On the Complexity of Rationalizing Behavior

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  - Rationalization by multiple rationales (Kalai, Rubinstein, and Spiegler 2002; KRS): behavior is rationalized through a collection of linear orders. For every choice problem there is a linear order that rationalizes it.
  - It is as if the DM had in mind a partition of the set of choice problems, and applies one rationale to each element of the partition.

RATIONALIZATION BY MULTIPLE RATIONALES

- Definition (CC, CF)

  Given a set of elements X and a domain $\mathcal{D} \subseteq \mathcal{U}$, a map $c : \mathcal{D} \rightarrow \mathcal{U}$ is a **choice correspondence** if for every $A \in \mathcal{D}$, $c(A) \subseteq A$. If for every $A \in \mathcal{D}$, $c(A)$ is a singleton, we say that c is a **choice function**.

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- Definition (RMR)

  A K-tuple of complete preorders $(\succ_k)_{k=1,\ldots,K}$ on X is a **rationalization by multiple rationales** (RMR) of choice correspondence c if for every $A \in \mathcal{D}$, the set of elements $c(A)$ is $\succ_k$-maximal in A for some k.

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- There are multiple books of rationales that can rationalize a given choice behavior. KRS propose to focus on those that use the **minimal** number of rationales.

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- Restriction of choice behavior. The choice correspondence satisfies the well-known consistency property known as the *weak axiom of revealed preference* (WARP). In other words, the minimal number of rationales is 1 with certainty. The problem is polynomial.

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- We will be able to draw a connection with a natural graph theory problem.
- This is especially useful since there is a wealth of algorithms for graph problems that may be used to solve the problem of rationalization of certain choice structures.

THE MOST GENERAL CASE

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Partition into Cliques (PIC): Given a graph $G = (V, E)$, can the vertices of G be partitioned into $k \leq K$ disjoint sets $V_1, V_2, \ldots, V_k$ such that for $1 \leq i \leq k$ the subgraph induced by V_i is a complete graph?

RESTRICTION OF CHOICE BEHAVIORS

- c-**Maximal Sets**: A subset $S \in \mathcal{D}$ is said to be c-maximal if for all $T \in \mathcal{D}$, with $S \subset T$, it is the case that $c(S) \neq c(T)$. Denote the family of c-maximal sets under the choice domain $\mathcal{D}$ by $M_c^{\mathcal{D}}$.

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- **Weak Axiom of Revealed Preference (WARP)**: Let $A, B \in \mathcal{D}$ and assume $x, y \in A \cap B$; if $x = c(A)$ then $y \neq c(B)$.

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Let the choice function c be a rational procedure on $\mathcal{D}$. Then $|M_c^{\mathcal{D}}| \leq |X| - 1$ and the problem of finding the linear order $\succ$ that rationalizes c is polynomial.

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This result remains an open question for the case of Choice Correspondences (RCP-$\mathcal{U}$).

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Cycle: The collection $\{A_t\}_{t=1}^{n} \in M_c^{\mathcal{D}}$, $n \geq 2$, is a cycle if $A_1 = A_n$ and for every $i \in \{1, \ldots, n-1\}$, $A_i \rightarrow A_{i+1}$.

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Partition into DAGs A partition of $M_c^{\mathcal{D}}$ $\{V_p\}_{p=1,\ldots,P}$ is said to be a Partition into DAGs if every class V_p is a DAG, i.e., it admits no cycle. It is said to be minimal if any other Partition into DAGs has at least P classes.

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-If $\{\succeq_p\}_{p=1,\dots,P}$ *is a minimal RMR, then there is a minimal Partition into DAGs* $\{V_p\}_{p=1,\dots,P}$ *of* $M_c^{\mathcal{D}}$ *where all the choice problems explained by any rationale are grouped together.*

-If $\{V_p\}_{p=1,\dots,P}$ *is a minimal Partition into DAGs of* $M_c^{\mathcal{D}}$*, then there is a minimal RMR* $\{\succeq_p\}_{p=1,\dots,P}$ *where all the choice problems in the same equivalence class are explained by the same rationale.*

Conclusions

- In the case of single-valued choice functions, it is the conjunction of unstructured choice and unrestricted domain that drives the intractability result.

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- In the choice correspondences case, it may well be the case that the difficulty in finding a minimal book is triggered by choice behavior per se.
- Graph Theory has mainly focused on the relevance of the Maximal DAG problem. This paper provides an intuitive application of the Partition into DAGs problem. Literature?