Comparing players in simple games

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Significance of computing influence

The mathematical study (under different names) of pivotal agents and influences is quite basic in percolation theory and statistical physics, as well as in probability theory and statistics, reliability theory, distributed computing, complexity theory, game theory, mechanism design and auction theory, other areas of theoretical economics, and political science.

- G. Kalai and S. Safra. (Threshold phenomena and influence. In Computational Complexity and Statistical Physics. Oxford University Press, 2006.)

Simple Games

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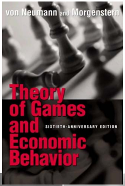
Simple Games

Background: Von Neumann and Morgenstern, *Theory of Games and Economic Behavior, 1944*



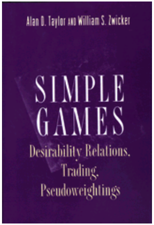
John Von Neumann
1903-1957

Oskar Morgenstern
1902-1976.

Simple Games

Reference: A. Taylor and W. Zwicker, *Simple Games: Desirability Relations, Trading, Pseudoweightings*, New Jersey: Princeton University Press, 1999.

> *...few structures arise in more contexts and lend themselves to more diverse interpretations than do simple games.*



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- Notation: $[q; w_1, \ldots, w_n]$

MWVG

Definitions

An **multiple weighted voting game** (MWVG) is the simple game $(N, v_1 \wedge \cdots \wedge v_m)$ where the games (N, v_t) are the WVGs $[q^t; w_1^t, \ldots, w_n^t]$ for $1 \leq t \leq m$. Then $v = v_1 \wedge \cdots \wedge v_m$ is defined as:

$$v(S) = \begin{cases} 1, & \text{if } v_t(S) = 1, \forall t, 1 \leq t \leq m. \\ 0, & \text{otherwise.} \end{cases}$$

Key Concepts

Being critical for a coalition

A player, i is *critical* for a losing coalition C if the player's inclusion results in the coalition winning.

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Banzhaf Value

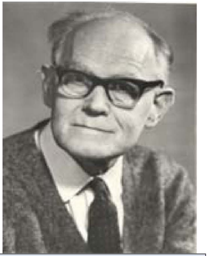
Banzhaf Value, η_i of a player i is the number of coalitions for which i is critical.

Banzhaf Index

Banzhaf Index, β_i is the ratio of the *Banzhaf value* of the player i to sum of the *Banzhaf value* of all players.

Banzhaf Index

John Banzhaf III



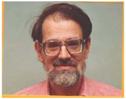
Lionel Penrose

Shapley-Shubik index

Depends on permutations instead of coalitions.

Definitions

The **Shapley-Shubik value** is the function κ that assigns to any simple game (N, v) and any voter i a value $\kappa_i(v)$ where $\kappa_i = \sum_{X \subseteq N}(|X|-1)!(n-|X|)!(v(X)-v(X-\{i\}))$. The **Shapley-Shubik index** of i is the function ϕ defined by $\phi_i = \frac{\kappa_i}{n!}$




Lloyd Shapley Martin Shubik

Player types

A player in a simple game may be of various types depending on its level of influence.

Definitions

For a simple game v on a set of players N, player i is

- **dummy** if and only if $\forall S \subseteq N$, if $v(S) = 1$, then $v(S \setminus \{i\}) = 1$;

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- **dictator** if and only if $\forall S \subseteq N$, $v(S) = 1$ if and only if $i \in S$.

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Dictator

- If a dictator exists, it is unique and all other players are dummies.
- This means that a dictator has voting power one, whereas all other players have zero voting power.

Dummy

- We already know that for the case of WVGs, it is NP-hard to identify dummy players. [Matsui and Matsui, 2000]
- It follows that it is NP-hard to identify dummies in MWVGs.

Dummy

Lemma

A player i in a simple game v is a dummy if and only if it is not present in any minimal winning coalition.

Proof.

- Let us assume that player i is a dummy but is present in a minimal winning coalition.
- That mean that it is critical in the minimal winning coalition which leads to a contradiction.
- Now let us assume that i is critical in at least one coalition S such that $v(S \cup \{i\}) = 1$ and $v(S) = 0$.
- In that case there is a $S' \subset S$ such that $S' \cup \{i\}$ is a MWC.

$\square$

Dummy

Proposition

For a simple game v,

1. *Dummy players can be identified in linear time if v is of the form (N, W^m).*
2. *Dummy players can be identified in polynomial time if v is of the form (N, W).*

Proof.

We examine each case separately:

1. If a player is not critical for any MWC, then it is a dummy.
2. Initialize a set of dummy players as N. For each coalition $S \in W$, check for each player i in S whether the defection of player i leads to S becoming losing. If yes, remove i from the set of dummy players.

Vetoers

Proposition

Vetoers can be identified in linear time for a simple game in the following representations: (N, W), (N, W^m), WVG and MWVG.

Proof.

We examine each of the cases separately:

1. (N, W): Initialize all players as vetoers. For each winning coalition, if a player is not present in the coalition, remove him from the list of vetoers.
2. (N, W^m): If there exists a winning coalition which does not contain player i, there will also exist a minimal winning coalition which does not contain i.
3. WVG: For each player i, i has veto power if and only if $w(N \setminus \{i\}) < q$.
4. MWVG: For each player i, i has veto power if and only if $N \setminus \{i\}$ is losing.

□

Passers & Dictators

Proposition

For a simple game represented by (N, W), (N, W^m), WVG or MWVG, it is easy to identify the passers and the dictator.

Proof.

We check both cases separately:

1. Passers: This follows from the definition of a passer. A player i is a passer if and only if $v(\{i\}) = 1$.
2. Dictator: It is easy to see that if a dictator exists in a simple game, it is unique. It follows from the definition of a dictator that a player i is dictator in a simple game if $v(\{i\}) = 1$ and $v(N \setminus \{i\}) = 0$.

□

Complexity of player types

Table: Complexity of player types

Input $\rightarrow$	(N, W)	(N, W^m)	WVG	MWVG
IDENTIFY-DUMMIES	P	Linear	NP-hard	NP-hard
IDENTIFY-VETOERS	linear	linear	linear	linear
IDENTIFY-PASSERS	linear	linear	linear	linear
IDENTIFY-DICTATOR	linear	linear	linear	linear

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Desirability relation

In a simple game (N, v),

- A player i is **more desirable/influential** than player j ($i \succeq_D j$) if $v(S \cup \{j\}) = 1 \Rightarrow v(S \cup \{i\}) = 1$ for all $S \subseteq N \setminus \{i, j\}$.

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- Players i and j are **equally desirable/influential** or *symmetric* ($i \sim_D j$) if $v(S \cup \{j\}) = 1 \Leftrightarrow v(S \cup \{i\}) = 1$ for all $S \subseteq N \setminus \{i, j\}$.

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- A player i is **strictly more desirable/influential** than player j ($i \succ_D j$) if i is more desirable than j, but if i and j are not equally desirable.
- A player i and j are **incomparable** if there exist S, $T \subseteq N \setminus \{i, j\}$ such that $v(S \cup \{i\}) = 1$, $v(S \cup \{j\}) = 0$, $v(T \cup \{i\}) = 0$ and $v(T \cup \{j\}) = 1$.

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Linear

Linear simple games are a natural class of simple games:

Definitions

A simple game is **linear** whenever the desirability relation $\succeq_D$ is complete that is any two players i and j are comparable ($i \succ j$, $j \succ i$ or $i \sim j$).

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For linear games, the relation $R_\sim$ divides the set of voters N into equivalence classes $N/R_\sim = \{N_1, \ldots, N_t\}$ such that for any $i \in N_p$ and $j \in N_q$, $i \succ j$ if and only if $p < q$.

Linear games

Proposition

A simple game with three or fewer players is linear.

Proof.

- For a game to be non-linear, we want to player 1 and 2 to be incomparable i.e. there exists coalition $S_1, S_2 \subseteq N \setminus \{1,2\}$ such that $v(\{1\} \cup S_1) = 1$, $v(\{2\} \cup S_1) = 0$, $v(\{1\} \cup S_2) = 0$ and $v(\{2\} \cup S_2) = 1$.
- This is not clearly not possible for $n = 1$ or 2. For $n = 3$, without loss of generality v is non-linear only if $v(\{1\} \cup \emptyset) = 1$, $v(\{2\} \cup \emptyset) = 0$, $v(\{1\} \cup \{3\}) = 0$ and $v(\{2\} \cup \{3\}) = 1$.
- However the fact that $v(\{1\} \cup \emptyset) = 1$ and $v(\{1\} \cup \{3\}) = 0$ leads to a contradiction.

□

Desirability ordering

A desirability ordering on linear games is any ordering on players such that $1 \succeq_D 2 \succeq_D \ \ldots \ \succeq_D n$. A **strict** desirability ordering is the following ordering on players: $1 \ \circ \ 2 \ \circ \ \ \ldots \ \ \circ \ n$ where $\circ$ is either $\sim_D$ or $\succ_D$.

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Proposition

For a WVG:

1. *A desirability ordering of players can be computed easily.*
2. *It is NP-hard to compute the strict desirability ordering of players.*

Desirability ordering

Proof.

We check both cases separately:

1. WVGs are linear games. When $w_i = w_j$, then we know that $i \sim j$. Moreover, if $w_i > w_j$, then we know that i is at least as desirable as j, that is $i \succeq j$.

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Proof.

We check both cases separately:

1. WVGs are linear games. When $w_i = w_j$, then we know that $i \sim j$. Moreover, if $w_i > w_j$, then we know that i is at least as desirable as j, that is $i \succeq j$.
2. Follows from the fact that it is NP-hard to check whether two players are symmetric. (Matsui and Matsui [2000])

$\square$

Linearity of MWVGs

The following is an example of a small non-linear MWVG:

Example

- In game $v = [10; 10, 9, 1, 0] \wedge [10; 9, 10, 0, 1]$,
- $\{1\} \cup \{4\}$ wins, $\{2\} \cup \{4\}$ loses, $\{2\} \cup \{3\}$ wins and $\{1\} \cup \{3\}$ loses.
- Players 1 and 2 are incomparable.
- So, whereas simple games with 3 players are linear, it is easy to construct a 4 player non-linear MWVG.

Linearity of MWVGs

Proposition

It is NP-hard to verify whether a MWVG is linear or not.

Proof:
We prove this by a reduction from an instance of the classical NP-hard PARTITION problem.
Name: PARTITION
Instance: A set of k integer weights $A = \{a_1, \ldots, a_k\}$.
Question: Is it possible to partition A, into two subsets $P_1 \subseteq A$, $P_2 \subseteq A$ so that $P_1 \cap P_2 = \emptyset$ and $P_1 \cup P_2 = A$ and $\sum_{a_i \in P_1} a_i = \sum_{a_i \in P_2} a_i$?

Linearity of MWVGs-Proof

- Given an instance of PARTITION $\{a_1, \ldots, a_k\}$, we may as well assume that $\sum_{i=1}^{k} a_i$ is an even integer, $2t$ say.

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- **Reduction**: We can transform the instance into the multiple weighted voting $v = v_1 \wedge v_2$ where
$v_1 = [q; 20a_1, \ldots, 20a_k, 10, 9, 1, 0]$ and
$v_2 = [q; 20a_1, \ldots, 20a_k, 9, 10, 0, 1]$ for $q = 10 + 20t$ and $k + 4$ is the number of players.

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- **If** A **is a 'no' instance of PARTITION**, then we see that a subset of weights $\{20a_1, \ldots, 20a_k\}$ cannot sum to $20t$. This implies that players $k+1$, $k+2$, $k+3$, and $k+4$ are not critical for any coalition. Since players $1, \ldots, k$ have the same desirability ordering in both v_1 and v_2, v is linear.

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Linearity of MWVGs-Proof

- **If A is a 'yes' instance of PARTITION** with a partition (P_1, P_2). In that case players $k+1$, $k+2$, $k+3$, and $k+4$ are critical for certain coalitions. We see that $v(\{k+1\} \cup (\{k+4\} \cup P_1)) = 1$, $v(\{k+2\} \cup (\{k+4\} \cup P_1)) = 0$, $v(\{k+1\} \cup (\{k+3\} \cup P_1)) = 0$ and $v(\{k+2\} \cup (\{k+3\} \cup P_1)) = 1$. Therefore, players $k+1$ and $k+2$ are not comparable and v is not linear.

Makino's result

Proposition

(Makino, 2002) For a simple game $v = (N, W^m)$, it can be verified in $O(n(|W^m|))$ time if v is linear or not.

Proof.

Makino [2002] proved that for a positive boolean function on n variables represented by a set of all minimal true vectors $minT(f)$, it can be checked in $O(n|minT(f)|)$ whether the function is *regular*(linear) or not. Makino's algorithm CHECK-FCB takes $minT(f)$ as input and outputs 'Yes' if f is regular and 'No' otherwise. □

Corollary

Corollary

For a simple game $v = (N, W)$, it can be verified in polynomial time if v is linear or not.

Proof.

We showed earlier that (N, W) can be transformed in to (N, W^m) in polynomial time. After that we can use Makino's method to verify whether the game is linear or not. $\square$

Linearity

Proposition

Let $v = (N, W^m)$ *be a linear simple game and let* $d_{k,i} = |\{S : i \in S, S \in W^m, |S| = k\}|$*. Then for two players* i *and* j*,*

1. $i \sim_D j$ *if and only if* $d_{k,i} = d_{k,j}$ *for* $k = 1, \ldots n$*.*
2. $i \succ_D j$ *if and only if for the smallest* k *where* $d_{k,i} \neq d_{k,j}$*,* $d_{k,i} > d_{k,j}$*.*

Proof part 1

($\Rightarrow$)

- Let us assume $i \sim_D j$.
- Then by definition, $v(S \cup \{j\}) = 1 \Leftrightarrow v(S \cup \{i\}) = 1$ for all $S \subseteq N \setminus \{i, j\}$.
- So $S \cup \{i\} \in W^m$ if and only if $S \cup \{j\} \in W^m$.
- Therefore, $d_{k,i} = d_{k,j}$ for $k = 1, \ldots n$.

Proof part 1

($\Leftarrow$)

- Let us assume that $i \not\sim_D j$. Since v is linear, i and j are comparable.
- Without loss of generality, we assume that $i \succ_D j$.
- Then there exists a coalition $S \setminus \{i, j\}$ such that $v(S \cup \{i\}) = 1$ and $v(S \cup \{j\}) = 0$ and $|S| = k - 1$.
- If $S \cup \{i\} \in W^m$, then $d_{k,i} > d_{k,j}$.
- If $S \cup \{i\} \notin W^m$ then there exists $S' \subset S$ such that $S' \cup \{i\} \in W^m$.
- Thus there exists $k' < k$ such that $d_{k',i} > d_{k',j}$.

Proof part 2

($\Rightarrow$)

- Let us assume that $i \succ_D j$ and let k' be the smallest integer where $d_{k',i} \neq d_{k',j}$.
- If $d_{k',i} < d_{k',j}$, then there exists a coalition S such that $S \cup \{j\} \in W^m$, $S \cup \{i\} \notin W^m$ and $|S| = k' - 1$.
- $S \cup \{i\} \notin W^m$ in only two cases.
- The first possibility is that $v(S \cup \{i\}) = 0$ but this is not true since $i \succ_D j$.
- The second possibility is that there exists a coalition $S' \subset S$ such that $S' \cup \{i\} \in W^m$.
- But that would mean that $v(S' \cup \{i\}) = 1$ and $v(S' \cup \{j\}) = 0$.
- This also leads to a contradiction since k' is the smallest integer where $d_{k',i} \neq d_{k',j}$.

Proof part 2

($\Leftarrow$)

- Let us assume that for the smallest k where $d_{k,i} \neq d_{k,j}$, $d_{k,i} > d_{k,j}$.
- This means there exists a coalition S such that $S \cup \{i\} \in W^m$, $S \cup \{j\} \notin W^m$, $|S| = k - 1$.
- This means that either $v(S \cup \{j\}) = 0$ or there exists a coalition $S' \subset S$ such that $S' \cup \{i\} \in W^m$.
- If $v(S \cup \{j\}) = 0$, that means $i \succ_D j$.
- If there exists a coalition $S' \subset S$ such that $S' \cup \{j\} \in W^m$, then $d_{k',j} > d_{k',i}$ for some $k' < k$.
- This leads to a contradiction.

Linearity

Proposition

Let $v = (N, W^m)$ be a linear simple game and let $d_{k,i} = |\{S : i \in S, S \in W^m, |S| = k\}|$. Then for two players i and j,

1. *$i \sim_D j$ if and only if $d_{k,i} = d_{k,j}$ for $k = 1, \dots n$.*
2. *$i \succ_D j$ if and only if for the smallest k where $d_{k,i} \neq d_{k,j}$, $d_{k,i} > d_{k,j}$.*

Algorithm

Algorithm 1 Strict-desirability-ordering-of-simple-game

Input: Simple game $v = (N, W^m)$ where $N = \{1, \ldots, n\}$ and $W^m(v) = \{S_1, \ldots, S_{|W^m|}\}$.

Output: NO if v is not linear. Otherwise output desirability equivalence classes starting from most desirable in case v is linear.

X = CHECK-FCB(W^m)
if $X = NO$ **then**
    return NO
else
    Initialize a $n \times n$ matrix D where entries $d_{i,j} = 0$ for all i and j in N
    for $i = 1$ to $|W^m|$ **do**
        for each player x in S_i **do**
            $d_{|S_i|,x} \leftarrow d_{|S_i|,x} + 1$
        end for
    end for
    return classify($N, D, 1$)
end if

Algorithm

Algorithm 2 classify

Input: set of integers $\text{class}_{\text{index}}$, $n \times n$ matrix D, integer k.
Output: subclasses.

if $k = n + 1$ or $|\text{class}_{\text{index}}| = 1$ **then**
    return $\text{class}_{\text{index}}$
end if
$s \leftarrow |\text{class}_{\text{index}}|$
mergeSort($\text{class}_{\text{index}}$) in descending order such that $i > j$ if $d_{k,i} > d_{k,j}$.
for $i = 2$ to s **do**
    $\text{subindex} \leftarrow 1$; $\text{class}_{\text{index.subindex}} \leftarrow \text{class}_{\text{index}}[1]$
    if $d_{k,\text{class}_{\text{index}}[i]} = d_{k,\text{class}_{\text{index}}[i-1]}$ **then**
        $\text{class}_{\text{index.subindex}} \leftarrow \text{class}_{\text{index.subindex}} \cup \text{class}_{\text{index}}[i]$
    else if $d_{k,\text{class}_{\text{index}}[i]} < d_{k,\text{class}_{\text{index}}[i-1]}$ **then**
        $\text{subindex} \leftarrow \text{subindex} + 1$; $\text{class}_{\text{index.subindex}} \leftarrow \{\text{class}_{\text{index}}[i]\}$
    end if
end for
$\text{Returnset} \leftarrow \emptyset$; $A \leftarrow \emptyset$
for $j = 1$ to subindex **do**
    $A \leftarrow \text{classify}(\text{class}_{\text{index.j}}, D, k+1)$; $\text{Returnset} \leftarrow A \cup \text{Returnset}$
end for
return Returnset

Time complexity

The time complexity of Algorithm 1 is $O(n.|W^m| + n^2 log(n))$

Proof.

- The time complexity of $CHECK - FCB$ is $O(n.|W^m|)$.
- The time complexity of computing matrix D is $O(\mathrm{Max}(|W^m|, n^2)$.
- For each iteration, sorting of sublists requires at most $O(nlog(n))$ time.
- There are at most n loops.
- Therefore the total time complexity is $O(n.|W^m|) + O(\mathrm{Max}(|W^m|, n^2) + O(n^2 log(n)) = O(n.|W^m| + n^2 log(n))$.

$\square$

Linearity

Corollary

The strict desirability ordering of players in a linear simple game $v = (N, W)$ can be computed in polynomial time.

Proof.

The proof follows directly from the Algorithm. Moreover, we know that the set of all winning coalitions can be transformed into a set of minimal winning coalitions in polynomial time. $\square$

Summary

Table: Summary

Input $\longrightarrow$	(N, W)	(N, W^m)	WVG	MWVG
IS-LINEAR	P	P	(Always linear)	NP-hard
DESIRABILITY-ORDERING	P	P	P	NP-hard
STRICT-DESIRABILITY-ORDERING	P	P	NP-hard	NP-hard

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Holler index and Deegan Packel index

Definitions

We define the **Holler value** M_i as $\{S \in W^m : i \in S\}$. The **Holler index** which is called the *public good index* is defined by $H_i(v) = \frac{|M_i|}{\sum_{j \in N} |M_j|}$.

Definitions

The **Deegan Packel index** for player i in voting game v is defined by $D_i(v) = \frac{1}{|W^m|} \sum_{S \in M_i} \frac{1}{|S|}$.

Complexity to compute power indices

- It is NP-hard to compute the Banzhaf index, Shapley-Shubik index and Deegan-Packel index of a player [Matsui and Matsui,2000].

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- Similarly, one can prove that it is NP-hard to compute the Holler index of players in a WVG. This follows directly from the fact that it is NP-hard to decide whether a player is dummy or not.
- Prasad and Kelly [1990] and Deng and Papadimitriou [1994] proved that for WVGs, computing the Banzhaf values and Shapley-Shubik values respectively is #P-complete.

Complexity

Proposition

For a simple game (N, W^m), the Holler index and Deegan-Packel index for all players can be computed in linear time.

Proof.

We examine each of the cases separately:

- Initialize M_i to zero. Then for each $S \in W^m$, if $i \in S$, increment M_i by one.

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- Initialize d_i to zero. Then for each $S \in W^m$, if $i \in S$, increment d_i, by $\frac{1}{|S|}$. Then $D_i = \frac{d_i}{|W^m|}$.

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$\square$

Complexity of Power indices

Proposition

For a simple game $v = (N, W)$, Banzhaf index, Shapley Shubik index, Holler index and Deegan-Packel index can be computed in polynomial time.

Proof The proof follows from the definitions. We examine each of the cases separately:

- Holler index: Transform W into W^m. This can be done in polynomial time.
- Deegan-Packel: Transform W into W^m. This can be done in polynomial time.

Complexity of Power indices

- Banzhaf index: Initialize Banzhaf values of all players to zero. For each $S \in W$, check if the removal of a player results in S becoming losing (not a member of W). In that case increment the Banzhaf value of that player by one. The time complexity of the algorithm is polynomial in the order of input.
- Shapley-Shubik index: Initialize Shapley value of all players to zero. For each $S \in W$, check if the removal of a player result in S becoming losing (not member of W). In that case increment the Shapley value of the player by $(|S|-1)!(n-|S|)!$. The time complexity of the algorithm is polynomial in the order of input.

Complexity of computing Banzhaf values in (N, W^m)

Proposition

For a simple game $v = (N, W^m)$, the problem of computing the Banzhaf values of players is #P-complete.

Proof:

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- It is known that counting the number of vertex covers is #P-complete.

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- Now we take a game $v = (N, W^m)$ and convert it into another game $v' = (N \cup \{n+1\}, W^m(v'))$ where for each $S \in W^m(v)$, $S \cup \{n+1\} \in W^m(v')$.

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Conclusion

- We examined the complexity of comparison of influence of players from different angles.
- For a simple game represented by minimal winning coalitions, although it is easy to verify whether a player has zero or one voting power, computing the Banzhaf value of the player is #P-complete.
- For a simple game with a set W^m of minimal winning coalitions, an algorithm to compute desirability ordering is presented.
- MWVGs are the only representations for which it is NP-hard to verify whether the game is linear or not.

Summary

Table: Summary of results

Input $\longrightarrow$	(N, W)	(N, W^m)	WVG	MWVG
IDENTIFY-DUMMIES	P	Linear	NP-hard	NP-hard
IDENTIFY-VETOERS	linear	linear	linear	linear
IDENTIFY-PASSERS	linear	linear	linear	linear
IDENTIFY-DICTATOR	linear	linear	linear	linear
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BANZHAF-INDICES	P	?	NP-hard	NP-hard
SHAPLEY-SHUBIK-VALUES	P	?	#P-complete	#P-complete
SHAPLEY-SHUBIK-INDICES	P	?	NP-hard	NP-hard
HOLLER-INDICES	P	Linear	NP-hard	NP-hard
DEEGAN-PACKEL-INDICES	P	Linear	NP-hard	NP-hard

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STRICT-DESIRABILITY-ORDERING	P	P	NP-hard	NP-hard
BANZHAF-VALUES	P	#P-complete	#P-complete	#P-complete
BANZHAF-INDICES	P	?	NP-hard	NP-hard
SHAPLEY-SHUBIK-VALUES	P	?	#P-complete	#P-complete
SHAPLEY-SHUBIK-INDICES	P	?	NP-hard	NP-hard
HOLLER-INDICES	P	Linear	NP-hard	NP-hard
DEEGAN-PACKEL-INDICES	P	Linear	NP-hard	NP-hard

Conclusion

It is conjectured that computing Shapley values and Shapley-Shubik indices is #P-complete and it is NP-hard to compute Banzhaf indices for a simple game represented by (N, W^m).

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