Products or Sums?

In the standard formulation of quantum mechanics, projection operators play a vital role. The spectral theorem describing self-adjoint operators, Wigner’s theorem describing unitary and anti-unitary operators, and Gleason’s theorem describing states, place the lattice of projection operators at the center of descriptions of states, observables, and evolutions of quantum systems. Further, projections have direct interpretation as the yes/no questions of a system, hence are tied to the logic of the system, and are the correct vehicle for techniques in representation theory. Projections also retain a link to classical notions of projective geometry and a type of geometric view of Hilbert space theory.

Projections correspond to sums for Hilbert spaces, and the idea of considering sums of more general structures as vehicles for studies in quantum mechanics is an old, and unsuccessful one, essentially ended by the Amemiya-Araki theorem. One cannot move more than incrementally from Hilbert space and retain appropriate structure in the sums.

Projections also correspond to direct products for Hilbert spaces. Here the situation is far different. The direct product factorizations of any set, group, ring, vector space, topological space, uniform space, etc. have a structure of basically the appropriate form for use in quantum mechanics, that of an OMP.

We discuss the program of replacing the projections of a Hilbert space with the direct product factorizations of more general objects in treatments of quantum mechanics. In particular, we discuss the central themes of the first paragraph in this context.