of $\mathcal{N}$ agents. We represent a **vote** by a linear order of the m alternatives. We define a **profile** to be an array of n votes, one for each agent. Let $\mathcal{P} = (P_1, P_2, \ldots, P_n)$ be our profile. If a linear order $P_i \in \mathcal{L}(A)$ represents the preferences of the i^{th} agent, then by aP_ib, where $a, b \in \mathcal{A}$, we denote that this agent prefers a to b.

A multi-set of linear orders of $\mathcal{A}$ is called a **voting situation**. A voting situation specifies which linear orders were submitted and how many times they were submitted but not who submitted them.

The Impartial Culture (IC) assumption is that each profile is equally likely. The Impartial Anonymous Culture (IAC) assumption is that each voting situation is equally likely. To understand the difference, consider a two alternative election with billions of agents; under IC it is almost certain that each alternative will get 50% (±0.5%) of the vote; under IAC, 50.0% is no more likely than any other value.

Let $\mathcal{P} = (P_1, P_2, \ldots, P_n)$ be our profile. We define n_{xy} to be the number of linear orders in $\mathcal{P}$ that rank x above y, i.e. $n_{xy} \equiv \#\{i \mid xP_iy\}$.

Definition 2.1. *The **advantage** of a over b is defined as follows:*

$$adv(a, b) = \max(0, n_{ab} - n_{ba})$$

*A **Condorcet winner** is an alternative a for which $adv(a, b) > 0$ for all other alternatives b. We define a **Condorcet-tie winner**, to be an alternative a such $adv(b, a) = 0$ for all other alternatives a. A Condorcet winner or Condorcet-tie winner does not always exist.*

It is traditional to define the Dodgson score of an alternative as the terms of the minimum number of swaps of neighbouring alternatives required to make that alternative *defeat* all others in pairwise elections, i.e. make the alternative a Condorcet winner. When not requiring solutions to be integer this becomes undefined, as if we defeat an alternative by $\epsilon > 0$ then there exists a better solution where we defeat the alternative by only $\epsilon/2$.

For this reason, when defining the Dodgson scores we only require that the alternative defeat *or tie* other alternatives, i.e. make the alternative a Condorcet-tie winner. For better consistency with the more traditional Dodgson rule we could define the Condorcet winner as an alternative a for which $\text{adv}(a, b) \geq 1$. However this would mean that the Dodgson Clone rule would not be resistant to cloning of the electorate.

This difference in definition does not affect convergence. Our proof of convergence relies only on fact that Dodgson, D&, DR and DC scores differ by at most a fixed amount

($\mathcal{O}(m!)$) when the number of alternatives is fixed. To convert a Condorcet-tie winner c into a Condorcet winner c we need to swap c over at most $(m-1)$ alternatives, each requiring at most $(m-1)$ swaps of neighbouring alternatives. Hence the difference between the score according to these different definitions of Dodgson is at most $(m-1)^2$.

We will now define a number of rules in terms of scores. The winner of each rule below is the alternative with the lowest score.

The **Dodgson score** (Dodgson 1876, see e.g. Black 1958; Tideman 1987), which we denote as $\mathrm{Sc}_{\mathbf{D}}(a)$, of an alternative a is defined as the minimum number of swaps of neighboring alternatives required to make a a Condorcet-tie winner. We call the alternative(s) with the lowest Dodgson score(s) the **Dodgson winner**(s). (Bartholdi et al., 1989)

The **Tideman score** $\mathrm{Sc}_{\mathbf{T}}(a)$ of an alternative a is:

$$\mathrm{Sc}_{\mathbf{Q}}(a) = \sum_{b \neq a} \mathrm{adv}(b, a).$$

The **Dodgson Quick (DQ) score** $\mathrm{Sc}_{\mathbf{Q}}(a)$, of an alternative a is

$$\mathrm{Sc}_{\mathbf{Q}}(a) = \sum_{b \neq a} F(b, a), \text{ where } F(b, a) = \left\lceil \frac{\mathrm{adv}(b, a)}{2} \right\rceil.$$

Although the definitions of $\mathrm{Sc}_{\mathbf{Q}}$ and $\mathrm{Sc}_{\mathbf{T}}$ are very similar, the Dodgson Quick rule converges exponentially fast to Dodgson's rule under the Impartial Culture assumption, where as Tideman's rule does not (M^cCabe-Dansted et al., 2007). This is because Dodgson and DQ are more sensitive to a large number of alternatives defeating a by a small odd margin (e.g. 1) than Tideman is.

We define the k-Dodgson score $\mathrm{Sc}^k_{\mathbf{D}}(d)$ of a as being the Dodgson score of a in a profile where each agent has been replaced with k clones, divided by k. That is, where $\mathcal{P}$ is our fixed profile, $\mathcal{P}^k$ is the profile with each agent replaced with k clones, and $\mathrm{Sc}_{\mathbf{D}}[\mathcal{P}](a)$ is the Dodgson score of a in the profile $\mathcal{P}$, then

$$\mathrm{Sc}^k_{\mathbf{D}}(a) = \mathrm{Sc}^k_{\mathbf{D}}[\mathcal{P}](a) = \frac{\mathrm{Sc}_{\mathbf{D}}[\mathcal{P}^k](a)}{k}$$

We define the **Dodgson Clone (DC) score** $\mathrm{Sc}_{\mathbf{C}}(a)$ of an alternative a as $\min_k \mathrm{Sc}^k_{\mathbf{D}}(a)$. The DC score can be equivalently defined by modifying the Dodgson rule to allow votes to be split into rational fractions and allowing swaps to be made on those fractions of a vote. Note that like the Tideman approximation, the DC score is less sensitive than Dodgson to a large number of alternatives defeating a by a margin of 1, so the DC score is unlikely to converge to Dodgson as quickly as DQ under the Impartial Culture assumption.

We define the **Dodgson Relaxed (DR) score** $\mathrm{Sc}_{\mathbf{R}}$ as with the Dodgson score, but allow votes to be split into rational fractions. However we require that a be swapped over b at least $F(b, a)$ times. Thus $\mathrm{Sc}_{\mathbf{R}}(d) \geq \mathrm{Sc}_{\mathbf{Q}}(d)$ and the Dodgson Relaxed rule will converge at least as quickly as DQ. The DR rules thus sacrifices independence to cloning of the electorate to be closer to the Dodgson rule than DC.

The **Dodgson Relaxed and Rounded (D&) score** $\mathrm{Sc}_{\&}$ is the DR score rounded up, i.e. $\mathrm{Sc}_{\&}(d) = \lceil \mathrm{Sc}_{\mathbf{R}}(d) \rceil$.

3 Dodgson Linear Programmes

The Dodgson Clone scores can be computed by relaxing the integer constraints on the Integer Linear Programme (ILP) for Dodgson's rule (Rothe et al., 2003). In this section we will show that the difference between the solutions of the ILP and LP are $\mathcal{O}(m!)$. Bartholdi et al. (1989) note that there are only $m!$ orderings of the alternatives and thus no more than $m!$ vote types. However, since we never swap d down the profile (M^c^Cabe-Dansted, 2006), the ordering of the candidates below d are irrelevant. We will formalise this notion as d-equivalence:

Definition 3.1. *Where* $\mathbf{v}$ *is a linear order on* m *alternatives, let* $\mathbf{v}_i$ *represent the* i^{th} *highest ranked alternative and* $\mathbf{v}_{\leq i}$ *represent the sequence of* i^{th} *highest ranked alternatives. Where* d *is an alternative, we say* $\mathbf{v}$ *and* $\mathbf{w}$ *are d-equivalent* ($\mathbf{v} \equiv_d \mathbf{w}$) *iff there exists* i *such that* $\mathbf{v}_i = d$ *and* $\mathbf{v}_{\leq i} = \mathbf{w}_{\leq i}$.

Lemma 3.2. *Let* S_d *be the set of d-equivalence classes. Then* $|S_d|$ *is less than* $(m-1)!e$ *where* $e = 2.71\ldots$ *is the exponential constant.*

Proof. We see that there is one equivalence class where d is ranked in the top position, $m-1$ equivalence classes where d is ranked in the second highest position, and in general $\prod_{k=m-i+1}^{m-1} k$ when d is ranked i^{th} from the top. We note that:

$$\prod_{k=m-i+1}^{m-1} k = \frac{(m-1)!}{(m-i)!}$$

We see that

$$\begin{aligned} |S_d| &= \frac{(m-1)!}{(m-m)!} + \frac{(m-1)!}{(m-(m-1))!} + \cdots + \frac{(m-1)!}{(m-1)!} \\ &< (m-1)!\left(\frac{1}{0!} + \frac{1}{1!} + \cdots\right) = (m-1)!e \end{aligned}$$ □

Corollary 3.3. *If we categorise votes into type based on d-equivalence classes (instead of linear orders), the ILP below has less than* $m\,(m-1)!e = m!e$ *variables.*

Note that there are less than $(m-1)!e$ choices for i, no more than m choices for j and thus less than $m\,(m-1)!e = m!e$ variables (each of the form y_{ij}).

Lemma 3.4. *We can transform the ILP of Bartholdi et al. (1989) into the following form (M^c^Cabe-Dansted, 2008, 2006):*
$\min \sum_i \sum_{j>0} y_{ij}$ *subject to*
$y_{i0} = N_i$ *(for each type of vote* i*)*
$\sum_{ij}(e_{ijk} - e_{i(j-1)k})y_{ij} \geq D_k$ *(for each alternative* k*)*
$y_{ij} \leq y_{i(j-1)}$ *(for each* i *and* $j > 0$*)*
$y_{ij} \geq 0$*, and each* y_{ij} *must be integer.*

Proof. For each i and j variable y_{ij} represents the number of times that the candidate d is swapped up *at least* j positions. In the LP D_k may be defined as $\text{adv}(k,d)/2$ to compute the DC score or $\lceil \text{adv}(k,d)/2 \rceil$ to compute the DR score (under the ILP these are equivalent). □

Theorem 3.5. *The DR* ($Sc_{\mathbf{R}}$)*, DC* ($Sc_{\mathbf{C}}$) *and D&* ($Sc_{\&}$) *scores are bounded as follows:*

$$Sc_{\mathbf{D}}(d) - (m-1)!(m-1)e < Sc_{\mathbf{C}}(d) \leq Sc_{\mathbf{R}}(d) \leq Sc_{\&} \leq Sc_{\mathbf{D}}(d)$$

Proof. Every solution to the Integer Linear Program for the Dodgson score is a solution to the Linear Program for the DR score. Every solution to the LP for the DR score is a solution to the LP for the DC score. Thus the DC score cannot be greater than the DR score, which cannot be greater than the Dodgson score ($\mathrm{Sc}_{\mathbf{C}}(d) \leq \mathrm{Sc}_{\mathbf{R}}(d) \leq \mathrm{Sc}_{\mathbf{D}}(d)$). Since $\mathrm{Sc}_{\mathbf{D}}$ is integer, it follows that $\mathrm{Sc}_{\mathbf{R}}(d) \leq \lceil \mathrm{Sc}_{\mathbf{R}}(d) \rceil = \mathrm{Sc}_{\&}(d) \leq \mathrm{Sc}_{\mathbf{D}}(d)$. Also note that given a solution y to either LP, we can produce a solution y' to the ILP simply by rounding up each variable ($y'_{ij} = \lceil y_{ij} \rceil$), hence

$$\mathrm{Sc}_{\mathbf{D}}(d) - \mathrm{Sc}_{\mathbf{C}}(d) \leq \sum_i \sum_{j>0} \lceil y_{ij} \rceil - y_{ij} < \sum_i \sum_{j>0} 1 \leq (m-1)!e$$

Since i can take less than $(m-1)!e$ values, and j can vary from 1 to $(m-1)$, it follows that

$$\mathrm{Sc}_{\mathbf{D}}(d) - (m-1)!(m-1)e < \mathrm{Sc}_{\mathbf{C}}(d) \leq \mathrm{Sc}_{\mathbf{R}}(d) \leq \mathrm{Sc}_{\&} \leq \mathrm{Sc}_{\mathbf{D}}(d) \qquad \square$$

These results can also be used to find tighter bounds on the complexity of solving the ILP and LPs (M^c^Cabe-Dansted, 2006, 2008).

4 Counting Proof of Convergence under IAC

For a voting situation U and linear order $\mathbf{v}$, we represent the number of linear orders of type $\mathbf{v}$ in U by $\#_U(\mathbf{v})$.

Where $X \in \{D, T\}$, let $\Delta_{\mathbf{X}}(a, z)$ be equivalent to $\mathrm{Sc}_{\mathbf{X}}(a) - \mathrm{Sc}_{\mathbf{X}}(z)$. Given an arbitrary pair of alternatives (a, z) we pick an arbitrary linear order $ab \ldots z$ with a ranked first and z ranked last and call it $\mathbf{v}$. We also define the reverse linear order $\tilde{\mathbf{v}} = z \ldots ba$.

Lemma 4.1. *Replacing a vote of type $\tilde{\mathbf{v}}$ with a vote of type $\mathbf{v}$ will increase $\Delta_{\mathbf{T}}(a, z)$ by at least one.*

Proof. We see that replacing a vote of type $\tilde{\mathbf{v}}$ with a vote of type $\mathbf{v}$ will increase $\mathrm{adv}(a, z)$ by one, or decrease $\mathrm{adv}(z, a)$ by one. $\square$

Lemma 4.2. *Replacing a vote of type $\tilde{\mathbf{v}}$ with a vote of type $\mathbf{v}$ will increase $\Delta_{\mathbf{D}}(a, z)$ by at least one.*

Proof. Say a is not a Condorcet-tie winner, but is a Condorcet-tie winner after some minimal set S of swaps is applied to the profile P. Let P' be the profile P after one vote of type $\tilde{\mathbf{v}}$ has been replaced with a vote of type $\mathbf{v}$. If any swaps were applied to the vote those swaps are no longer required, and $\mathrm{Sc}_{\mathbf{D}}[P'](a) < \mathrm{Sc}_{\mathbf{D}}[P](a)$. Otherwise we can apply the set of swaps S to P' resulting in $\mathrm{adv}(a, k)$ being at least 2 for all other alternatives k. Hence we can remove one of the swaps, and still result in a being a Condorcet-tie winner after the swaps have been applied to P'.

Say a is a Condorcet-tie winner in P. Then z is not a Condorcet-tie winner in P'. As in the previous paragraph we can conclude that $\mathrm{Sc}_{\mathbf{D}}[P](z) < \mathrm{Sc}_{\mathbf{D}}[P'](z)$. $\square$

Lemma 4.3. *For a fixed integer k, and a fixed ordered pair of alternatives (a, z) the proportion of voting situations, with n agents and m alternatives, for which $\Delta_{\mathbf{X}}(a, z) = k$ is no more than:*

$$\frac{(m! - 2)}{(n + m! - 2)}$$

Proof. Say $\mathbf{v} = ab \ldots z$ is some fixed linear order and $\tilde{\mathbf{v}} = z \ldots ba$ is the reverse order. We define an equivalence relation $\sim$ on the set of voting situations, as follows: say U, V are two voting situations, then

$$U \sim V \iff \forall_{\mathbf{w} \neq \mathbf{v},\, \mathbf{w} \neq \tilde{\mathbf{v}}} \#_V(\mathbf{w}) = \#_U(\mathbf{w}).$$

From Lemma 4.1 and 4.2 we see that in each equivalence class, there can be at most one voting situation for which $\Delta_{\mathbf{X}}(a,z) = k$. Also note that whereas there are

$$|\mathcal{S}^n(A)| = \binom{n+m!-1}{n}$$

distinct voting situations there are at most

$$\binom{n+(m!-1)-1}{n}$$

equivalence classes under $\sim$. Hence the proportion of voting situations for which $\Delta_{\mathbf{X}}(a,z) = k$ is no more than:

$$\frac{(n+m!-1)!}{n!(m!-1)!}\frac{n!(m!-2)!}{(n+m!-2)!} = \frac{(n+m!-1)!}{(n+m!-2)!}\frac{(m!-2)!}{(m!-1)!} = \frac{(m!-2)}{(n+m!-2)} \qquad \square$$

Lemma 4.4. *If $\Delta_{\mathbf{T}}(a,z) = k$ and a is a Tideman winner and z is a DQ winner, then $0 \le k < m$.*

Proof. As a is a Tideman winner and z is a DQ winner, then

$$\mathrm{Sc}_{\mathbf{T}}(a) \le \mathrm{Sc}_{\mathbf{T}}(z), \quad \mathrm{Sc}_{\mathbf{Q}}(z) \le \mathrm{Sc}_{\mathbf{Q}}(a)$$

Recall that

$$\mathrm{Sc}_{\mathbf{Q}}(x) = \sum_{y \ne x} \left\lceil \frac{\mathrm{adv}(y,x)}{2} \right\rceil, \quad \mathrm{Sc}_{\mathbf{T}}(x) = \sum_{y \ne x} \mathrm{adv}(y,x).$$

We see that $\mathrm{adv}(y,x) \le 2\lceil \mathrm{adv}(y,x)/2 \rceil \le \mathrm{adv}(y,x) + 1$ and so $\mathrm{Sc}_{\mathbf{T}}(x) \le 2\mathrm{Sc}_{\mathbf{Q}}(x) < \mathrm{Sc}_{\mathbf{T}}(x) + m$ for all alternatives x. Thus

$$\mathrm{Sc}_{\mathbf{T}}(z) \le 2\mathrm{Sc}_{\mathbf{Q}}(z) \le 2\mathrm{Sc}_{\mathbf{Q}}(a) < \mathrm{Sc}_{\mathbf{T}}(a) + m.$$

And so $\mathrm{Sc}_{\mathbf{T}}(a) \le \mathrm{Sc}_{\mathbf{T}}(z) < \mathrm{Sc}_{\mathbf{T}}(a) + m$. Let $k = \mathrm{Sc}_{\mathbf{T}}(a) - \mathrm{Sc}_{\mathbf{T}}(z)$. Then $0 \le k < m$, and so there are no more than m ways of choosing k if we wish the DQ and Tideman winners to differ. $\square$

Recall that Theorem 3.5 states:

$$\mathrm{Sc}_{\mathbf{D}}(d) - (m-1)!(m-1)e < \mathrm{Sc}_{\mathbf{C}}(d) \le \mathrm{Sc}_{\mathbf{R}}(d) \le \mathrm{Sc}_{\&} \le \mathrm{Sc}_{\mathbf{D}}(d)$$

where $e = 2.71\ldots$ is the exponential constant. Given that there are only m ways of choosing k such that the Tideman and DQ winners differ, and less than $(m-1)!(m-1)e$ ways of choosing k such that the Dodgson, DC, DR and/or D& winners differ, we get the following theorem.

Theorem 4.5. *The proportion of voting situations, with n agents and m alternatives, for which a is a Tideman winner and z is a DQ winner is no more than:*

$$\frac{(m!-2)}{(n+m!-2)}m.$$

The proportion for which a is Dodgson winner and z is a DC, DR and/or D& winner is less than

$$\frac{(m!-2)}{(n+m!-2)}(m-1)!(m-1)e.$$

As there are only $m(m-1)$ ways of choosing a and z from the set of alternatives, we get the following corollary

Corollary 4.6. *The probability that the DQ and Tideman rule pick the same winners converges to* 1 *as $n \to \infty$, under the Impartial Anonymous Culture assumption. Likewise the probability that the DR, DC, D& and Dodgson rules pick the same winner converges to 1 as $n \to \infty$.*

In other words the DR, DC and D& winners (and scores) provide "deterministic heuristic polynomial time" (Procaccia and Rosenschein, 2007) algorithms for the Dodgson winner (and score) with the IAC distribution.We can use the same technique to show that the Greedy Algorithm proposed by Homan and Hemaspaandra (2005) converges to the DQ and Tideman rules under IAC, as the `GreedyScore` differs from the DQ score by less than m.By setting k to 0 we may likewise prove that the probability of a non-unique Tideman or Dodgson winner converges to 0 under IAC.

We extend the concept of a "frequently self-knowingly correct algorithm" (Homan and Hemaspaandra, 2005) such that we can specify a distribution over which the algorithm is frequently self-knowingly correct.

Definition 4.7. *A self-knowingly correct (Homan and Hemaspaandra, 2005) algorithm A is a "frequently self-knowingly correct algorithm over a distribution μ" for $g \colon \Sigma^* \to T$ iff*

$$\lim_{n\to\infty} \sum_{x\in\Sigma^n,\, A(x)\in T\times\{maybe\}} P_\mu(x) = 0$$

where $P_\mu(x)$ is the probability that $X = x$ when X is chosen from Σ^n under the μ distribution and for all x we have $(A(x))_1 = g(x)$ or $(A(x))_2 =$ "maybe".

Using the set of linear orders $\mathcal{L}(\mathcal{A})$ as Σ we may construct a frequently self-knowingly correct algorithm from DC (or DR, or D&) as the algorithm can output "definitely" whenever the DC winner has DC score that is at least $(m-1)!(m-1)e$ less than any other alternative.

5 Non-convergence of Tideman Based rules

Definition 5.1. *A "voting ratio" is a function $f \colon \mathcal{L}(\mathcal{A}) \to [0,1]$ such that*

$$\sum_{\mathbf{v}\in\mathcal{L}(\mathcal{A})} f(\mathbf{v}) = 1.$$

We say that a profile $\mathcal{P}$ reduces to a voting ratio f if

$$\forall_{\mathbf{v}\in\mathcal{L}(\mathcal{A})} \#\{i : \mathcal{P}_i = \mathbf{v}\} = nf(\mathbf{v})$$

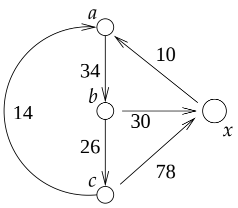
Note 5.2. *A voting ratio is similar to a voting situation, but unlike a voting situation does not contain any information about the total number of agents.*

Definition 5.3. *We say that a voting ratio f is "bad" if for every profile $\mathcal{P}$ that reduces to f and has an even number of agents, the DQ winner of $\mathcal{P}$ differs from the Dodgson winner.*

Example 5.4. *The following voting ratio is bad.*

$$h(\mathbf{v}) = \begin{cases} 16/39 \text{ if} & \mathbf{v} = abcx \\ 12/39 \text{ if} & \mathbf{v} = cxab \\ 10/39 \text{ if} & \mathbf{v} = bcxa \\ 1/39 \text{ if} & \mathbf{v} = cbax \\ 0 & otherwise \end{cases}$$

We see that for a profile that reduces to the above voting ratio, we have the following advantages and scores per 78 agents:



	a	b	c	x
DQ score	12	17	13	54
Dodgson score	14	17	13	54

The Dodgson score of a is higher than the DQ score of a as to swap a over c we must first swap a over b or a over x. We have to swap a over c at least 7 times. Hence we must use a total of at least 14 swaps to make a a Condorcet-tie winner. We see that a is the DQ winner, but c is the Dodgson winner.

We will now show that there exists a neighbourhood around the voting ratios g and h that is bad.

Lemma 5.5. *Altering a single vote will change the Dodgson scores and DQ scores by at most $m-1$.*

Proof. Recall that

$$\mathrm{Sc}_{\mathbf{Q}}(a) = \sum_{b \neq a} F(b,a), \text{ where } F(b,a) = \left\lceil \frac{\mathrm{adv}(b,a)}{2} \right\rceil .$$

We see that for each other alternative b changing a single vote can change $F(b,a)$ by at most one, and there are $m-1$ such alternatives.

Say that P and R are two profiles that differ only in a single vote. Let P' and R' be P and R respectively after some arbitrary d has been swapped to the top of the vote that differs, which requires no more than $m-1$ swaps. We see that

$$\mathrm{Sc}_{\mathbf{D}}[P](d) \leq \mathrm{Sc}_{\mathbf{D}}[P'](d) + m - 1 \leq \mathrm{Sc}_{\mathbf{D}}[R](d) + m - 1$$
$$\mathrm{Sc}_{\mathbf{D}}[R](d) \leq \mathrm{Sc}_{\mathbf{D}}[R'](d) + m - 1 \leq \mathrm{Sc}_{\mathbf{D}}[P](d) + m - 1 \ . \qquad \square$$

Corollary 5.6. *For any positive integer k, alternative d, profile P and rule $X \in \{D, Q\}$ (i.e. Dodgson or DQ), if $Sc_{\mathbf{X}}(d) < Sc_{\mathbf{X}}(a) - 2k(m-1)$ for all other alternatives a, then d will remain the unique X winner in any profile that results from changing k or less votes.*

Lemma 5.7. *There is a neighbourhood of bad voting ratios around the voting ratio h from Example 5.4.*

Proof. We see that in a profile with n agents that reduces to the voting ratio h the Dodgson and DQ winners have scores that are at least $\frac{n}{78}$ lower than the other alternatives. Thus if we alter less than $\frac{n}{78}\frac{1}{2(1)(4-1)} = \frac{n}{468}$ votes, the DQ winner will remain different from the Dodgson winner. $\square$

Thus we may now state the following theorem:

Theorem 5.8. *If we generate profiles randomly according to the IAC distribution, with $m \geq 4$, the DQ, Greedy and Tideman winners do not converge to the Dodgson winner as the number of agents tends to infinity.*

It is easy to generalise this result to non-trivial Pólya-Eggenberger distributions. See (M^{c}Cabe-Dansted, 2006) for details.

Under the IAC, the Tideman rule (M^{c}Cabe-Dansted et al., 2007) and the greedy algorithm proposed by Homan and Hemaspaandra (2005) converges to the DQ rule, which does not converge to Dodgson's rule. It follows that the Tideman rule and greedy algorithm do not converge to Dodgson's rule under the IAC.

As the difference between the DQ scores and the `GreedyScores` (Homan and Hemaspaandra, 2005) is less than m, the DQ winner and `GreedyWinner` will converge under the IAC.

6 Conclusion

We have previously (M^{c}Cabe-Dansted et al., 2007) proved that Tideman's rule does converge under IC, and presented a refined rule (DQ) which converged exponentially quickly. Another approximation with exponentially fast convergence was independently proposed by Homan and Hemaspaandra (2005).

Unfortunately, real voters are not independent, and so the Impartial Culture assumption is not realistic. This paper investigates the asymptotic behaviour of approximations to the Dodgson rule under other assumptions of voting behaviour, particularly the Impartial Anonymous Culture (IAC) assumption, that each multiset of votes is equally likely.

We found that the approximations converge to each other under IAC, but they do not converge to Dodgson. Hence the DQ rule is not asymptotically closer to the Dodgson rule than the Tideman rule is, although DQ converges faster under the IC.

It is not realistic to assume that voters' behaviour will precisely follow any mathematical model, including IAC. Fortunately the proof that our new approximations converge to Dodgson does not depend on the details of the IAC. Indeed, the DC, DR, D& and Dodgson winners will all be the same if the scores of the two leading alternatives differ by less than $(m-1)!(m-1)e$. This assumption is realistic for a sufficiently large number of voters. Even in the neck and neck 2000 US presidential elections, the popular vote for the leading two alternatives differed by half a percent — over half a million votes. If we used the Dodgson rule to choose between the top five alternatives, the difference between the Dodgson scores of the leading two alternatives would have to be less than 261 for the winners to differ. Even if the entire electorate conspired to cause the winners to differ, minor inaccuracies such as hanging chads could frustrate their attempt. Even in cases where the Dodgson Relaxed does differ from the Dodgson, the difference seems to be primarily that the Dodgson Relaxed rule picks a smaller set of tied winners as it is able to split ties in favour of alternatives that have fractionally better DR scores. This is in some sense more democratic than tie breaking procedures such as breaking ties in favour of the preferences of the first voter.

Thus determining that an algorithm is "frequently self-knowingly correct", as defined by Homan and Hemaspaandra (2005), is insufficient to conclude that the algorithm will converge in practice. The DQ and `GreedyWinner` provide frequently self-knowingly correct algorithms, but do not converge under other assumptions of voter behaviour such as IAC. We have extended the definition to allow a distribution to be specified. So, in other words, DQ and `GreedyWinner` provide algorithms that are "frequently self-knowingly correct over IC", but unlike DC, DR, and D&, do not provide algorithms that are "frequently self-knowingly correct algorithms over IAC".

We have previously shown that the Dodgson scores and winners of a voting situation can be computed from a voting situation with $\mathcal{O}(f_1(m)\ln n)$ arithmetic operations of $\mathcal{O}(f_2(m)\ln n)$ bits of precision (M^{c}Cabe-Dansted, 2006) for some pair of functions f_1 and f_2. However we did not find a good upper bound on f_1 or f_2, even $f_1(4)$ may be unreasonably large. This suggests that it may be important to use a variant of Dodgson's rule that is truely polynomial, such as DC, DR or D&. See (McCabe-Dansted, 2006) for a discussion of the complexity of these rules.

We have found that the scores of the most of the approximations we have studied form a hierarchy of increasingly tight lower bounds on the Dodgson score:

$$\frac{\mathrm{Sc}_{\mathbf{S}}(x)}{2} \leq \frac{\mathrm{Sc}_{\mathbf{T}}(x)}{2} \leq \mathrm{Sc}_{\mathbf{Q}} \leq \mathrm{Sc}_{\mathbf{R}} \leq \mathrm{Sc}_{\&} \leq \mathrm{Sc}_{\mathbf{D}} \leq \mathrm{Sc}_{\mathbf{R}} + (m-1)!(m-1)e$$

The Dodgson Clone rule does not fit in that hierarchy, although it is the case that

$$\frac{\mathrm{Sc}_{\mathbf{T}}(x)}{2} \leq \mathrm{Sc}_{\mathbf{C}} \leq \mathrm{Sc}_{\mathbf{R}} \leq \mathrm{Sc}_{\mathbf{D}} \leq \mathrm{Sc}_{\mathbf{C}} + (m-1)!(m-1)e \ .$$

Despite the great accuracy of the D& approximation, there are good reasons to pick other approximations. The difference between the DR rule and the D& rule is that the DR rule can split ties based on fractional scores, so the DR rule may be considered superior to D& and Dodgson's rule. The DC rule is resistant to cloning of the electorate. The DQ rule is very simple to compute, and very easy to write in any programming language. As the DQ rule is known to converge exponentially fast under IC, this makes the DQ rule very appropriate for cases where the data is known to distributed according to IC. This is the case for studies that have randomly generated data according to the IC (see e.g. McCabe-Dansted and Slinko, 2006; Shah, 2003; Nurmi, 1983). The Tideman rule is no more easy to compute than the DQ rule. However, the mathematical definition of the Tideman rule is simpler than the DQ rule. This makes the Tideman rule useful for theoretical studies of the Dodgson rule where the speed of convergence is not important. Also, like the DC rule, the Tideman rule is resistant to cloning the electorate.

We conclude that the DC and DR rules are superior, for social choice, to the traditional definition of the Dodgson rule. DC provies resistance to cloning the electorate, and both are better at splitting ties than the traditional definition of Dodgson rule. We know of no advantage of the traditional definition over these rules. Even if the traditional Dodgson winner is preferred, it may be hard to justify the computational complexity of the traditional Dodgson rule, especially since it is almost certain that these rules would pick the same result. If, despite all this, the traditional Dodgson rule is still chosen, these new rules provide frequently self-knowingly correct algorithms for the Dodgson winner for any reasonable assumption of voter behaviour, including IAC.

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